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Many-to-Many Graph Matching

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Synonyms

Error-correcting graph matching; Error-tolerant graph matching; Inexact matching; Transportation problem

Related Concepts

►Graph Matching; ►Many-to-Many Feature Correspondence; ►Object Categorization

Definition

When objects exhibit large within-class variation and/or when image features are under- or over-segmented, the image features extracted from two exemplars belonging to the same category may no longer be in one-to-one correspondence but, in general, many-to-many correspondence. If the features are structured, i.e., captured in a graph, then computing the correct correspondence can be formulated as a many-to-many graph matching problem.

Background

The matching of image features to object models is typically formulated as a one-to-one assignment problem, based on the assumption that for every salient image feature belonging to the object to be matched, e.g., SIFT feature, image patch, contour fragment, there exists a single corresponding feature on the model (and vice versa). While the one-to-one correspondence assumption has been prevalent in the object recognition community throughout its entire evolution, including the paradigms of graph matching [9], alignment [13], geometric invariants [11], local appearance [14], and a recent return to local contour-based features [8], one-to-one feature correspondence is a highly restrictive assumption that breaks down as within-class variation increases and as the segmentation and extraction of more abstract image features suffer from over- or under-segmentation [7]. In the more general case, feature correspondence is not one-to-one, but rather *many-to-many*. If a feature set is described by a graph, with nodes representing features and edges capturing pairwise relations between features, computing the correct many-to-many feature correspondence can be formulated as *many-to-many graph matching*.

Consider two simple examples, shown in Fig. 1. In Fig. 1a, a set of multiscale blobs and ridges have been extracted from two exemplars (humans) belonging to the same category. In the top image, the straight arm yields a single elongated ridge, while in the bottom image, the bent arm yields two smaller and coterminating elongated ridges. In this case, simple object articulation (a form of within-class variation) has led to a violation of the one-to-one correspondence assumption. Instead, the correspondence is clearly

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two-to-one; enforcing one-to-one correspondence will lead to an incorrect matching of the entire arm to either the upper or lower arm, e.g., the red highlighted features. In Fig. 1b, two region segmentations of two exemplars belonging to the same class yield a set of region correspondences that are rarely one-to-one, but more typically many-to-many. Once again, enforcing a one-to-one feature correspondence will ensure an incorrect matching, and will miss the correct correspondence.

The problem of computing a one-to-one correspondence between a model feature graph and a cluttered image graph can be formulated as a largest isomorphic subgraph problem, whose complexity is NP-hard. The complexity of the many-to-many matching problem is even more prohibitive, for the space of possible correspondences is greater (any subset of features in the image may match any subset of features on the model). The intractable complexity of the many-to-many matching problem can only be reduced by exploiting the types of regularities suggested by the perceptual grouping community, such as proximity, continuity, conservation of mass, etc. In what follows, a formal statement of the problem is introduced, and a number of approaches to its solution is reviewed.

Theory

The main objective of the many-to-many graph matching problem is to establish a minimum cost mapping between the vertices of two attributed, edge-weighted graphs. In an attribute-weighted graph $G = (V, E)$, let $\mathbb{L}(v)$ denote the set of attributes associated with $v \in V$. Given a subset $U \subset V$, let $\mathbb{L}(U) = \cup_{u \in U} \mathbb{L}(u)$. For a set $U \subset V$, let $G|_U$ denote the subgraph of G induced on the vertices in U, and let $w(u, v)$ denote the weight of an edge $(u, v) \in E$. Finally, let $\mathbb{P}(G)$ denote the power-set 2^V for the vertex set of G. A *many-to-many mapping* between two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a mapping among power-sets $\mathbb{P}(G_1)$ and $\mathbb{P}(G_2)$ and can be characterized as a function:

$$\mathcal{M} : (\mathbb{P}(G_1) \times \mathbb{P}(G_2)) \to \{0, 1\}. \quad (1)$$

