Worth: 8% **Due:** By 5:59pm on Tuesday 17 November

Remember to write the *full name* and *student number* of *every group member* prominently on your submission.

---

*Please read and understand the policy on Collaboration given on the Course Information Sheet. Then, to protect yourself, list on the front of your submission **every** source of information you used to complete this homework (other than your own lecture and tutorial notes). For example, indicate clearly the **name** of every student from another group with whom you had discussions, the **title and sections** of every textbook you consulted (including the course textbook), the **source** of every web document you used (including documents from the course webpage), etc.*

*For each question, please write up detailed answers carefully. Make sure that you use notation and terminology correctly, and that you explain and justify what you are doing. Marks **will** be deducted for incorrect or ambiguous use of notation and terminology, and for making incorrect, unjustified, ambiguous, or vague claims in your solutions.*

---

1. In this question, we will use a graph algorithm to solve a problem in philosophy.

   A *paradox* is a group of statements that lead to a contradiction. For example, the following two statements form a famous paradox:

   ```
   1. Statement 2 is FALSE.
   2. Statement 1 is TRUE.
   ```

   If we assume Statement 1 is TRUE, then Statement 2 is FALSE, which in turn means Statement 1 is FALSE. But if we assume Statement 1 is FALSE, then Statement 2 is TRUE, which in turn means Statement 1 is TRUE. So Statement 1 is TRUE iff Statement 1 is FALSE, a blatant contradiction.

   Now, suppose you are given a group of N statements, numbered from 1 to N. Each statement has the form: "`Statement X is TRUE/FALSE`," where `X` is a number between 1 and N. Your task is to figure out whether this group of statements forms a paradox. In particular, answer the following questions.

   (a) Describe how to construct a graph to solve this problem. Be precise: state clearly what vertices your graph contains (and what each vertex represents) and what edges your graph contains (and what each edge represents).

   (b) Give a **necessary and sufficient** condition for the N statements to form a paradox. Justify your answer.

   (c) How do you efficiently detect whether your graph from part (a) satisfies your condition described in part (b)? Describe your algorithm in concise (but precise) English.

   (d) Analyse the worst-case runtime of your algorithm.

2. In this question, you will use a graph algorithm to solve the general form of a well-known brain-teaser.

   You are given two **initially empty** buckets A and B, with capacities of m litres and n litres, respectively. Your goal is to measure exactly k litres of water using these two buckets. Assume that m, n and k are positive **integers** and that $k \leqslant \max(m, n)$. You want to achieve this goal by performing a sequence of **moves** until one of the buckets has exactly k litres of water. Each move can be one of the following:

- Fill a bucket until it's full.
- Empty a bucket.
- Use the water in one bucket to fill the other until one of the buckets is full or empty.

Now, you must devise an algorithm BUCKETMEASURE(m,n,k), which takes m, n and k as inputs and outputs a sequence of moves that results in one bucket (it does not matter which one) having exactly k litres of water. The number of moves in the returned sequence must be the **smallest possible** number of moves that are needed to achieve the goal. If it is impossible to measure k litres of water using the two buckets, the algorithm returns NIL.

Answer the following questions.

(a) How do you construct a graph for solving this problem? Describe the vertices and edges in your graph clearly and precisely.

(b) How does your algorithm work? Give a detailed description and justify its correctness.

(c) Analyse the worst-case runtime of your algorithm.