Appendix: Proof of Theorems 1 and 2

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In this appendix, we will use p to denote the desired candidate the manipulating coalition wishes to install.

1 Theorem 1

We will start by proving Theorem 1 for a special case of k-approval voting rule, called *the veto rule*. Veto is defined to be $(m-1)$-approval. In other words, each voter specifies which candidate he dislikes the most. We start from the case where we have only 3 candidates and then prove the general case by induction.

Let n be the number of sincere voters, c be the number of manipulators and $A = \{a, b, p\}$. The balanced strategy BAL in this case is $\lfloor c/2 \rfloor$ manipulators veto a and $\lceil c/2 \rceil$ manipulators veto b. We have the following lemma.

Lemma 1 *Assume impartial culture, the balanced strategy BAL has the highest probability of uniquely electing p.*

Proof: Consider an alternative strategy ALT where without loss of generality[1] $\lfloor c/2 \rfloor - d$ manipulators veto a and $\lfloor c/2 \rfloor + d$ manipulators veto b. Because we are assuming impartial culture, it is sufficient to count the number of preference profiles for the sincere voters where, together with the manipulators' votes, p is vetoed strictly less times than both a and b. Note that each preference profile naturally induces a vector of length n of vetoed alternatives, which we will refer to as the *veto vector*. Each veto vector corresponds to 2^n preference profiles with the same outcome (by arbitrarily selecting alternatives for the first and second positions in the sincere voters' preferences), hence it is sufficient to count veto vectors.

Fix the identities of s_p sincere voters who veto p. We will show that for every such choice there are (weakly) more veto vectors such that p is the unique winner under BAL than under ALT. We consider *veto pairs* (s_a, s_b), where s_x is the number of sincere voters who veto x for $x \in \{a, b\}$. Note that $s_a = s_b = n - s_p$, and each veto pair corresponds to $\binom{n-s_p}{s_a} = \binom{n-s_p}{s_b}$ veto vectors. Letting $t = s_p + 1$ be the *threshold*, we have that p is a unique winner under BAL if and only if

$$s_a \geq t - \lfloor c/2 \rfloor \ \wedge \ s_b \geq t - \lceil c/2 \rceil. \tag{1}$$

[1] a and b are symmetric and it is obviously suboptimal to veto p.

Similarly, p is a unique winner under ALT if and only if

$$s_a \geq t - \lfloor c/2 \rfloor + d \ \wedge \ s_b \geq t - \lceil c/2 \rceil - d. \tag{2}$$

To complete the proof we will construct a one-to-one mapping between pairs (s_a, s_b) satisfying (2) and pairs (s'_a, s'_b) satisfying (1) such that

$$\binom{n - s_p}{s_a} \leq \binom{n - s_p}{s'_a}. \tag{3}$$

Let (s_a, s_b) be a veto pair satisfying (2). If it also satisfies (1), we set $s'_a = s_a$, $s'_b = s_b$, i.e., we map the pair to itself. If not then it must be the case that $s_b \in \{t - \lceil c/2 \rceil - d, \ldots, t - \lceil c/2 \rceil\}$. Let $s_b = t - \lceil c/2 \rceil - b$, for $b \in \{1, \ldots, d\}$. We set $s'_a = t - \lfloor c/2 \rfloor + d - b$, $s'_b = n - s_p - s'_a$. The pair (s'_a, s'_b) does not satisfy (2). To see that it does satisfy (1), note that $s_a + s_b \geq 2t - c + d - b$, and hence

$$s'_b = (n - s_p) - s'_a = (s_a + s_b) - s'_a \geq t - \lceil c/2 \rceil.$$

It follows that the mapping defined above is one-to-one. Furthermore, it is easy to verify that s'_a is at least as close to $(n - s_p)/2$ as s_b. This implies (3), and hence the theorem. ■

We now extend the lemma above to handle any number of candidates.

Lemma 2 *Let $A = \{a_1, a_2, \ldots, a_m, p\}$, assume impartial culture, and consider the veto rule. Then the balanced strategy BAL where c/m[2] manipulators veto $a_i, (i \in [m])$ has the highest probability of uniquely electing p.*

Proof: For each *compact veto vector* $(s_{a_1}, s_{a_2}, \ldots, s_{a_m})$ of sincere voters, where s_{a_i} is the number of manipulators veto a_i and $\sum_i s_{a_i} = n - s_p$, it corresponds to $\binom{n-s_p}{s_{a_1}, s_{a_2}, \ldots, s_{a_m}}$ [3] veto vectors of sincere voters.

