CSC263 Week 12

Announcements

➔No tutorial this week

➔No office hours today (but the usual ones on Friday and Monday)

➔Extra office hours for final exam

Lower Bounds

So far, we have mostly talked about **upper-bounds** on algorithm complexity,
i.e., **O(n log n)** means the algorithm takes **at most** **cn log n** time for some **c**.

However, sometime it is also useful to talk about **lower-bounds** on algorithm complexity, i.e., how much time the algorithm **at least** needs to take.

Scenario #1

You, implement a sorting algorithm with worst-case runtime **O(n log log n)** by next week.

Okay Boss, I will try to do that ~

You try it for a week, cannot do it, then you are fired...

Scenario #2


You, implement a sorting algorithm with worst-case runtime **O(n log log n)** by next week.
No, Boss. **O(n log log n)** is below the **lower bound** on sorting algorithm complexity , I can't do it, **nobody** can do it!

Why learn about lower bounds

➔Know your limit

- we always try to make algorithms faster, but if there is a limit that you cannot exceed, you want to know

➔Approach the limit

- Once you have an understanding about of limit of the algorithm's performance, you get insights about how to approach that limit.

Lower bounds
on sorting algorithms

Upper bounds: We know a few sorting algorithms with worst-case **O(n log n)** runtime.

Is **O(n log n)** the best we can do?

Actually, yes, because the lower bound on sorting algorithms is **Ω(n log n)**, i.e., a sorting algorithm needs **at least** **cn log n** time to finish in worst-case.

actually, more precisely ...

The lower bound **n log n** applies to only all **comparison based** sorting algorithms, with **no assumptions** on the values of the elements.

It is possible to do faster than **n log n** if we make **assumptions** on the values.

Example: sorting with assumptions

Sort an array of **n** elements which are either **1** or **2**.

2	1	1	2	2	2	1

➔Go through the array, count the number of 1's, namely, **k**

➔Then output an array with **k** 1's followed by **n-k** 2's

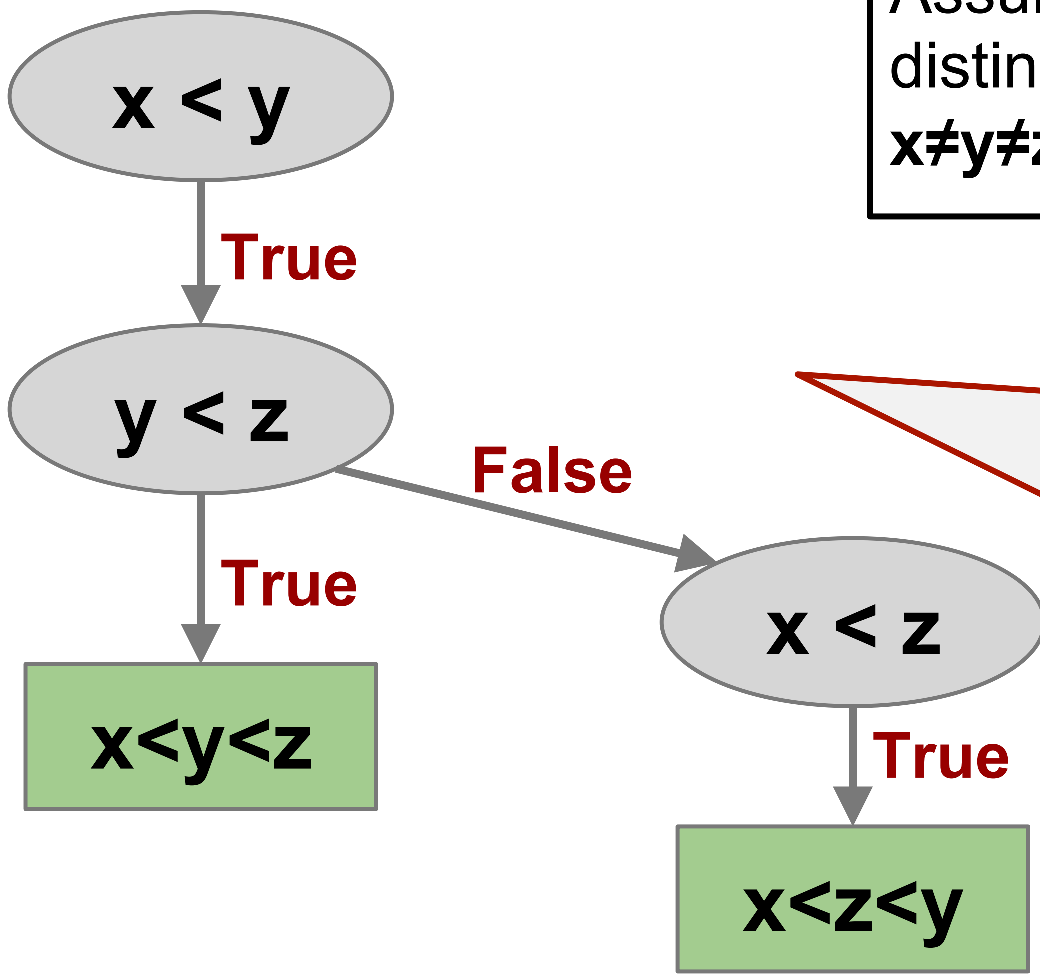
➔This takes **O(n)**.

