

FIRST-ORDER RESOLUTION

(See Buss, Intro to Proof Theory)

We now consider another proof system for predicate logic that is also sound + COMPLETE and very popular in theorem proving.

FO RES will be an extension of (propositional) RES.

How to use Resolution as a FIRST ORDER theorem prover?

- Convert an arbitrary formula into a generalized CNF formula. Terms "stand in" for quantifiers (Normal Form)
- generalize Resolution to apply to CNF formulas, but now containing terms (Herbrand Thesis)
(unification)

Conversion to Normal Form

① Convert A to prenex normal form, $A' = Q_1 x_1 \dots Q_n x_n B$
 Q_i is $\exists$ or $\forall$ quantifier-free CNF

- use unique variable names in quantification

- move negation inwards
 $\neg \forall x B \Leftrightarrow \exists x \neg B$
 $\neg \exists x B \Leftrightarrow \forall x \neg B$

- move quantifiers to left
 $(\forall x B \wedge C) \Leftrightarrow \forall x (B \wedge C)$
 $(\forall x B \vee C) \Leftrightarrow \forall x (B \vee C)$
 $(\exists x B \wedge C) \Leftrightarrow \exists x (B \wedge C)$
 $(\exists x B \vee C) \Leftrightarrow \exists x (B \vee C)$

② Convert B to CNF by distributivity

② Skolemize to eliminate all existentially quantified variables. } functional form

Remove each $\exists y$ in prefix

Replace y in formula by $f_y(x_1 \dots x_k)$

where $x_1 \dots x_k$ are the universally quantified vars preceding $\exists y$; f_y a new function symbol

③ Apply a variable renaming so that for any 2 clauses C_1, C_2 , the variables in C_1 are disjoint from variables in C_2 .
Remove universal quantification (assumed implicitly)

Example

$$\forall x ((A(x) \vee \neg \exists y B(x, y)) \wedge \exists z C(z, x))$$

$$(1) \quad \forall x \exists z \forall y [(A(x) \vee \neg B(x, y)) \wedge C(z, x)]$$

prenex
and CNF

$$(2) \quad \forall x \forall y \underbrace{[(A(x) \vee \neg B(x, y))]}_{\text{Clause 1}} \wedge \underbrace{C(fx, x)]}_{\text{Clause 2}}$$

Diagram showing the transformation of the formula into clauses. Arrows point from the underbraced parts of the formula in (2) to the word "Clauses".

$$(3) \quad (A(x) \wedge \neg B(x, y)) \wedge C(fz, z)$$

Theorem Let A be a FO formula, A' the normal form of A . Then A is satisfiable iff A' is satisfiable.

Same holds for Φ a set of formulas.

Unification (see Buss chapter 1, 2.6)

The job of unification is to take two atomic formula p, q and return the most general substitution that makes p and q syntactically identical.

t is a term containing function + constant symbols and variables.

A substitution σ is a partial map from variables to terms.

Let $A_1, \dots, A_k$ be atomic formulas. A unifier for $\{A_1, \dots, A_k\}$ is a substitution σ such that

$$A_1\sigma = A_2\sigma = \dots = A_k\sigma$$

syntactic equivalence

A unifier σ for $A_1, \dots, A_n$ is a most general unifier if $\forall$ unifiers τ for $A_1, \dots, A_n$, $\exists \rho$ s.t. $\tau = \sigma\rho$
(apply σ then ρ)

σ most general if $\forall$ unifiers τ for same set, $\exists \rho$ s.t. $\tau = \sigma\rho$.

Claim up to var renaming, a most general unifier is unique

UNIFICATION ALGORITHM

Input : Atomic formulas $A(\bar{p})$, $A(\bar{q})$ in

Output : the most general unifier, σ , of $\bar{p}$ and $\bar{q}$ if one exists

At stage s , algorithm maintains

E_s : set of equations to unify

G_s : substitution

(i) Initially $E_0 = \{R=q_1, P_2=q_2, \dots, P_n=q_n\}$

$$G_0 = \{x_i = x, y_i = y, \dots\} \quad (\text{identity})$$

(2) stage $s+1$ (given E_s, ϵ_s)

