University of Toronto
Faculty of Arts and Science
December 2008 Examinations

CSC 438H1F/2404H

Duration - 3 hours
No Aids Allowed

There are 9 questions worth a total of 122 marks.
Answer all questions on the question paper, using backs of pages for scratch work.
Check that your exam book has 12 pages (including this cover page).

PLEASE COMPLETE THIS SECTION:

Name ______________________________

(Please underline your family name.)

Student Number ______________________________

FOR USE IN MARKING:

1. ____________/ 6
2. ____________/ 6
3. ____________/20
4. ____________/10
5. ____________/10
6. ____________/24
7. ____________/ 6
8. ____________/16
9. ____________/24

Total:__________/122

[6] 1. Recall that $Con(PA)$ is a true formula of $\mathcal{L}_A$ asserting the consistency of **PA**. Prove that ther is a consistent extension $\mathbf{PA}'$ of **PA** such that $\mathbf{PA}' \vdash \neg con(PA)$.

[6] 2. If $a, b \in \mathbf{N}$, we say that a is a *quadratic residue* of b if there is $x \in \mathbb{N}$ such that $x < b$ and

$$x^2 \equiv a \ (\mathrm{mod} b)$$

(this means $x^2 - a$ is divisible by b). Let the relation $QR(a, b)$ hold iff a is a quadratic residue of b. Prove that $QR(a, b)$ is a primitive recursive relation. You may use any results in the Notes.

[20] 3. Define the set $EVEN$ to be the set of even numbers in $\mathbb{N}$. Let

$$A = \{x \mid dom(\{x\}_1) = EVEN\}$$

Is A r.e.? Is $\overline{A}$ r.e. Justify your answers (but do not use Rice's Theorem).
You may continue your solution on the next page.

(Continue your solution to Question 1 here).

[10] 4. Give an LK proof of the sequent

$$\forall x(x + 0 = x) \rightarrow \forall x \forall y(x + (y + 0) = x + y)$$

You do not need to put in weakenings or exchanges. Indicate which LK equality axioms you use.

[10] 5. Let $R(x, y)$ be a recursive relation. Give a primitive recursive relation $S(x, z)$ such that for all $x \in \mathbb{N}$

$$\exists y R(x, y) \Leftrightarrow \exists z S(x, z)$$

Justify our answer briefly.

6. Let $\mathcal{L}_s$ (the vocabulary of successor) be the vocabulary $[0, s; =]$. Let $Th(s)$ (theory of successor) be the set of all sentences over this vocabulary which are logical consequences of the following infinite set Ψ_s of axioms:

P1) $\forall x(sx \neq 0)$
P2) $\forall x \forall y(sx = sy \supset x = y)$
Q) $\forall x(x = 0 \vee \exists y(x = sy))$ (every nonzero element has a predecessor)

S1) $\forall x(sx \neq x)$
S2) $\forall x(ssx \neq x)$
S3) $\forall x(sssx \neq x)$
.
.
.

[10] (a) Prove that for each $n \geq 1$ the axiom Sn is not a logical consequence of {P1,P2,Q,S1,S2,...S$n-1$}. (Do this by giving a model.)

[8] (b) Prove using (a) that $Th(s)$ is not finitely axiomatizable. That is, show that there is no finite set Γ of sentences in $Th(s)$ such that every sentence in $Th(s)$ is a logical consequence of Γ. (Note that the sentences in Γ are not necessarily among the original set Ψ_s of axioms.)

[6] (c) Use the fact that every sentence true in the standard model $\underline{\mathbb{N}}_s$ for the language $\mathcal{L}_s$ is in $Th(s)$ to show that $Th(s)$ is decidable.

[6] 7. Let $F(x,y)$ be a total computable function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. For each $e \in \mathbb{N}$ let $g_e(x) = F(e,x)$ for all $x \in \mathbb{N}$. Show that there is a total unary computable function $f(x)$ such that f is not in the list $g_0, g_1, g_2, \ldots$.

8. Let Σ be a theory over the vocabulary $\mathcal{L}_A$. We say that a function $f : \mathbb{N} \to \mathbb{N}$ is *provably total* in Σ if there is an $\exists\Delta_0$ formula $A(x,y)$ which represents the relation $(y = f(x))$ (the graph of f) and further

$$\Sigma \vdash \forall x \exists ! y A(x,y)$$

Here $\exists ! y A(x,y)$ stands for the formula $\exists y (A(x,y) \wedge \forall z (A(x,z) \supset z = y))$, and it intuitively means there is a unique y satisfying $A(x,y)$.

[8] (a) Show that if a function $f : \mathbb{N} \to \mathbb{N}$ is provably total in **TA** then f is computable. Show this by giving a high level algorithm for computing f. Refer to the $\exists\Delta_0$ formula $A(x,y)$ which represents the graph of f.

[8] (b) Prove that not all total computable functions $f : \mathbb{N} \to \mathbb{N}$ are provably total in **PA**. You may use Church's Thesis.
(Hint: Use the results of questions 7 and 8a, and the completeness of LK.)

9. Let $\mathcal{L}$ be a language which includes the constant symbol 0 and the unary function symbol s, so that the numerals s_n are $\mathcal{L}$-terms. ($\mathcal{L}$ may have other function symbols.) Let Γ be a *finite* set of $\forall$-sentences in $\mathcal{L}$, where a $\forall$-sentence is one of the form $\forall x_1 \cdots \forall x_n B$, where B is a quantifier-free formula. Let

$$\Sigma = Th(\Gamma) = \{A \mid \Gamma \models A \text{ and } A \text{ is an } \mathcal{L} \text{ sentence}\}$$

Suppose that every r.e. relation $R(x)$ is representable in Σ by a formula of the form $\exists y A(x, y)$, where $A(x, y)$ is quantifier-free.

[8] (a) Prove that Σ is undecidable and incomplete.

[8] (b) Prove from part (a) that the set of valid sentences in $\mathcal{L}$ is undecidable.

[8] (c) Prove that the set of satisfiable $\forall$-sentences in $\mathcal{L}$ is undecidable.