Now prove it

the worst-case runtime of comparison based sorting algorithms is in $\Omega(n \log n)$

Sort {x, y, z} via comparisons

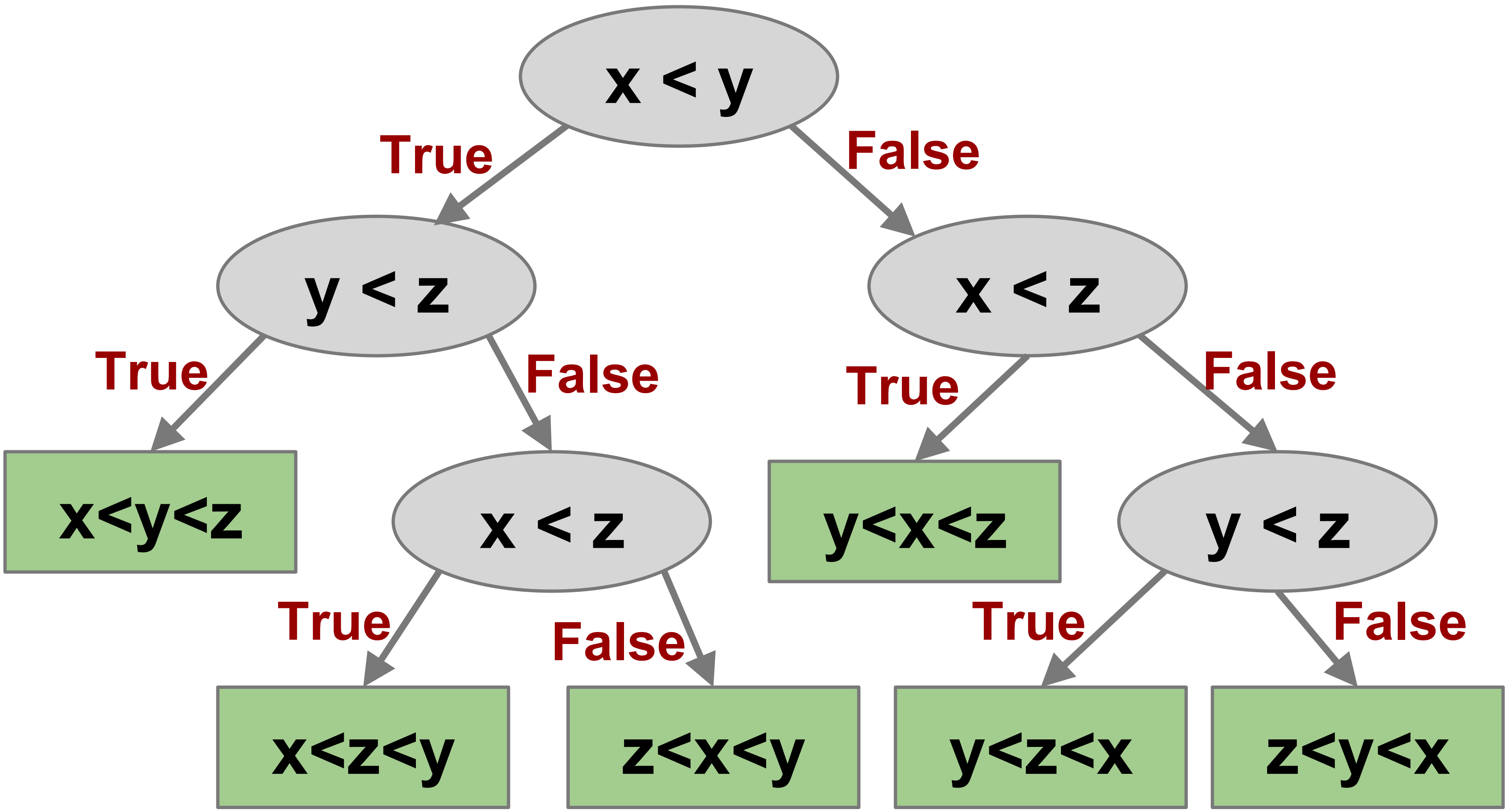
Assume **x, y, z** are distinct values, i.e., **x≠y≠z**



A tree that is used to **decide** what the sorted order of **x, y, z** should be ...

