CSC263 Fall 2011 Homework Assignment 4

due Thursday, December 1, 2011, 10:15am

1. (20 marks) A node in a directed graph is a *sinkhole* if every other node has an edge to it, but it has no edge to any node.

   (a) Prove that every directed graph has at most one sinkhole.

   (b) Suppose you are given the adjacency matrix representation of a directed graph with n nodes. Give an algorithm that determines whether the graph has a sinkhole and, if so, returns it. Your algorithm must run in $O(n)$ time. Prove that your algorithm is correct and runs in the desired time.

   (c) Suppose you are given the edge list representation of a directed graph with n nodes and m edges. Give an algorithm that determines whether the graph has a sinkhole and, if so, returns it. Your algorithm must run in $O(n+m)$ time. Prove that your algorithm is correct and runs in the desired time.

2. (15 marks)

   (a) Suppose that, during the BFS of an undirected graph $G = (V, E)$, node a is first visited before node b, which is first visited before node c. Prove that if $\{a, c\} \in E$, but $\{a, b\} \notin E$, then there exists a neighbour d of b which is visited before node a.

   (b) Suppose that, during the DFS of an undirected graph $G = (V, E)$, node a is first visited before node b, which is first visited before node c. Complete the following sentence and prove it is correct: If $\{a, c\} \in E$, but $\{a, b\} \notin E$, then there exists a neighbour d of b which is visited ...

3. (5 marks) Two trees are edge-disjoint if there is no edge appearing in both of them.
   Let G be a graph with 2 edge-disjoint spanning trees. What is the least number of vertices, n, that G can have? Give an example of a graph on n vertices which has 2 edge-disjoint spanning trees. Justify your answers.

4. (5 marks) Suppose that T is a minimum spanning tree of a graph G with weight function w. Let $e' \in T$ and let w' be the weight function obtained from w by decrementing the weight of edge e', i.e.

$$w'(e) = \begin{cases} w(e) & \text{if } e \neq e' \\ w(e) - 1 & \text{if } e = e' \end{cases}$$

   Prove that T is a minimum spanning tree of G with weight function w'.

5. (25 marks) Consider the following logic puzzle:

   > Four friends are going to a movie. Alice is in front of Bob, who is behind Cho. Cho is in front of Dai who is ahead of Alice. What order are they in line?

   (a) How could such a puzzle be represented as a graph?

   (b) What property must this graph have for an ordering of the friends to exist that satisfies all the given conditions?

(c) Describe an algorithm that, given such a puzzle, decides whether there is an order that satisfies all given conditions and, if so, outputs the friends, in such an order. If there is such an order, also decide whether the order is unique.
The running time of your algorithm should be linear in the size of the input.

(d) Explain why your algorithm is correct.

(e) Explain why the running time of your algorithm is linear in the size of the input.

(f) Implement your algorithm (in Python, Java, C or C++). Sample starter code will be available on the course website.

(g) Thoroughly test your algorithm. Explain your choice of test cases.

(h) Submit your code as a single file using the `submit` command on cdf.