For two sets, $U \in \mathbb{P}(G_1)$ and $V \in \mathbb{P}(G_2)$, there will be a cost $C(\mathbb{L}(U), \mathbb{L}(V))$ associated with mapping the labels in set $\mathbb{L}(U)$ to those in $\mathbb{L}(V)$. An example of a common cost function is the edit-distance between the labels in sets $\mathbb{L}(U)$ and $\mathbb{L}(v)$. Let $\mathcal{S}(G_1|_U, G_2|_V)$ denote the structural distance between induced subgraphs $G_1|_U$ and $G_2|_V$. Observe that every mapping $\mathcal{M}$ has a natural representation as a matrix, with $\mathcal{M}_{U,V} = 1$ if the sets $U \in \mathbb{P}(G_1)$ and $V \in \mathbb{P}(G_2)$ are mapped to each other under $\mathcal{M}$, and $\mathcal{M}_{U,V} = 0$ otherwise. Combining these two cost functions will result in the cost function $C(\mathcal{M})$ associated with the mapping $\mathcal{M}$:

$$\mathcal{C}(M) = \sum_{U \in \mathbb{P}(G_1), V \in \mathbb{P}(G_2)} \mathcal{M}_{U,V} \quad (2)$$

$$\times\, C(\mathbb{L}(U), \mathbb{L}(V)) \times \mathcal{S}(G_1|_U, G_2|_V).$$

In defining an optimal many-to-many matching between two attributed graphs, G_1 and G_2, a many-to-many mapping $\mathcal{M}^*$ of minimum cost $C(\mathcal{M}^*)$ subject to specific requirements on the structure or cardinality of $\mathcal{M}^*$ will be obtained. For example, to prevent a trivial solution that sets $\mathcal{M}_{U,V} = 0$, for all U and V, one can require a matching such that its cardinality, i.e., $\sum_{U,V} \mathcal{M}_{U,V}$, exceeds a threshold while minimizing $C(\mathcal{M})$. Other functions, such as maximizing the number of vertices from V_1 and V_2 that participate in $\mathcal{M}$, can be used to evaluate the quality of the mapping. Note that cost functions $C(\mathbb{L}(U), \mathbb{L}(V))$ and $\mathcal{S}(G_1|_U, G_2|_V)$ may be used to enforce constraints such as consistency of mapped labels, limits of feasible label mappings, or allowed structural mapping of induced graphs $G_1|_U$ and $G_2|_V$ by imposing arbitrary large values or by being ill-defined.

The above description of the many-to-many matching results in an intractable computational problem. First, due to the exponential size of power-sets $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$ in terms of number of vertices in G_1 and G_2, the size of the search space for the many-to-many matching problem is exponential. Even simplifying the problem to one-to-one mappings, by replacing the power-sets $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$ with sets V_1 and V_2, respectively, will result in the multidimensional matching problem that is known to be NP-complete for arbitrary labeled graphs.

Related Work

Many-to-many graph matching has been studied extensively in a variety of contexts, including graph edit distance [2, 16], spectral methods [4, 18], optimization problems [20], metric embedding [6], abstract models [10], and grammars [1, 21]. The classical formulation of graph edit distance introduces a set of graph edit operations, such as insertion, deletion, merging, splitting, and substitution of nodes and edges. Given a set of graph edit operations and a cost function, the objective is to find the lowest cost sequence of graph edit operations that transform one graph into the other. The edit distance between two graphs critically depends on the costs of the underlying edit operations; typically, lower costs are assigned to the most frequent edit operations. A number of approaches have addressed the problem of defining an appropriate cost, e.g., [3].

Many-to-many graph matching has also been studied in the context of spectral methods by examining the spectral properties of graph adjacency matrices. In [4], the authors present an approach based on renormalization projections of vertices into a common eigensubspace of two graphs. Instead of finding the overall similarity of two graphs from the positions of vertex projections, this approach uses an agglomerative hierarchical clustering technique to produce many-to-many vertex correspondences.

Another spectral method is due to [18, 19], which constructs a low-dimensional "signature" of a directed graph's "shape" from the magnitudes of the eigenvalues of the graph's adjacency matrix. The eigenvalues are invariant to the reordering of a graph's branches and are shown to be robust under minor structural perturbation of the graph. This vector can be used for both structural indexing and for matching in the presence of noise and occlusion. If two signatures (vectors) are close, their corresponding (sub)graphs, possibly having different cardinalities, are in many-to-many correspondence.

Recently, the approach presented in [20] formulates the many-to-many graph matching problem as a discrete optimization problem. The algorithm starts by extending the optimization problem for one-to-one matching to the case of many-to-one matching. The algorithm then obtains many-to-many vertex correspondences through two many-to-one mappings. Since this formulation of the many-to-many matching requires the solution of a hard optimization problem, the authors propose an approximate algorithm based on a continuous relaxation of the combinatorial problem.