The **decision tree** for sorting {x, y, z}

a tree that contains a complete set of decision sequences

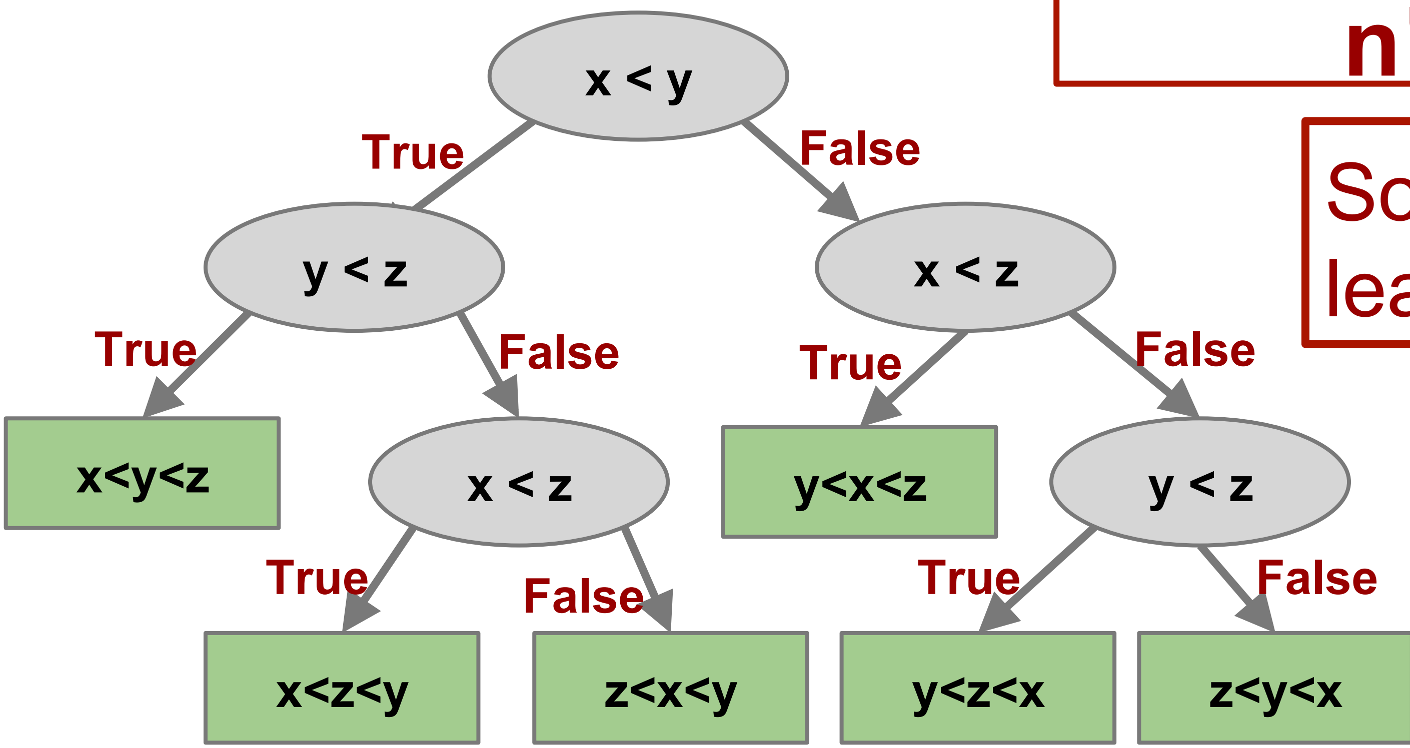


Each **leaf node** corresponds to a possible **sorted order** of {x, y, z}, a decision tree need to contain **all possible orders**.

How many possible orders for n elements?