- If $E_s = \phi$, halt and return G_s

- If E_s contains an equation

(*) $F(t_1, \dots, t_i) = F(t'_1, \dots, t'_i)$, then
 ↗ function symbol

$$b_{s+1} = b_s$$

E_{s+1} : remove (*) from E_s

and add $t = t'$

$$t_2 = t'_2$$
$$t_i = t'_i$$

(3) If E_s contains an equation

$$(**) \quad x = t$$

↑ ↖
variable term

If $t = x$, $E_{s+1} = E_s$ but with $(**)$ removed

If $t \neq x$ but contains x , halt + return FAIL

otherwise

E_{s+1} is the set of equations $s[x/t] = s[x/t]$

where $s = s'$ is in E_s

$$G_{s+1} = G_s[x/t]$$

↑
substitution mapping x to t
 $x := t$

Example

$$\text{Unify} [P(x, F(x, A)), P(H(y), z)]$$

$$1. E_0 = P(x, F(x, A)) = P(H(y), z) \\ \sigma = \text{identity} \quad x := x \quad y := y \quad z := z$$

$$2. E_1 = \{x = H(y), F(x, A) = z\} \\ \sigma_1 = \sigma.$$

$$3. E_2 = \{F(H(y), A) = z\} \\ \sigma_2 = x := H(y) \quad y := y \quad z := z$$

$$4. E_3 = \{\} \\ \sigma_3 = x := H(y) \quad y := y \quad z := F(H(y), A)$$

Generalized Resolution

Let A be a FO formula, and let A' be the normal form of A .

$$A' = \forall B_1 \wedge \dots \wedge B_m$$

↑
"clauses" or disjunctions of atomic formulas

A Resolution refutation of A' is a sequence of atomic formulas $L_1, L_2, \dots, L_q$ where each L_i is either a clause of A or follows from two previous L_k 's by the generalized Resolution rule:

Let B be a clause containing atomic formulas $P(S_1), \dots, P(S_k)$ (and possibly other atomic formulas)

Let C be a clause containing atomic formulas $\neg P(E_1), \dots, \neg P(E_\ell)$

Let σ be a most general unifier for $\{P(S_1), \dots, P(S_k), P(E_1), \dots, P(E_\ell)\}$

Derive clause $D = (B\sigma \setminus P(S_1)\sigma) \cup (C\sigma \setminus \neg P(E_1)\sigma)$
↗
resolve away all $P(S_1)\sigma, P(E_1)\sigma$

FO completeness Theorem for Resolution

A is unsatisfiable iff there is a First order Resolution refutation of A' (the normal form of A).

Completeness Proof

Defn A ground literal is an atomic formula or the negation of an atomic formula in which no variables occur.

A ground clause is the disjunction of a set of ground literals.

FACT By completeness of Propositional Resolution, a set of ground clauses is unsatisfiable iff it has a (ground) Resolution refutation.

Now let A be a set of unsatisfiable FO clauses. (A' are the clauses in the normal form of A , and may not consist of ground literals.)

By Herbrand's Theorem there is a set of substitutions $\sigma_1, \dots, \sigma_r$ s.t. each $A'\sigma_i$ is a set of ground clauses and so that $\bigcup_i A'\sigma_i = \Pi$ is propositionally unsatisfiable.

This, there is a ground resolution refutation of Π .

Show Any ground RES refutation can be lifted to a FO RES refutation of A' .

Lifting Ground Resolution to FO Resolution

Ground Literal: no variables

Ground Clause: a set of ground literals

By completeness of Propositional Resolution, a set of ground clauses is unsat. iff it has a ground Res refutation

- Let f be an unsat. FO formula, f' the normal form of f . By Herbrand's Theorem, there is a set of substitutions $\sigma_1, \sigma_2, \dots, \sigma_r$ such that $\Pi = \bigcup_i f' \sigma_i$ is an unsatisfiable set of ground clauses
- By prop. Res completeness, Π has a Res refutation
- show any ground Res refutation of Π can be lifted to a FO Res ref of f' .

Show If $C_1, C_2, \dots, C_n = \phi$ is a Res refutation of Π ,
 $\exists D_1, D_2, \dots, D_m = \phi$ which forms a FO Res refutation of f'
and there are substitutions $\sigma_1, \sigma_2, \dots, \sigma_n$ such that $D_i \sigma_i = C_i$

Pf by induction on i

① $C_i \in \Pi$. Then $C_i = D_i \sigma_i$ for some $D_i \in f'$ and some σ_i

② $\frac{C_j \quad C_k}{C_i}$ on literal $P(\bar{r})$.

Define E_j to be the subset of D_j which mapped to $P(\bar{r})$ by σ_j . Likewise for E_k .

then unify E_j, E_k and resolve to get D_i