Letting $t = s_p + 1$ be the threshold, we have that p is a unique winner under BAL if and only if

$$s_{a_1} \geq t - c/m \wedge, \ldots, s_{a_m} \geq t - c/m \tag{4}$$

p is a unique winner under ALT if and only if

$$s_{a_1} \geq t - c/m + d_1 \wedge, \ldots, s_{a_m} \geq t - c/m + d_m, \tag{5}$$

where $\sum_i d_i = 0$.

We now prove there is a bijection between compact veto vectors $(s'_{a_1}, s'_{a_2}, \ldots, s'_{a_m})$ that satisfy constraint (5) and compact veto vectors $(s_{a_1}, s_{a_2}, \ldots, s_{a_m})$ that satisfy constraint (4), such that

$$\binom{n - s_p}{s_{a_1}, s_{a_2}, \ldots, s_{a_m}} \geq \binom{n - s_p}{s'_{a_1}, s'_{a_2}, \ldots, s'_{a_m}} \tag{6}$$

We do so by induction on the number of nonzero d_i's. Denote this number by j.

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[2]For the proof, we assume for simplicity c/m is an integer. For the general case where $c = i \times m + j$, the proof is similar.

[3]$\binom{n-s_p}{s_{a_1}, s_{a_2}, \ldots, s_{a_m}} = \binom{n-s_p}{s_{a_1}}\binom{n-s_p-s_{a_1}}{s_{a_2}} \ldots, \binom{s_{a_m}}{s_{a_m}}$

- Base step, when $j = 0$, the bijection is a self-mapping and (6) holds with equality. $j \neq 1$ because it will violate $\sum_i d_i = 0$. When $j = 2$, let $d_{i_1} = -d_{i_2} \neq 0$. The bijection is constructed as follows:
  - for $i \neq i_1$ and $i \neq i_2$, let $s_{a_i} = s'_{a_i}$
  - for each (s'_{i_1}, s'_{i_2}) pair, we map it to (s_{i_1}, s_{i_2}) according to the bijection in 2-candidate case.

  Clearly, the one above is a bijection and one can easily verify that it satisfies (6).

- Inductive hypothesis, suppose there is such a bijection when $j \leq k$.

- Inductive case, now consider $j = k + 1$. WLOG, we pick the d_{i_1} with the smallest absolute value, there must exist d_{i_2} with the opposite sign and $|d_{i_2}| \geq |d_{i_1}|$. The bijection is constructed as follows:
  - For each (s'_{i_1}, s'_{i_2}), we map it to (s''_{i_1}, s''_{i_2}) according to the bijection in 2-candidate case where $s''_{i_1} \geq t - c/m$ and $s''_{i_2} \geq t - c/m + d_1 + d_2$.
  - Map $(s'_{a_1}, s'_{a_2}, \ldots, s''_{i_1}, \ldots, s''_{i_2}, \ldots, s'_{a_m})$ to $(s_{a_1}, s_{a_2}, \ldots, s_{a_m})$ according to the bijection in inductive hypothesis.

  Clearly, the resulting mapping is a compound of two bijections, which is still a bijection. Also, it is easy to verify that both bijections (weakly) increases the multinomial quantity. Thus, (6) holds as well.

■

We now extend the lemmas above to $k - veto$ where a voter can veto k candidates. Note that $k - veto$ is equivalent to $(m - k) - approval$, so we conclude the proof of Theorem 1 for the IC case.

Theorem 1 *Let $A = \{a_1, \ldots, a_m, p\}$. Assume impartial culture and consider $k - veto$ rule. The balanced strategy BAL where manipulator i votes $\{a_\ell \mid l \in \{ik + 1 \pmod m), \ldots, (i + 1)k \pmod m)\}\}$ provides the highest probability of manipulation.*

Proof: Let $V = (v_1, \ldots, v_m)$ be a vote profile and $(s_p, s_{a_1}, \ldots, s_{a_m})$ be the compact veto vector. Let $t = s_p + 1 - ck/m$ be the threshold such that p uniquely wins under BAL iff $s_{a_i} \geq t$ for all i. For alternative strategy ALT, p uniquely wins iff $s_{a_i} \geq t + d_i$ for all i, where $\sum_i d_i = 0$. We prove ALT has probability of manipulation no larger than BAL by induction on the number of non-zero d_i's, denote this number by j.

