CSC 438F/2404F **Problem Set 6 solutions** **Fall, 1999**

1. Let $A(z)$ be the formula $x \cdot (y+z) = x \cdot y + x \cdot z$. We are going to use the following induction axiom:

$$\underbrace{\forall x \forall y}_{\text{parameters}} \quad \Big[\big[A(0) \wedge \forall z(A(z) \supset A(sz))\big] \supset \forall z A(z)\Big]$$

To prove the basis case $A(0)$, that is $x \cdot (y+0) = x \cdot y + x \cdot 0$, we use P3 and P5.

Now we prove the induction step $\forall z(A(z) \supset A(sz))$, that is

$$\forall z(x \cdot (y+z) = x \cdot y + x \cdot z \supset x \cdot (y+sz) = x \cdot y + x \cdot sz)$$

We have

$$\begin{aligned} x \cdot (y+sz) &= x \cdot s(y+z) \quad \text{(P4)} && (1)\\ &= x \cdot (y+z) + x \quad \text{(P6)} && (2)\\ &= (x \cdot y + x \cdot z) + x \quad \text{(IH)} && (3)\\ &= x \cdot y + (x \cdot z + x) \quad \text{(Associativity of +)} && (4)\\ &= x \cdot y + x \cdot sz \quad \text{(P6)} && (5) \end{aligned}$$

2. *Proof of Corollary* 1. In a bounded sentence you just have to check finitely many tuples of numbers to see if it is true, so from this alone we can conclude that the set of bounded sentences of TA is decidable. On the other hand, we can argue as follows: if A is a bounded sentence, then A or $\neg A$ is true, so by the Main Lemma either A or $\neg A$ is in RA. Since RA is r.e. we can list all the theorems of RA until A or $\neg A$ appears.

   Proof of Corollary 2. Suppose that $\exists y A(y) \in$ TA. Then $A(s_a) \in$ TA for some s_a. Since $A(s_a)$ is a Δ_0-formula, it follows that it is also in RA (Main Lemma). Since RA $\vdash A(s_a)$ it follows that RA $\vdash \exists y A(y)$.

   Proof of Corollary 3. By Corollary 2 and the soundness of RA, if follows that the $\exists\Delta_0$ sentences of TA are the same as the $\exists\Delta_0$ sentences of RA. Since RA is axiomatizable, it is r.e. Thus we can enumerate the $\exists\Delta_0$ sentences of RA by enumerating all sentences of RA and outputting those which are of the form $\exists\Delta_0$.

   The $\exists\Delta_0$ sentences of TA are not decidable because by the Exists Delta Theorem $K(x)$ is represented in TA by a $\exists\Delta_0$ formula.

3. Assume that there is a recursive set S satisfying the conditions. Define the relation R as $R(x) \iff \neg S(d(x))$. Then $R(x)$ is a recursive relation, and hence it is strongly represented in RA by some formula $A(x)$ (Theorem 4).

   Let $e = \#A(x)$, so $d(e) = \#A(s_e)$. Then

$$S(d(e)) \overset{1}{\iff} \neg A(s_e) \in \text{RA} \overset{2}{\iff} A(s_e) \in \neg\text{RA} \overset{3}{\iff} \neg S(d(e))$$

Contradiction, hence there is no such set S. To see 1, note that $\neg R(e) \iff \neg A(s_e) \in \mathrm{RA}$. To see 2, just use the definition of $\neg\mathrm{RA}$. The right direction of 3 is clear, and for the left direction note that $\neg S(d(e))$ implies $A(s_e) \in S^c$, but $A(s_e) \in \mathrm{RA}$ or $A(s_e) \in \neg\mathrm{RA}$, so $A(s_e) \in \neg\mathrm{RA}$.