

RESOLUTION (RES)

Whereas PK is a propositional proof system (proves tautologies)
RES is a propositional refutation system (proves unsatisfiability)

Defn A literal is an atom P or its negation $\bar{P}$

A clause $C = (P \vee \neg Q \vee R \vee S)$ is a disjunction of literals.

A conjunctive normal form (CNF) formula

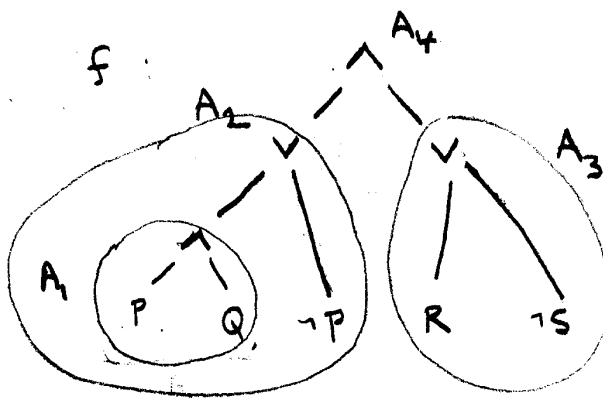
$f = C_1 \wedge C_2 \wedge \dots \wedge C_m$ is a conjunction of clauses.

3SAT

Theorem Let f be any propositional formula.

Then there exists a CNF formula f' such that

(1) $|f'| \leq 100 \cdot |f|$, and (2) $f' \leftrightarrow f$.



$|f|$ = number of $\wedge, \vee$ connectives in f

$$\begin{aligned}
 f' : & (A_1 \leftrightarrow P \wedge Q) \wedge (A_2 \leftrightarrow A_1 \vee \neg P) \wedge (A_3 \leftrightarrow R \vee \neg S) \wedge (A_4 \leftrightarrow A_2 \wedge A_3) \wedge (A_4) \\
 \equiv & (\neg A_1 \vee P) \wedge (\neg A_1 \vee Q) \wedge (\bar{P} \vee \bar{Q} \vee A_1) \wedge (\bar{A}_2 \vee A_1 \vee \neg P) \wedge (\bar{A}_1 \vee A_2) \wedge (P \vee A_2) \wedge (\bar{A}_3 \vee R \vee \neg S) \wedge (\neg R \vee A_3) \wedge (S \vee A_3) \wedge (\bar{A}_4 \vee A_2) \wedge (\bar{A}_4 \vee A_3) \wedge (\bar{A}_2 \vee \bar{A}_3 \vee A_4) \wedge A_4
 \end{aligned}$$

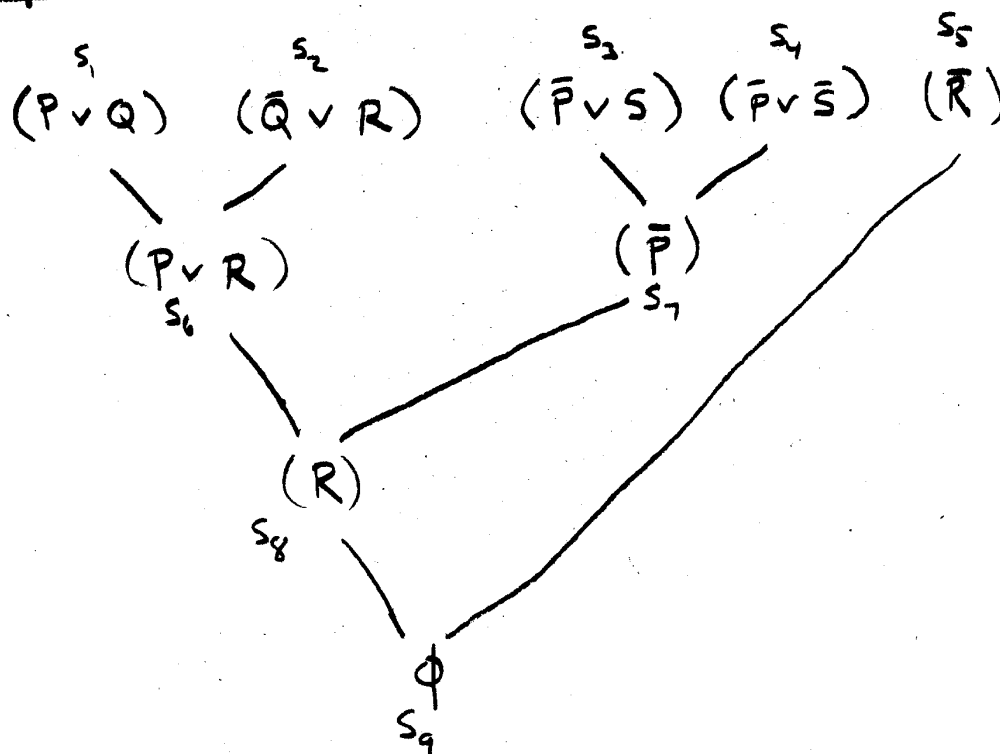
Resolution Rule Let $C_1 = (l \vee A)$, $C_2 = (\bar{l} \vee B)$