Base step: $j = 2$ and suppose WLOG $d = -d_1 = d_2 > 0$. We construct a one-to-one mapping of profiles V where p wins under ALT to profiles V' where p wins under BAL. Map all V that is manipulable in both ALT and BAL to itself. Otherwise we map by swapping a_1 and a_2 sequentially in V while ignoring other candidates, that is, we replace any occurrence of a_1 by a_2 and a_2 by a_1 in the votes until the score of a_1, $s'_{a_1} = s_{a_2} - d$. This must be feasible since every swap leaves the score of a_1 and a_2 either unchanged (i.e. when $a_1, a_2 \in v_i$ or $a_1, a_2 \notin v_i$) or increase the score of a_1 by one and decrease the score of a_2 by one, or vice versa; and if we were to swap all votes the score of a_1 is $s_{a_2} > s'_{a_1}$ so we must reach a first index i such that swapping in v_i results in the score of a_1 equal to s'_{a_1}. The resulting profile V' has $s'_{a_1} \geq t$ and $t \leq s'_{a_2} = s_{a_1} + d < t + d$ so that V' is manipulable under BAL.

This mapping is one-to-one. Any V' that is manipulable in both ALT and BAL is mapped to itself. Now suppose profiles X and Y maps to V' through swapping. X and Y must have the same veto score vector and suppose swapping stops after the i-th vote in X and the i'-th vote in Y. We have $i = i'$ otherwise say $i < i'$. This implies the l-th votes in X and Y are equal, for all $l \leq i$, but this means the veto vectors of X and Y must equal after removing the first i votes. Hence swapping the first i votes in Y must reach veto score s'_{a_1} for a_1, contradiction. So $i = i'$ and it follows that $X = Y$.

Inductive hypothesis: suppose that ALT has no larger probability of manipulation than BAL, for all $j \leq r$.

Inductive step: we have $j = r + 1$. WLOG suppose d_1 has the smallest absolute value and d_2 has the opposite sign. We map a profile V in the same way as in the base case except that a_1's target score becomes $s_{a_2} - d_2$. This is a one-to-one mapping to manipulable profiles of another strategy ALT$'$ where the new d'_1 is zero and the new d'_2 is $d_1 + d_2$, while d'_i for $i \geq 3$ equals d_i. By the inductive hypothesis ALT$'$ has r non-zero d'_i's and therefore has probability of manipulation no larger than that of BAL. Thus, ALT has probability of manipulation no larger than that of BAL. ■

We now show that, BAL is still the optimal manipulation strategy for veto when we replace the IC assumption with IAC.

Lemma 3 *Let $A = \{a_1, a_2, \ldots, a_m, p\}$, assume impartial anonymous culture, and consider the veto rule. Then the balanced strategy BAL where c/m manipulators veto $a_i, (i \in [m])$ has the highest probability of uniquely electing p.*

Proof: Given our construction of the bijection between compact veto vectors that select p under BAL and compact veto vectors that select p under ALT, it suffices to show that the number of voting situations that correspond to compact veto vectors under BAL is no less than that under ALT.

Consider a compact veto vector $(s_{a_1}, s_{a_2}, \ldots, s_{a_m})$, the number of voting situations corresponding to it is

$$x = \binom{s_{a_1} + (m-1)! - 1}{s_{a_1} - 1}\binom{s_{a_2} + (m-1)! - 1}{s_{a_2} - 1}\cdots\binom{s_{a_m} + (m-1)! - 1}{s_{a_m} - 1}.$$

Similarly, the number of voting situations corresponding to compact veto vector $(s_{a_1} - 1, s_{a_2} + 1, \ldots, s_{a_m})$ is

$$x' = \binom{s_{a_1} + (m-1)! - 2}{s_{a_1} - 2}\binom{s_{a_2} + (m-1)!}{s_{a_2}}\cdots\binom{s_{a_m} + (m-1)! - 1}{s_{a_m} - 1}.$$

$x/x' = \frac{(s_{a_1}+(m-1)!-1)s_{a_2}}{(s_{a_2}+(m-1)!)(s_{a_1}-1)} = \frac{s_{a_1}s_{a_2}+s_{a_2}(m-1)!-s_{a_2}}{s_{a_1}s_{a_2}+(s_{a_1}-1)(m-1)!-s_{a_2}}$

When $s_{a_1} > s_{a_2}$, we have $x/x' \leq 1$. This shows that the number of voting situations correspond to a more balanced compact veto vector is greater. From previous proof of the impartial culture case, we know that under the bijection, the compact veto vector under BAL is more balanced. This , in turn, implies that it yields more voting situations.

