CSC265 Fall 2011 Homework Assignment 3

due Thursday, November 24, 2011, 11:00am

1. (10 marks) The dealer at a casino has a new deck of (52) cards in her hand. The deck has never been played with and is in sorted order. Sadly the dealer does not know how to shuffle cards: All she can do is pick up the top card look at it and put it somewhere (chosen uniformly at random) in the deck. For example, the probability that she puts it back on top again is $1/52$. She keeps doing this until she picks up the card that was originally at the bottom of the deck and puts it somewhere in the deck.

   (a) Prove that when the dealer stops, all permutations of the 52 cards are equally likely.

   (b) What is the expected number of times that the dealer picks a card and puts it somewhere in the deck? Justify your answer.

2. (10 marks) Construct an algorithm, using a priority queue which, given a sequence $[a_1, b_1], \ldots, [a_n, b_n]$ of closed intervals on the real line, output the intervals in their union in $O(n \log n)$ time.
   For example, given the sequence of intervals [1,4], [7,12], [4,6], [14,16], [11,15], your algorithm should output the intervals [1,6] and [7,16].
   Explain why your algorithm is correct and runs in the required time.

3. (10 marks) An array $A[1..n]$ for all $i \in \{1, \ldots, n\}$, the rank of element $A[i]$ is between $i - k$ and $i + k$. Describe an algorithm that sorts any k-roughly sorted array A in $O(n \log k)$ time. Explain why your algorithm is correct and runs in the required time.

4. (15 marks) Suppose the numbers 1, 2, 3, 4, 5, 6, 7 are inserted into an empty heap H, where every order of insertion is equally likely. Compute the expected depth of the number 5 in the resulting heap. Explain your computation.

5. (20 marks) Suppose B is a set of 2^k elements that is represented by a binomial tree in the disjoint-set forests described in section 21.3 of the text. Let b be the element of B at the root of the tree and let c be the element of B with greatest depth.

   Let A be a set containing the single element a.

   (a) Prove by induction that the tree resulting from performing Union(a,b) followed by Find-Set(c) has the same shape as the tree resulting from performing Union(b,a,). Assume that path compression, as defined in the text, is used, but the union by rank heuristic is not used.

   (b) How much time does it take to perform Union(a,b) followed by Find-Set(c)? Justify your answer.

   (c) Use this to prove that the worst case time to perform a sequence of n Make-Set, Union, and Find-Set operations is in $\Omega(n \log n)$ if path compression, as defined in the text, is used, but the union by rank heuristic is not used.