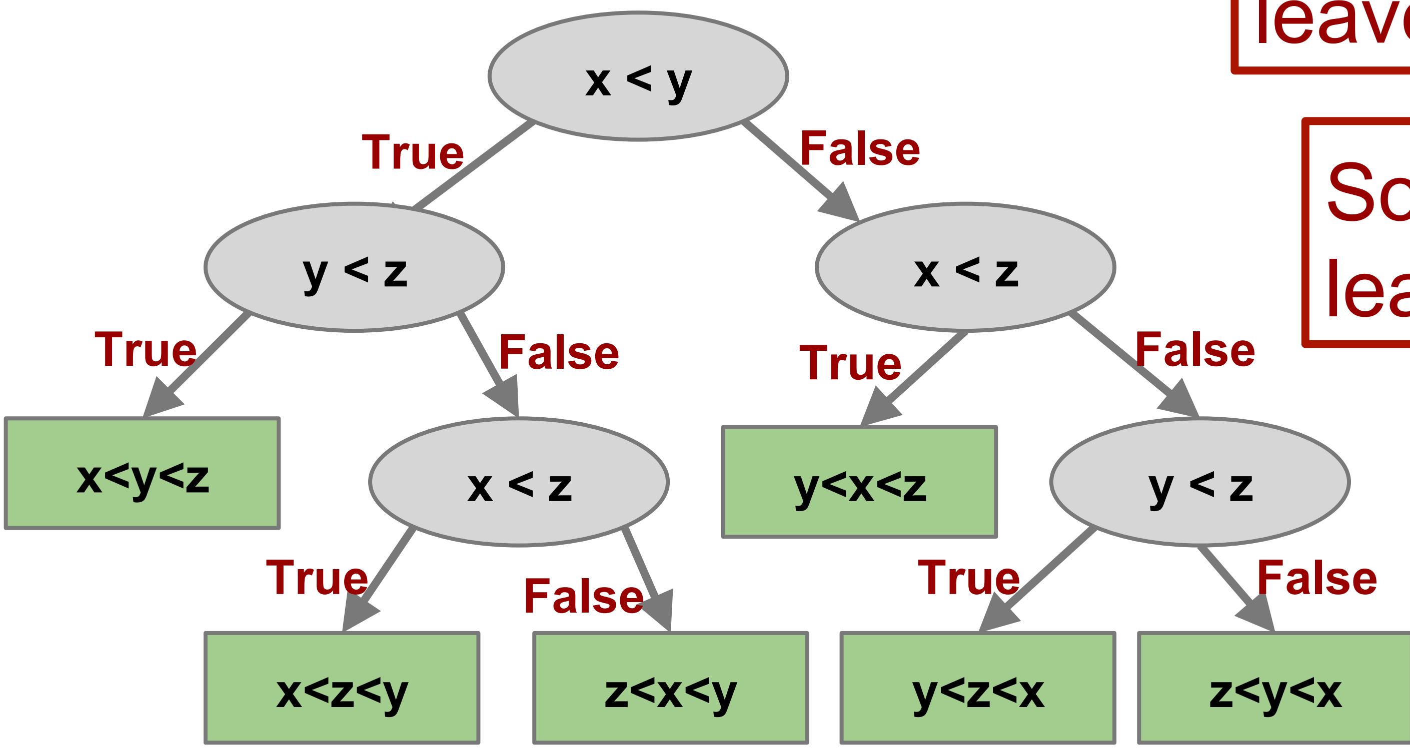
n!