■

The extension of the above lemma from veto to k veto is the same as that in the IC case. Thus, we complete the proof of Theorem 1.

2 Theorem 2

In this section, we use an equivalent, alternative description of Borda score, which assigns a score of $i-1$ to candidate ranked i-th and selects score minimizer (in a tie-breaker selects a candidate other than p). According to this description, BAL always assigns p with a score of zero. The IC part of Theorem 2 follows from the two lemmas below.

Lemma 4 *Let $A = \{x, y, p\}$. Assume impartial culture, that c is even, and consider the Borda rule. Then the optimal manipulation is either the balanced strategy BAL, where manipulators alternatingly vote $p \succ x \succ y$ and $p \succ y \succ x$, or ALT[1], where in general manipulators give p a score of zero, and x, y scores such that their difference is two.*

Proof: Let ALT[d] denote the strategy where manipulators assign a score of zero to p, and scores to x, y such that their difference is $2d$ (assume WLOG that x's score is higher). We will construct a one-to-one mapping of profiles where p wins under ALT[d] to profiles where p wins under ALT[$d-2$] for any $d \geq 2$ (note that ALT[0] is BAL), showing that ALT[1] is better than ALT[d] for all odd $d \geq 3$ and BAL better than ALT[d] for all even $d \geq 2$. Consider a profile $V = (v_1, \ldots, v_n)$ where p wins under ALT[d], and let s_p, s_x, and s_y denote the corresponding scores for the candidates with respect to V.

If p also wins under ALT[$d-2$] then map V to itself. Otherwise map V to a profile V' where x has a score of $s'_x = s_x + 2d - 2$ as follows. Consider all votes $U = (v_{j_1}, \ldots, v_{j_l})$, $j_1 < \cdots < j_l$, where p is ranked second. Start swapping x and y for each vote in U in turn, and stop after swapping in v_r if the swap results in the score of x equal to s'_x. Note that we can never strictly exceed s'_x since each swap increases the score of x by two and $s'_x - s_x$ is even. If such a v_r does not exist in U, then finish swapping in U and start swapping votes in $V \backslash U$, from the smallest to the largest index. Observe that we must reach some vote where the swap would reach score target s'_x, otherwise we would have swapped x and y in all votes without score of x reaching s'_x, which is impossible since $s_y > s'_x$.

Now map V' to V'' by swapping x and y in all votes of V' so that $s''_y = s'_x$. Now V'' has p as the winner under ALT[$d-2$]. First note the score threshold for ALT[$d-2$] for x and y are $t_x = s_p - 3c/2 - d + 2$ and $t_y = s_p - 3c/2 + d - 2$, respectively. Since $s_x > s_p - 3c/2 - d$, the threshold for ALT[d], and $s_x \leq t_x$, then $s_p - 3c/2 + d - 2 = t_y < s''_y \leq s_p - 3c/2 + d$. Also note that $s''_x = s_y - 2d + 2 > s_p - 3c/2 - d + 2 = t_x$.

The mapping from V to V'' is one to one. Suppose there are two profiles X and Y that map to V''. If V'' has p as winner under both ALT[d] and ALT[$d-2$] then $X = Y = V''$, otherwise V'' must be mapped through swapping. First note that the mapping preserves the positions of p, therefore the sequence of vote indices for swapping must be the same. Now suppose the swapping stops at some vote index i in X and index i' in Y. It must be that $i = i'$ since if X and Y both map to V'' their score vectors (for x, y and p) must be identical, and the profile, say X, where the swapping ends earlier must have reached the score target s'_x (for V') earlier but that implies swapping the same sequence of i votes in Y results in reaching s'_x, because the swapped votes in X and Y must be identical as it corresponds to the same votes in V''. Since $i = i'$ it follows that $X = Y$.

