Soundness

Soundness is again proved by showing that the rules are sound. Now we want to prove that if A, B are valid then so is C, where C is derived from A and B.

- It is clear that if C is valid, then so is $C[x/t]$.
- We have already seen that the Resolution rule is sound.

Completeness

Assume that S is in normal form, and is unsatisfiable.

We want to prove that S has a generalized Resolution refutation.

A *ground literal* is a literal in which no variables occur; a *ground clause* is a set of ground literals. Ground literals are like ordinary literals in propositional logic.

- By Herbrand's Theorem, some set S' of ground instances is unsatisfiable. S' is just an ordinary set of propositional clauses.

- By the completeness of propositional Resolution, there exists a Resolution refutation of S'

- By the Lefting Lemma, if there is a propositional Resolution refutation of S', then there is a generalized Resolution refutation of S.

Lifting Lemma–Example

Lemma. Let C_1 C_2 be two clauses with no shared variables and let C_1', C_2' be ground instances of C_1 and C_2. If C' is a resolvent of C_1' and C_2' then there exists C such that C is a resolvent of C_1 and C_2 and C' is a ground instance of C.

$$C_1 = \neg P(x, F(x, A)) \vee \neg Q(x, A) \vee R(x, B)$$

$$C_2 = \neg N(G(y), z) \vee P(H(y), z)$$

$$C = \neg N(G(y), F(H(y), A)) \vee \neg Q(H(y), A) \vee R(H(y), B)$$

$$C_1' = \neg P(H(B), F(H(B), A) \vee \neg Q(H(B), A) \vee R(H(B), B)$$

$$C_2' = \neg N(G(B), F(H(B), A)) \vee P(H(B), F(H(B), A))$$

$$C' = \neg N(G(B), F(H(B), A)) \vee \neg Q(H(B), A) \vee R(H(B), B)$$

Resolution control strategies

We saw that first-order Resolution is complete, but what is a good way to mechanistically search for a proof? Here are some common heuristics.

- **Breadth-first.** A first-level resolvent is one whose parents are both original clauses. An ith level resolvent is one whose deepest parent is an $(i-1)$st level resolvent. Breadth-first computes all 1st level, then all 2nd level, and so on.

  This method is complete but sometimes inefficient.

- **Set of Support.** Start by identifying a subset of the clauses/sentences called the *set of support*.

  Every resolution combines a clause/sentence from the set of support with another clause/sentence and adds the resolvent to the set of support.

  Can be incomplete, depending on the initial set of support.

- **Unit Resolution.** Always resolve on a unit clause (if possible).

- **Input Resolution.** Every resolution combines one of the input sentences with another sentence.

  Incomplete in general.

  Theorem. A set of clauses has a unit refutation if and only if it has an input refutation.

- **Linear Resolution.** A linear resolution refutation of Γ is a refutation $A_1, A_2, ..., A_n$ such that each A_i is either in Γ or is obtained by resolution from A_{i-1} and A_j for some $j < i - 1$.

  Linear Resolution is complete.

Summary of Method

- Start with a knowledge base of first-order sentences $f_1, ..., f_k$. We want to know if another first-order sentence g logically follows from $f_1, ..., f_k$. (i.e., if for every model where $f_1, .., f_k$ are true, g is also true.) To show this, it suffices to show that $F = f_1 \wedge ... \wedge f_k \wedge \neg g$ is unsatisfiable.

  Recall unsatisfiable means in every model, the sentence is false.

- Convert F to normal form to obtain F^N. The conversion has the property that F is unsatisfiable if and only if F^N is unsatisfiable.

- F^N has the form $A(x_1, .., x_n)$ where the x_i's are assumed to be universally quantified. F^N is unsatisfibale if and only if in every model, F^N is false.

- Apply generalized Resolution to yield a refutation of F^N. Generalized Resolution is sound and complete. Soundness means that if we can derive a generalized Resolution refutation (of the empty clause) from F^N, then F^N is unsatisfiable; completeness means that if F^N is unsatisfiable, then there exists a generalized Resolution refutation of F^N.

Conversion to Normal Form

Let F be a first order formula, and let F^N be the normal form of F (after moving quantifiers to the left, skolemizing, and then putting in CNF form). F is unsatisfiable if and only if F^N is unsatisfiable.

Steps 1 and 3 are easy.

Step 2 (Skolemization):

$$\forall x \exists y A(x,y) \Rightarrow \forall x A(x, f(x))$$

F^N satisfiable $\Rightarrow$ F satisfiable:

If F^N is satisfiable, then there is some model such that F^N is true. But F is also true in this same model.

F satisfiable $\Rightarrow$ F^N satisfiable:

Suppose that M is a model such that F is satisfiable. Extend the model to f by letting $f(x)$ be the element y in M such that $A(x,y)$ is true. F^N will be true in this extended model.

Summary of FO Logic

First-order logic is very expressive.

Generalized Resolution is a simple, sound and complete proof system for FO logic.

Major drawback: There is no general algorithm to decide if a given formula in FO logic is valid. (First order validity is semi-decidable.)

Again there are special cases of the validity problem for FO logic that are decidable, and sometimes even polynomial-time computable.

Bottom Line: Applications must be very careful to limit the problem to a tractable/computable case.

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