So number of leaves **L ≥ n!**



Now think about the **height** of the tree

A **binary** tree with height **h** has at most **2^h leaves**



So number of leaves **L ≤ 2^h**

So number of leaves **L ≥ n!**

So,

$2^h \geq n!$

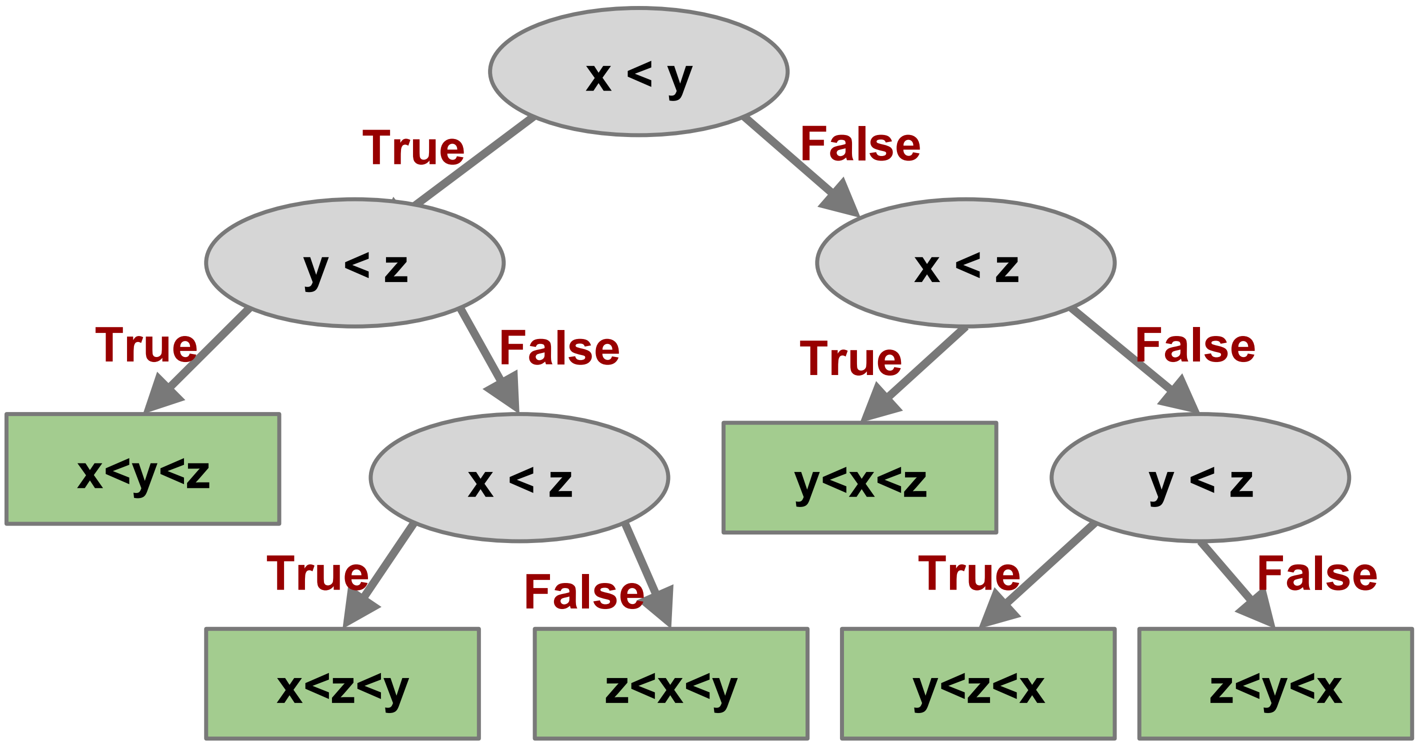
Not trivial, will show it later

So number of leaves **L ≤ 2^h**

$h \geq \log(n!) \in \mathbf{\Omega(n \log n)}$

So number of leaves **L ≥ n!**

$$\mathbf{h \in \Omega(n \log n)}$$



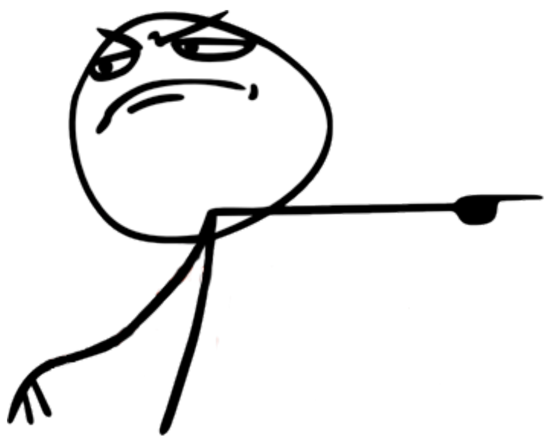
What does **h** represent, really?

The worst-case # of comparisons to sort!

$$\mathbf{h \in \Omega(n \log n)}$$

What did we just show?

The worst-case number of comparisons needed to sort **n** elements is in **Ω (n log n)**



Lower bound proven!

Appendix: the missing piece

Show that **log (n!)** is in **Ω (n log n)**

log (n!)

= log 1 + log 2 + … + log n/2 + … + log n

≥ log n/2 + … + log n **(n/2 + 1 of them)**

≥ log n/2 + log n/2 + … + log n/2 **(n/2 + 1 of them)**

≥ **n/2 · log n/2**

∈ **Ω (n log n)**

Often the number of possible solutions is small so we can't use the previous easy strategy.

A more general lower bound tool:

The Adversary Method

How does your opponent smartly **cheat** in this game?

➔ While you ask questions, the opponent alters their ships' positions so that they can "**miss**" whenever possible, i.e., construct the **worst possible input** (layout) **based on your questions**.

➔ They won't get caught as long as their answers are **consistent** with one possible input.

If we can prove that, no matter what sequence of questions you ask, the opponent can always craft an input such that it takes at least **42 guesses** to sink a ship.

Then we can say the **lower bound** on the complexity of the “sink-a-ship” problem is **42 guesses**, no matter what “guessing algorithm” you use.

more formally ...

To prove a lower bound **L(n)** on the complexity of problem **P**,

we show that for every algorithm **A** and arbitrary input size **n**, there exists some input of size **n** (picked by an imaginary adversary) for which **A** takes at least **L(n)** steps.

Example: search unsorted array

Problem:

Given an unsorted array of **n** elements, return the **index** at which the value is **42**.
(assume that **42** must be in the array)

3	**5**	**2**	**42**	**7**	**9**	**8**

Possible algorithms

- ➔Check through indices 1, 2, 3, …, n
- ➔Check from n, n-1, n-2, …., to 1
- ➔Check all odd indices 1, 3, 5, …, then check all even indices 2, 4, 6, …
- ➔Check in the order 3, 1, 4, 1, 5, 9, 2, 6, ...

Prove: the **lower bound** on this problem is **n-1**, no matter what algorithm we use.

3	**5**	**2**	**42**	**7**	**9**	**8**

Proof: (using adversarial argument)

➔Let **A** be an **arbitrary** algorithm in which the first **n-1** indices checked are **i_1, i_2, …, i_{n-1}**

➔Construct (adversarially) an input array **L** such that **L[i_1]**, **L[i_2], …, L[i_{n-1}]** are **not 42**, and **L[i_n]** is **42.**

➔Because **A** is arbitrary, therefore the lower bound on the complexity of solving this problem is **n**, no matter what algorithm is used.