■

Lemma 5 *Let $A = \{x, y, p\}$. Assume impartial culture, and consider the Borda rule. Then the optimal manipulation strategy under the condition that either (i) n is even and $c + 2$ divisible by four; or (ii) n is odd and c is divisible by four., is BAL.*

Proof: By Lemma 4, we just need to show BAL is better than ALT[1], which we do through a one-to-one mapping of manipulable profiles of ALT[1] to that of BAL. Again map any profile that is manipulable under both strategies to itself. Consider a profile $V = (v_1, \ldots, v_n)$ where p wins under ALT[1], but not BAL. Let s_p, s_x, and s_y denote the corresponding scores for the candidates with respect to V. It must be that $s_x = s_p - 3c/2 - 1$, we will map V to profile V' by first mapping to a profile V'' where $s''_x = s_x + 1$ and mapping V'' to V' by swapping x and y in every vote of V'' so that $s'_x = s''_y$ and $s'_y = s''_x$. Now consider two cases.

Case 1: there exists a v_i that ranks p in either first or last place, and i is the smallest such index. Mapping to V'': first look at v_i, (x, y are adjacent in rank positions) if x is above y then swap x, y to get V''. Otherwise consider all votes $U = (v_{j_1}, \ldots, v_{j_l})$, $j_1 < \cdots < j_l$, where p is ranked second. Start swapping x and y for each vote in U in turn, and stop at v_r if the swap at v_r results in the score of x being one more than s''_x. Then swap x and y in both v_r and v_i so we reach targets s''_x and s''_y (recall v_i has y above x). Note that it is impossible to reach targets s''_x and s''_y by only swapping in U since any swap increases or decreases the score of x by 2. If such a v_r does not exist in U, then finish swapping in U and start swapping votes in $V \backslash U$, from the smallest to the largest index. Observe that we must reach some vote where the swap would reach score targets s''_x and s''_y, otherwise we would have swapped x and y in all votes without score of x reaching or exceeding s''_x, which is impossible since $s_y > s_x$.

Case 2: all votes in V rank p second. Thus, $s_p = n$ and the scores of x and y must be even. But if conditions (1) or (2) in the theorem holds then the score thresholds for x and y under ALT[1], $t_x = s_p - 3c/2 - 1$ and $t_y = s_p - 3c/2 + 1$ must be even, so V must be either simultaneously manipulable or not manipulable under both ALT[1] and BAL since s_x is even and t_x is even.

The above mapping always has V' manipulable under BAL because in Case 1, $s'_y = s_x + 1 = s_p - 3c/2$ and $s'_x = s''_y = s_y - 1 > s_p - 3c/2$. Furthermore V' is not manipulable under ALT[1] so any profile that is mapped because it is manipulable under both strategies has a unique inverse mapping. Any V' mapped through Case 1 would also have a unique inverse mapping using similar arguments as in the proof of Theorem 5. ■

Up to now, we have showed that Theorem 2 holds for the IC case. We now show that it also holds for the IAC case.

Lemma 6 *Assume impartial anonymous culture and consider Borda where $A = \{x, y, p\}$, the optimal strategy for the manipulators is BAL or ALT[1].*

Proof: We now construct a one-to-one mapping of voting situations where p wins under ALT[d] to situations where p wins under ALT[d-2]. Consider a voting situation S where p wins under ALT[d], if it also wins under ALT[d-2], we map S to itself. Otherwise, we do the following:

1. Instantiate S to a voting profile V where, voters 1 to S_{xpy} vote $x \succ p \succ y$, followed by S_{ypx} voters, named $S_{xpy} + 1$ to $S_{xpy} + S_{ypx}$, vote $y \succ p \succ x$, followed, in turn, by voters that vote $x \succ y \succ p$, $y \succ x \succ p$, $p \succ x \succ y$ and $p \succ y \succ x$. Where $S_{\succ}$ is the number of voters that vote $\succ$ according to S.

2. Apply our one-to-one mapping of the impartial culture case, obtain a profile V'.

3. Reduce V' to its corresponding situation S'.

Feasibility follows immediately from that of the impartial culture case. To prove this is also a one-to-one, observe that if S_1 and S_2 are mapped to the same S' under our mapping. We claim that both S_1 and S_2 induce the same U, defined to be the set of votes that rank p second. Otherwise, it is not hard to verify that we can't reach the same U in S' by swapping x and y according to the order we defined earlier. We can similarly claim that both S_1 and S_2 induce the same U', the set of votes that rank p last, and also induce the same U'', the set of votes that rank p first. To sum up, we show that S_1 and S_2 must be the same voting situation.

■

The remainder (i.e, the "Furthermore" part in Theorem 2) of the IAC case is the same as the IC case. Therefore, we conclude that Theorem 2 holds for IAC case as well.