The concept of a low-distortion graph embedding has been used to obtain many-to-many vertex correspondences [6]. Specifically, low-distortion graph embedding is employed to transform the problem of many-to-many graph matching to a many-to-many point matching problem in a geometric space. This transformation maps nodes to points and edge weights to interpoint distances, not only simplifying the original graph representation (by removing the edges), but also retaining important local and global graph structure; moreover, the transformation is robust under perturbation. Representing two graphs as sets of points reduces the many-to-many graph matching problem to that of many-to-many point matching in the geometric space, for which a number of efficient distribution-based similarity measures are available. The authors use the Earth Mover's Distance [15] algorithm to find such correspondences and show that the resulting many-to-many point matching realizes the desired many-to-many matching between the vertices of the input graphs.

A number of researchers, e.g., [10, 12] and [5], have explored many-to-many graph matching in the context of model-based abstraction from images. The work presented in [10] starts by forming a region adjacency graph from each image. The approach then searches the space of pairwise region groupings in each graph, forming a lattice. Each input image yields a lattice such that its bottom node represents the original region adjacency graph and its top node represents the silhouette of the object. The framework defines a common abstraction as a set of nodes, one per lattice, such that for a pair of nodes, their corresponding graphs are isomorphic. The lowest common abstraction (LCA) is defined as the common abstraction whose underlying graph has the maximum number of nodes. Thus, the resulting LCA carries the most informative abstraction common to each image. Although effective, this technique does not find a match between two graphs whose common abstraction does not exist.

The two algorithms presented in [12] and [5] use the many-to-many graph matching technique of [6] for automatically constructing an abstract model from examples. The work in [12] computes the multiscale ridge/blob decomposition (AND-OR) graph for each input image and obtains the many-to-many node correspondences between each pair of graphs, yielding a matching matrix. By exploring this matrix, the algorithm first finds features that match one-to-one across many pairs of input images. The many-to-many matchings between these features are then analyzed to obtain the decompositional relations among them. The extracted features and their relations are used to construct the final abstract model.

After obtaining many-to-many node correspondences based on [6], the algorithm in [5] computes the abstracted medial axis graph by first computing the averages of the corresponding pairs of subgraphs to yield the nodes in the abstracted graph, and then defining the overall topology of the resulting abstract parts to yield the relations. Each matching pair of subgraphs corresponds to a single node in the abstracted graph, and two abstracted nodes are connected by an edge if the corresponding subgraphs are adjacent in the original graphs. This procedure forms the basis of an iterative framework in which pairs of similar medial axis graphs are clustered and abstracted, yielding a set of abstract medial axis graph class prototypes.

In the domain of grammars, objects are represented as variable hierarchical structures. Each part in this representation can be defined in terms of other parts, allowing an object to be modeled by its coarse-to-fine appearance. Overall, grammar-based models including AND-OR graphs support structural variability. To represent intra-category variation and to account for many-to-many correspondence, the grammar creates a large number of configurations from a small vocabulary set. To scale to a large number of object categories, the AND-OR graph, learning, and inference algorithms are defined recursively. Some examples of this type of approach include [1, 21].

Experimental Results

In this section, some example results from some of the many-to-many matching approaches described in the Related Work section are illustrated. After representing silhouettes as skeleton graphs in Fig. 2, the algorithm proposed in [6] obtains many-to-many node correspondences through metric embedding, as discussed earlier. Based on the many-to-many correspondences of this algorithm, Fig. 3 demonstrates an example for the abstract shape created by the approach presented in [5]. The left part presents input silhouettes, their skeleton graphs, and many-to-many correspondences. The right part presents the abstract skeleton graph and its shape reconstructed from this graph.

Graph edit distance is another important class of many-to-many graph matching algorithms. Figure 4 shows the result of matching the skeleton graphs for two input shapes using the graph edit distance algorithm described in [16]. Same colors indicate the matching skeleton parts while gray colors show spliced or contracted edges. Observe that the many-to-many correspondences are intuitive in these figures.