where l is a literal, A, B clauses.

Then C_1, C_2 derives $C_3 = (A \vee B)$.

A resolution refutation of an unsatisfiable CNF formula $f = C_1 \wedge C_2 \wedge \dots \wedge C_m$ is a sequence of clauses $s_1, s_2, \dots, s_q$ such that the final clause $s_q = \phi$ (the empty clause); and all other clauses $s_i, i < q$ are either a clause in f , or follow from two previous clauses by the Resolution Rule.

Example $f = (P \vee Q) \wedge (\bar{Q} \vee R) \wedge (\bar{P} \vee S) \wedge (\bar{P} \vee \bar{S}) \vee (\bar{R})$



Let f be a formula in CNF.

Defn

Let f be an unsatisfiable formula,
and let f' be the equivalent UNSAT formula in CNF
given by the 3SAT theorem.

Then a Resolution refutation of f' is a resolution
refutation of f .

Let g be a tautology. Let $f = \neg g$, and let
 f' be the equivalent 3CNF for f given by the
3SAT formula. Then a Resolution proof for g
is a Res refutation of f' .

RES SOUNDNESS THEOREM

Let f be a CNF formula.

If f has a RES refutation, then f is unsatisfiable.

Let $f = C_1 \wedge \dots \wedge C_m$, and let $P = C_1, C_2, \dots, C_m, D_1, \dots, D_q = \phi$ be a
RES refutation of f . Prove by induction on i that if
 α is a satisfying assignment to $C_1 \wedge \dots \wedge C_m$ then α is a
satisfying assignment to $C_1 \wedge \dots \wedge C_m \wedge D_1 \wedge \dots \wedge D_i$. When $i=0$, trivial.
For D_{i+1} , assume D_{i+1} derived from z previous
clauses $(x \vee E) (\bar{x} \vee F) \Rightarrow D_{i+1} = (E \vee \bar{F})$. Then
 α satisfies D_{i+1} . But D_q is unsatisfiable, thus
so is f .

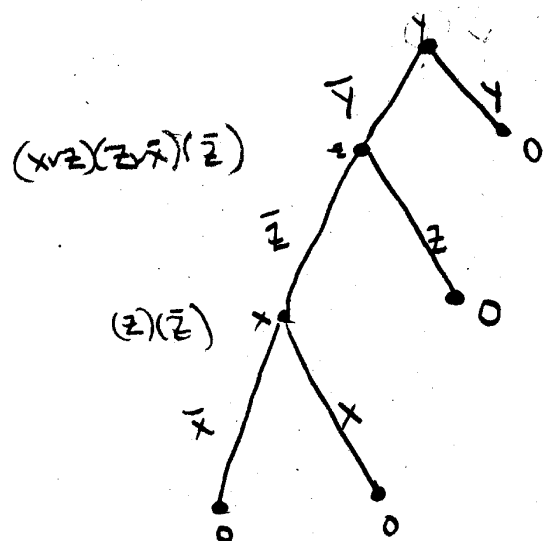
RES COMPLETENESS THEOREM

Let f be an unsatisfiable CNF formula. Then f has a RES refutation.