The problem

Given **n** elements, determine the **maximum** element.

How many comparisons are needed **at least**?

The problem

Given **n** elements, determine the **maximum** element.

How many comparisons are needed **at least**?

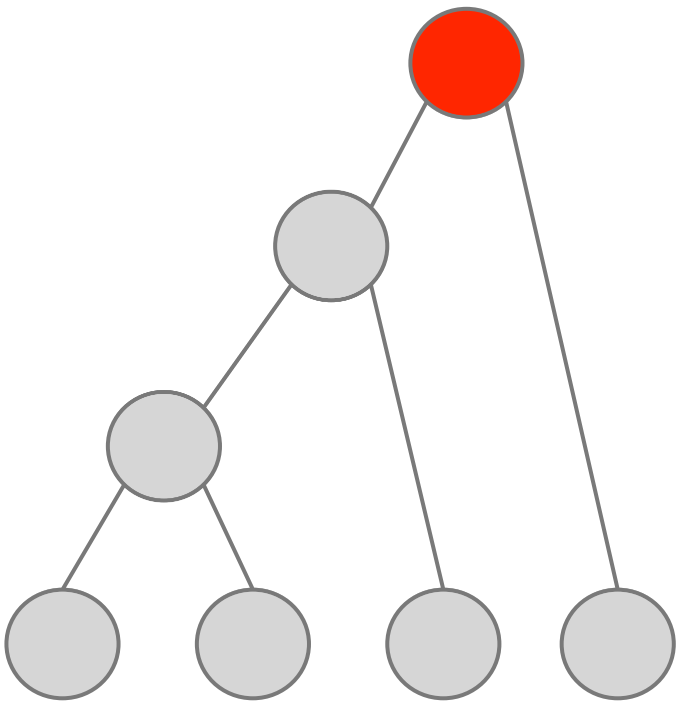
Answer: Need at least **n-1** comparisons

Insight: upper bound for max

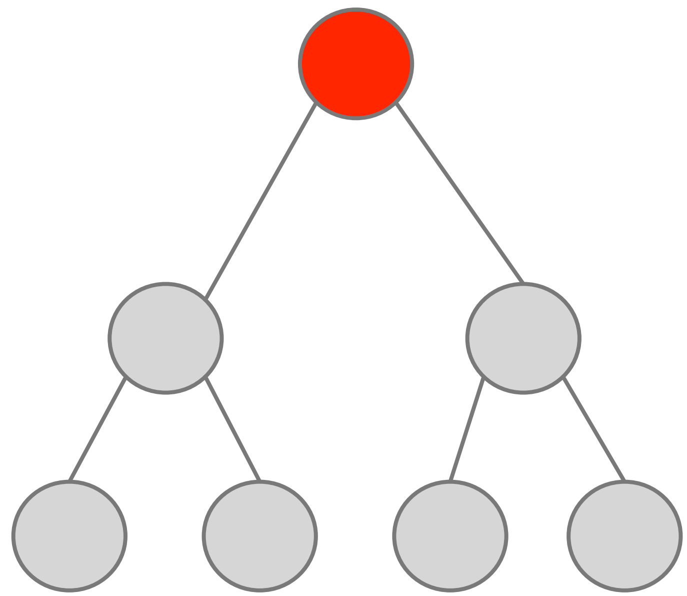
How to design a maximum-finding algorithm that reaches the lower bound **n-1** ?

- ➔ Make every comparison **count**, i.e., every comparison should guarantee to **eliminate a possible candidate** for maximum/champion.
- ➔ No match between losers, because neither of them is a candidate for champion.
- ➔ No match between a candidate and a loser, because if the candidate wins, the match makes no contribution (not eliminating a candidate)

These algorithms reach the lower bound



Linear scanning



Tournament

Adversary strategy for Max

Suppose Algorithm A claims to find the max of n elements using < n-1 comparisons (on some path)

Construct a graph in which we join two elements by an edge if they are compared (along this path) by A.

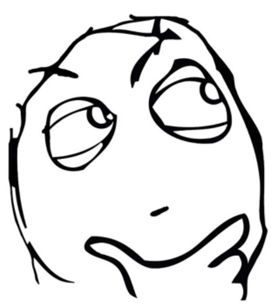
Since < n-1 comparisons on this path, the underlying graph has at least 2 components, C1 and C2

Suppose A outputs u in component C1 (as max)

Then we can fix values for elements in C1, C2 to be consistent with the comparisons, and where every element in C2 is larger than u. Contradiction!