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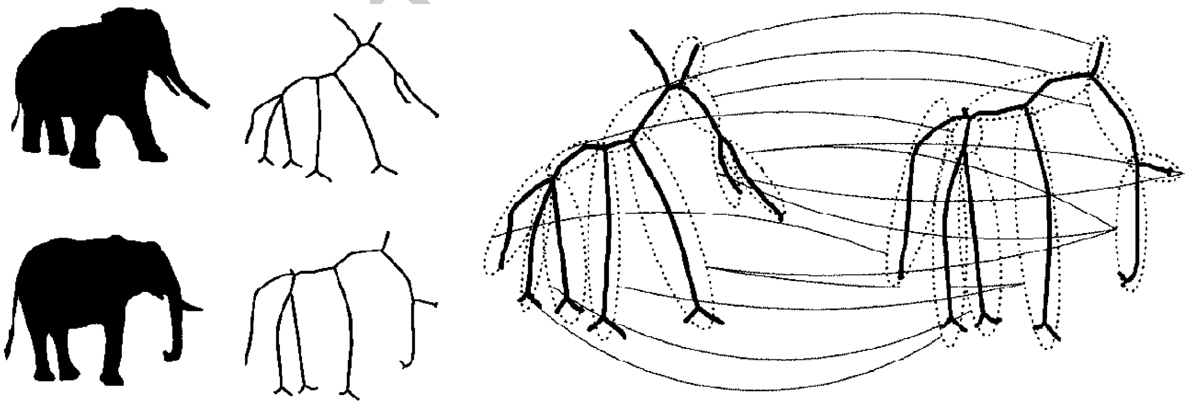
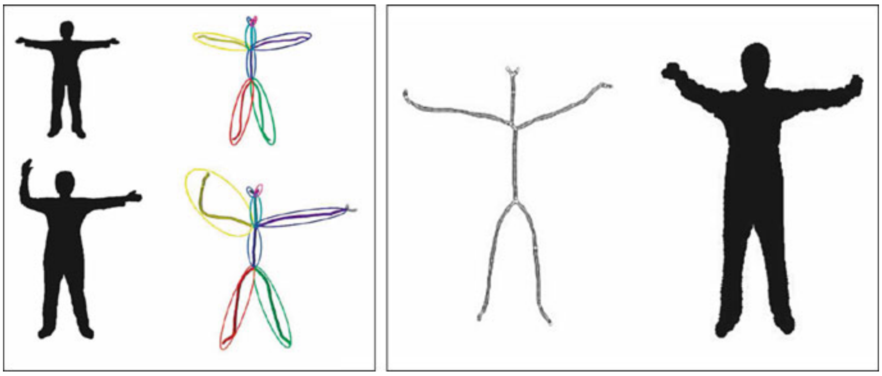
Many-to-Many Graph Matching, Fig. 1 Two graph matching problems in computer vision for which assuming a one-to-one feature correspondence will lead to incorrect correspondences, and which can only be solved if formulated as a many-to-many graph-matching problem. In (**a**), a multiscale blob and ridges decomposition [17] of the two humans yields a single ridge for the extended arm (*top*) and two coterminating ridges for the bent arm (*bottom*). In this example, articulation has violated the one-to-one feature correspondence assumption; if a one-to-one correspondence is enforced for the arm, e.g., the red highlighted features, it will be incorrect. In this case, the correspondence should be two-to-one (or more generally, many-to-many). In (**b**), two different cup exemplars (*bottom row*) have been region segmented (*top row*), yielding regions that are rarely in one-to-one correspondence (due to within-class variation or region over- and/or under-segmentation). Once again, the correct correspondence is not one-to-one, but rather many-to-many



Many-to-Many Graph Matching, Fig. 2 Example many-to-many correspondences computed by [6]. After representing two silhouettes as skeleton graphs, the graphs are embedded into geometric spaces of the same dimensionality. The embedded points are then matched using the Earth Mover's Distance algorithm. The *right part* illustrates the many-to-many correspondences between the vertices of the input graphs. Each *dashed ellipsoid* represents a set of vertices from the original graph

Many-to-Many Graph Matching, Fig. 3 A shape abstraction example of [5] based on many-to-many correspondences obtained by [6]. The *left image* shows input silhouettes and their skeleton graphs in which the same color is used to show the corresponding parts. Using these correspondences, the abstract skeleton graph and its silhouette are created as shown on the *right*



Many-to-Many Graph Matching, Fig. 4 Graph edit distance algorithms compute many-to-many correspondences of a pair of graphs by finding the lowest cost sequence of graph edit operations needed to transform one graph into another. In the example, same colors indicate the matching skeleton parts, while gray colors show spliced or contracted edges (The example is taken from Ref. [16])