Topic: analyzing logical statements

1. The statements

\[ \exists x \in D, (P(x) \land Q(x)) \]

and

\[ (\exists x \in D, P(x)) \land (\exists x \in D, Q(x)) \]

are not equivalent.

The first statement says that there is an object that has both property \( P \) and also property \( Q \). This single object has both properties.

The second statement says that there is an object that has property \( P \) and, in addition, there is an object that has property \( Q \).

We can give a concrete example by considering a domain \( D \) and predicates \( P \) and \( Q \) which are mutually exclusive. For example, consider the domain \( D \) to be the set of integers \( \mathbb{Z} \), and the predicates \( P(x) : "x \text{ is even}" \) and \( Q(x) : "x \text{ is odd}" \).

The statement \( \exists x \in \mathbb{Z}, (P(x) \land Q(x)) \) is false, since there is no integer that is both even and odd.

But the statement \( (\exists x \in D, P(x)) \land (\exists x \in D, Q(x)) \) is true, since 2 is even (proving that \( (\exists x \in D, P(x)) \) is true, and 3 is odd (proving that \( (\exists x \in D, Q(x)) \) is true).

2. \[
\neg(\forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon) \quad \text{(negation of given)}
\]

\[ \iff \exists \epsilon \in \mathbb{R}^+, \neg(\exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon) \quad \text{(negation of } \forall \text{)} \]

\[ \iff \exists \epsilon \in \mathbb{R}^+, \forall \delta \in \mathbb{R}^+, \neg(\forall x \in \mathbb{R}, 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon) \quad \text{(negation of } \exists \text{)} \]

\[ \iff \exists \epsilon \in \mathbb{R}^+, \forall \delta \in \mathbb{R}^+, \exists x \in \mathbb{R}, 0 < |x - a| < \delta \land \neg(|f(x) - L| < \epsilon) \quad \text{(negation of } \Rightarrow \text{)} \]

\[ \iff \exists \epsilon \in \mathbb{R}^+, \forall \delta \in \mathbb{R}^+, \exists x \in \mathbb{R}, 0 < |x - a| < \delta \land |f(x) - L| \geq \epsilon \quad \text{(negation of } < \text{)} \]

3. (a)

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<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( R )</th>
<th>( (P \lor Q) )</th>
<th>( (Q \land R) )</th>
<th>( [(P \lor Q) \land R] )</th>
<th>( [P \lor (Q \land R)] )</th>
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Since the last two columns are not the same, we conclude that indeed \[ ((P \lor Q) \land R) \] is not equivalent to \[ P \lor (Q \land R) \]. The parentheses matter!

(b)

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<th></th>
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<th>R</th>
<th>((P \lor Q))</th>
<th>((P \Rightarrow R))</th>
<th>((Q \Rightarrow R))</th>
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where (1) stands for \( (P \lor Q) \land (P \Rightarrow R) \land (Q \Rightarrow R) \) and (2) stands for \( ((P \lor Q) \land (P \Rightarrow R) \land (Q \Rightarrow R)) \Rightarrow R \). Since all of the entries in the last column are True we can conclude that the statement \( ((P \lor Q) \land (P \Rightarrow R) \land (Q \Rightarrow R)) \Rightarrow R \) is always True.

4. (a) “No student likes computer science” means that there does not exist an object that has the properties of both being a student and liking computer science. Or equivalently, that every student does not like computer science. That is, every object having the property of being a student has the property of not liking computer science. In symbols, we can write

\[-\exists x \in D, (S(x) \land C(x))]\]

or equivalently,

\[\forall x \in D, (\neg S(x) \lor \neg C(x))\]

or equivalently,

\[\forall x \in D, (S(x) \Rightarrow \neg C(x))\]

(b) William and Catherine are particular elements of \( D \). Since nothing is being said about a property of the domain, quantifiers are not needed. We can write:

\[S(w) \land S(c) \land C(w) \land C(c)\]

(c) “All students who like mathematics also like computer science” means that for every object that has the properties of both being a student and liking mathematics it follows that they also like computer science, We can write:

\[\forall x \in D, [(S(x) \land M(x)) \Rightarrow C(x)]\]

Alternatively, we could think of this as saying that if an object has the property of being a student, then if that object has the property of liking mathematics, then it has the property of liking computer science. We can write:

\[\forall x \in D, [S(x) \Rightarrow (M(x) \Rightarrow C(x))]\]

(d) “Some student likes mathematics and computer science” means that there is an object that has the properties of being a student, liking mathematics and liking computer science. We can write:

\[\exists x \in D, [S(x) \land M(x) \land C(x)]\]

2
(e) “Some student likes mathematics, and some student likes computer science” means that there is an object that has the property of being both a student and liking mathematics, and in addition, there is a (possibly different) object that has the property of being a student and liking mathematics. We can write:
\[
\exists x \in D, (S(x) \land M(x)) \land \exists x \in D, (S(x) \land C(x))
\]

(f) “Only students who like mathematics like computer science” tells us that if it is true that a student likes computer science, then it must be true that they like mathematics. We can write:
\[
\forall x \in D, [(S(x) \land C(x)) \Rightarrow M(x)]
\]
or equivalently, as in part (c),
\[
\forall x \in D, [S(x) \Rightarrow (C(x) \Rightarrow M(x))]
\]

(g) “Any student who likes every instructor also likes themself.” This one is tricky. We can think of this as: If an object is a student and the student likes every instructor, then the student likes themself.
Let us break it down into parts. Suppose we have an object x. We know S(x) is true when x has the property of being a student. And we know L(x, x) is true when x likes themself. Another object y is an instructor when I(y) is true, and x likes y when L(x, y) is true. We can write:
\[
\forall x \in D, [(S(x) \land (\forall y \in D, I(y) \Rightarrow L(x, y))) \Rightarrow L(x, x)]
\]

There are other equivalent solutions.