Challenge question

Given **n** elements, what is the lower bound on the number of comparisons needed to determine both the **maximum** element and the **minimum** element?



Hint: it is smaller than **2(n-1)**

proving lower bounds using Reduction

The idea

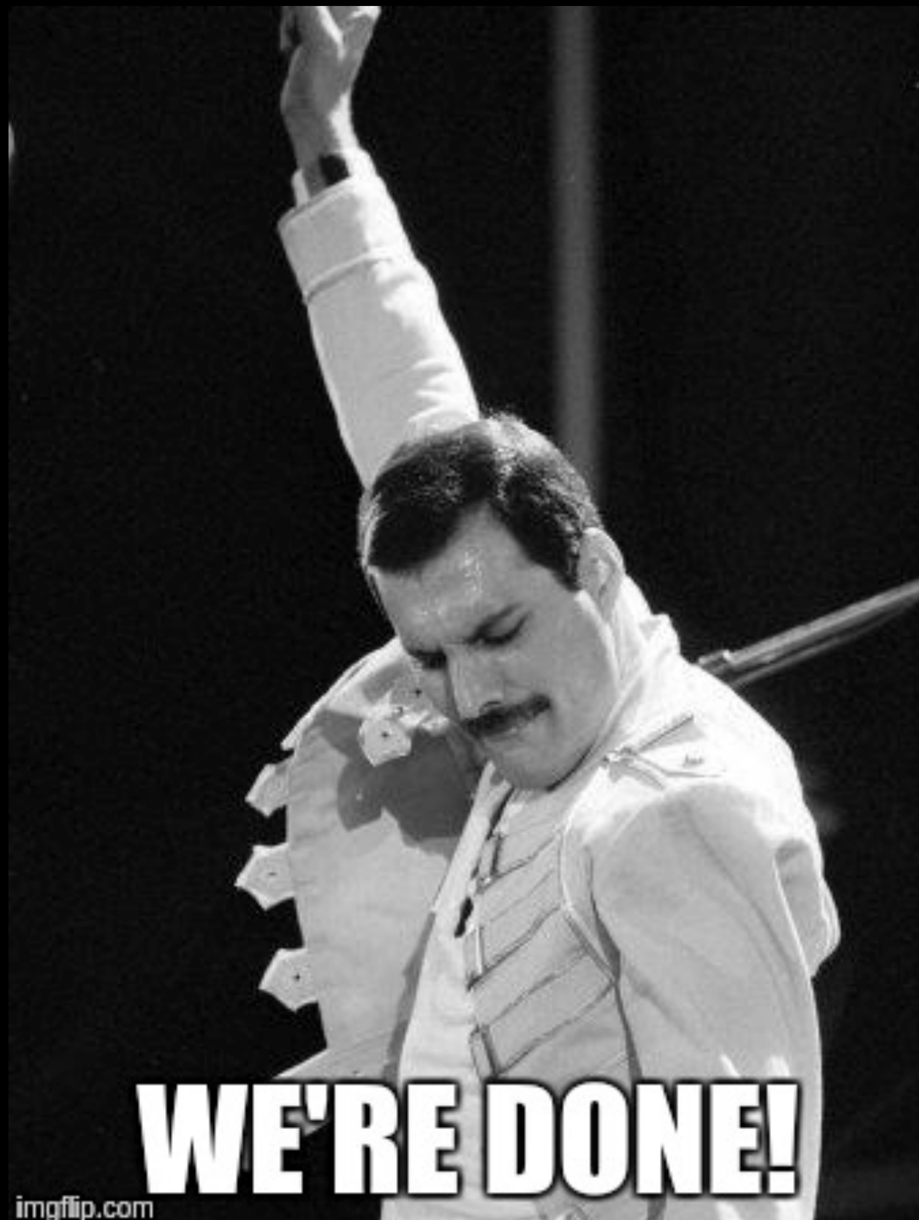
- ➔Proving one problem's lower bound using **another** problem's **known** lower bound.
- ➔If we know problem **B** can be solved by solving an instance of problem **A**, i.e., **A** is "harder" than **B**
- ➔and we know that **B** has lower bound **L(n)**
- ➔then **A** must also be lower-bounded by **L(n)**

Example:

Prove: **ExtractMax** on a binary heap is lower bounded by **Ω(log n)**.

Suppose ExtractMax can be done faster than **log n**, then HeapSort can be done faster than **n log n**, because HeapSort is basically ExtractMax **n** times

But HeapSort, as a comparison based sorting algorithm, has been proven to be lower bounded by **Ω(n log n).** Contrdiction, so ExtractMax must be lower bounded by **Ω(log n)**


WE'RE DONE!
imgflip.com

Final thoughts

what did we learn in CSC263

Data structures are the underlying skeleton of a good computer system.

If you will get to design such a system yourself and make fundamental decisions, what you learned from CSC263 should give you some clues on what to do.