Def'n A decision tree for $f(x_1, \dots, x_n)$ is a rooted tree, depth $\leq n$.

- nodes of T labelled with variables x_i
- If a ~~node~~^{node} labelled x_i , its two outedges labelled by literals x_i and $\bar{x}_i$.
- A path labelling corresponds to a partial truth assignment
- A node n is a leaf iff $f|_c = 0$ or 1 , where c is the path labelling to n .

Example $f = (x \vee z)(\bar{z} \vee \bar{x})(y \vee \bar{z})(\bar{y})$



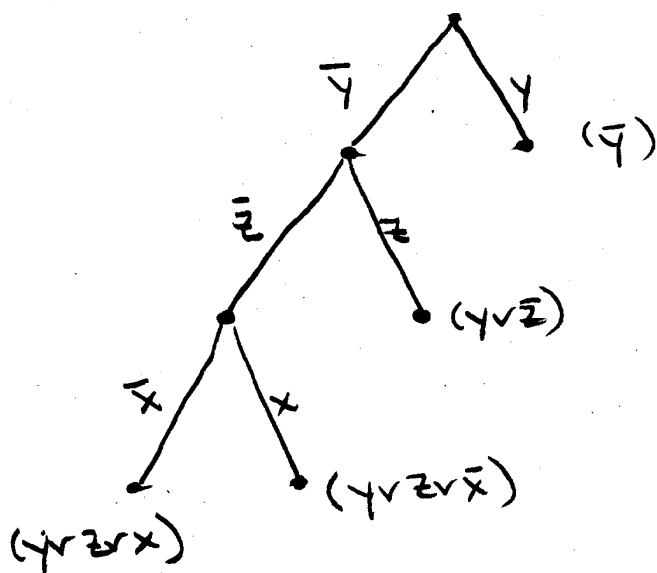
Claim f is unsatisfiable iff for any decision tree T for f , all leaves of T are labelled 0.

Let T be a decision tree for f , f an unsatisfiable CNF formula.

Then T can be converted into a RES refutation of a generalization of f , f' .

- If $l_1 \dots l_k$ labels path to n ,
 $\text{Clause}(n) = (\bar{l}_1 \vee \bar{l}_2 \vee \dots \vee \bar{l}_k)$.

Example (from previous page)



← RES refutation
of

$$f' = (y \vee z \vee x)(y \vee z \vee \bar{x})(y \vee \bar{z})(\bar{y})$$

$$f = (x \vee \bar{z})(z \vee \bar{x})(y \vee \bar{z})(\bar{y})$$

f' is a generalization of f if:

Def'n f' is a generalization of f iff
 for all clauses $C_i \in f$, $\exists C_j \in f'$ such that $C_i \leq C_j$.

Claim f' is a generalization of f .

Proof of claim

By construction of T for f , for each leaf node l of T , there exists a clause $c_l \in f$ that is falsified by the path to l .

By construction of T^* , for each leaf node l of T , the clause c_l^* associated with l is defined to be the maximum clause that is falsified by the path to l .

Thus, for each leaf l , $c_l \subseteq c_l^*$.

Proof of Resolution Completeness

1. Let $f(x_1, \dots, x_n)$ be an unsatisfiable CNF formula
2. Create a decision tree T for f . By claim, all leaves of T are labelled by 0.
~~For each leaf ℓ of T , there exists a clause $C_\ell \in f$ such that $\ell \models C_\ell$.~~
3. Convert T to a RES refutation Π^* of $f' = C'_1 \wedge \dots \wedge C'_m$, where f' is a generalization of f . That is, for all $C'_i \in f'$, there exists a clause $C_i \in f$ such that $C_i \leq C'_i$.
4. (Exercise) Show that for any f', f where f' is a generalization of f , a RES refutation of f' can be converted into a RES refutation of f .

~~Ordered~~
DLL Resolution Procedure

Order vars $x_1 - x_n$
Create tree, querying vars in order
unless there is a unit clause.