- ➔ Understand the nature of the system / problem, and model them into structured data
- ➔ Investigate the probability distribution of the input
- ➔ Investigate the real cost of operations
- ➔ Make reasonable assumptions and estimates where necessary
- ➔ Decide what you care about in terms of performance, and analyse it
  - ◆ "No user shall experience a delay more than 500 milliseconds" -- worst-case analysis
  - ◆ "It's ok some rare operations take a long time" -- average-case analysis
  - ◆ "what matter is how fast we can finish the whole sequence of operations" -- amortized analysis

In CSC263, we learned to be a computer scientist, not just a programmer.

Original words from lecture notes of Michelle Craig

what we did NOT learn

but are now ready to learn

Other (even better!) kinds of heaps

➔Sometimes we want to be able to **merge** two heaps into one heap, with binary heap we can do it in **O(n)** time worst-case.

➔Using **binomial heap**, we can do merge in **O(log n)** time worst-case

➔Using **Fibonacci heap**, we can do merge (as well as Max/Insert/IncreaseKey) in **O(1)** time amortized.

Even better kinds of search trees

- ➔We learned BST and AVL tree, and there are others called red-black tree, 2-3 tree, splay tree, AA tree, scapegoat tree, etc.
- ➔There is **B-tree**, optimized for accessing big blocks of data (like in a hard drive)
- ➔There is **B+ tree**, which is even better than B-tree (widely used in database systems).
- ➔You'll learn about these in CSC443.

Amazing applications of hashing

→**Perfect hashing** guarantees **worst-case** O(1) time for searching, instead of **average-case** O(1) time

→**Cuckoo hashing** (coolest thing ever)

Shortest paths in a graph

➔ We learned how to get shortest paths using BFS on a graph

➔ We did NOT learn how to get **shortest (weighted) paths** on a weighted graph.

- Dijkstra, Bellman-Ford, ...

➔ You'll learn about them in CSC 373

Greedy algorithms

➔We learned that Kruskal’s and Prim’s MST algorithms are greedy

➔What property is satisfied by the problems that can be perfectly solved by greedy algorithms?

➔Will learn in CSC373

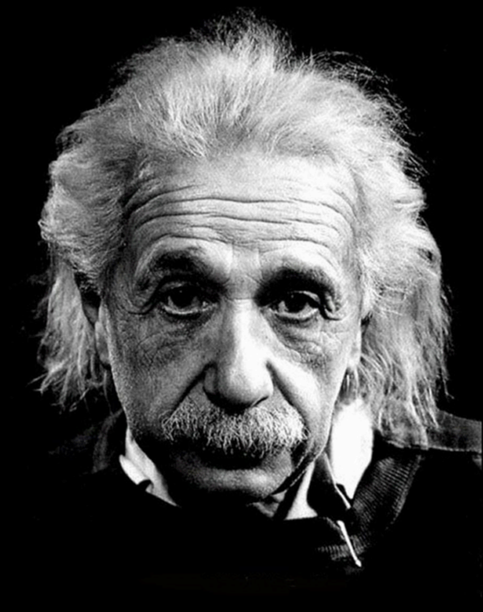
Dynamic programming

➔Pick an interesting algorithm design problem, very likely it involves dynamic programming

➔Will learn in CSC373

P vs NP, approximation algorithms

- ➔We learned a bit about lower bounds.
- ➔There are some problems, we can prove they cannot be perfectly solved in polynomial time.
- ➔For these problems, we have to design some **approximation algorithms**.
- ➔Will learn in CSC373 / 463



As our circle of knowledge expands, so does the circumference of darkness surrounding it.

Final Exam Prep

Topics covered: all of them

- ➔Heaps
- ➔BST, AVL tree, augmentation
- ➔Hashing
- ➔Randomized algorithms, Quicksort
- ➔Graphs, BFS, DFS, MST
- ➔Disjoint sets
- ➔Lower bounds
- ➔Analysis: worst-case, average-case, amortized.

Types of questions

- ➔ Short-answer questions testing basic understanding.
- ➔ Trace operations we learned on a data structure
- ➔ Implement an ADT using a data structure
- ➔ Analysis runtimes
  - ◆ best / worst-case
  - ◆ average-case
  - ◆ amortized cost
- ➔ Given a real-world problem, design data structures / algorithms to solve it.

Study for the exam

- ➔Review lecture notes/slides
- ➔Review tutorial problems
- ➔Review all problem sets / assignments
- ➔Practice with past exams (available at exam repository)
- ➔Come to **office hours** whenever confused.

Toni's pre-exam office hours

- ➔Monday Dec 7, 3-4pm
- ➔Wednesday Dec 9, 1-2pm

Exam Time & Location

Friday, Dec 11, 2:00 - 5:00 pm

No aid sheet

Go to the right location.

All the best!