Dichotomy for tree-structured trigraph list homomorphism problems

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Abstract

Trigraph list homomorphism problems (also known as list matrix partition problems) have generated recent interest, partly because there are concrete problems that are not known to be polynomial time solvable or NP-complete. Thus while *digraph* list homomorphism problems enjoy *dichotomy* (each problem is NP-complete or polynomial time solvable), such dichotomy is not necessarily expected for trigraph list homomorphism problems. However, in this paper, we identify a large class of trigraphs for which list homomorphism problems do exhibit a dichotomy. They consist of trigraphs with a tree-like structure, and, in particular, include all trigraphs whose underlying graphs are trees. In fact, we show that for these tree-like trigraphs, the trigraph list homomorphism problem is polynomially equivalent to a related *digraph* list homomorphism problem. We also describe a few examples illustrating that our conditions defining the tree-like trigraphs are necessary at least for some trigraphs.

Key words: trigraph, list homomorphism, matrix partition, trigraph homomorphism, surjective list homomorphism, dichotomy, trigraph tree

1. Introduction

A *trigraph* H consists of a set $V = V(H)$ of vertices, and two disjoint sets of directed edges on V – the set of *weak edges* $W(H) \subseteq V \times V$, and the set of *strong edges* $S(H) \subseteq V \times V$. If both edge sets $W(H), S(H)$, viewed as relations on V, are symmetric, we have a *symmetric*, or *undirected* trigraph. A weak, respectively strong, edge vv is called a *weak*, respectively *strong, loop* at v.

The *adjacency matrix* of a trigraph H, with respect to an enumeration $v_1, v_2, \ldots, v_n$ of its vertices, is the $n \times n$ matrix M over $0, 1, *$, in which $M_{i,j} = 0$ if $v_i v_j$ is not an edge, $M_{i,j} = *$ if $v_i v_j$ is a weak edge, and $M_{i,j} = 1$ if $v_i v_j$ is a strong edge. Note that a trigraph H is symmetric if and only if its adjacency matrix is symmetric.

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[1]This research was partially supported by P. Hell's NSERC Discovery Grant.

We consider the class of digraphs included in the class of trigraphs, by viewing each digraph H as a trigraph with the same vertex set $V(H)$, and with the weak edge set $W(H) = E(H)$ and strong edge set $S(H) = \emptyset$. Conversely, if H is a trigraph, the *associated digraph* of H is the digraph with the same vertex set $V(H)$, and with the edge set $E(H) = W(H) \cup S(H)$. Moreover, the *underlying graph* of the trigraph H is the underlying graph of the associated digraph, and the *symmetric graph* of the trigraph H is the symmetric graph of the associated digraph. To be specific, xy is an edge of the *underlying graph* of H just if $xy \in W(H) \cup S(H)$ or $yx \in W(H) \cup S(H)$, and xy is an edge of the *symmetric graph* of H just if $xy \in W(H) \cup S(H)$ and $yx \in W(H) \cup S(H)$. These conventions allow us to extend the usual graph and digraph terminology to trigraphs. We speak, for instance, of *adjacent vertices, components, neighbours, cutpoints, or bridges* of a trigraph H, meaning the corresponding notions in the associated digraph of H, or in its underlying graph; and we speak of *symmetric edges, symmetric neighbours, etc.* in a trigraph H, meaning the edges, neighbours, etc., in the symmetric graph of H.

Let G be a digraph and H a trigraph. A *homomorphism* of G to H is a mapping $f : V(G) \to V(H)$ such that the following two conditions are satisfied for each $u \neq v$:

– if $uv \in E(G)$ then $f(u)f(v) \in W(H) \cup S(H)$
– if $uv \notin E(G)$ then $f(u)f(v) \notin S(H)$.

In other words, edges of G must map to either weak or strong edges of H, and non-edges of G must map to either non-edges or weak edges of H.

If each vertex v of the digraph G has a *list* $L(v) \subseteq V(H)$, then a *list homomorphism* of G to H, with respect to the lists L, is a homomorphism f of G to H such that $f(v) \in L(v)$ for all $v \in V(G)$. Following standard practice [16], we also call a homomorphism of G to H an *H-colouring of* G, and a list homomorphism of G to H (with respect to lists L) a *list H-colouring* of G (with respect to L).

Suppose H is a fixed trigraph. The *H-colouring problem* HOM(H) has as instances digraphs G, and asks whether or not G admits an H-colouring. The *list H-colouring problem* L-HOM(H) has as instances digraphs G with lists L, and asks whether or not G admits a list H-colouring with respect to L. As noted earlier, H could be a digraph, viewed as a trigraph (with $W(H) = E(H)$, $S(H) = \emptyset$). Digraph homomorphism and list homomorphism problems have been of much interest [16]. If necessary, we will emphasize the distinction between *trigraph* list homomorphism problems and *digraph* list homomorphism problems, depending on whether H is a trigraph or a digraph respectively, i.e., whether H has any strong edges or not. However, note that the input G is always a digraph.

For a fixed trigraph H, the list H-colouring problem L-HOM(H) concerns the existence of vertex partitions of the input digraphs G. For instance, if H is the undirected trigraph with $V(H) = \{0, 1\}$, with a strong loop at 1 and a weak edge joining 0 and 1, then an H-colouring of G is precisely a partition of $V(G)$ into a clique and an independent set. Thus G is H-colourable if and only if G is a split graph. Many graph partition problems, especially those arising from the theory of perfect graphs, can be formulated as trigraph homomorphism (or list homomorphism) problems; this is discussed in detail in [7]. Equivalently, all these problems can be described in terms of the adjacency matrix of the trigraph, in the language of matrix partitions and list partitions (see [2, 7, 9, 10]). In this paper it will be more convenient to emphasize the trigraph (rather than the matrix) terminology, since we are dealing with the structure of the trigraph H.

It is generally believed [12] that for each digraph H the H-colouring problem HOM(H) is NP-complete or polynomial time solvable. (This is equivalent to the so-called *CSP Dichotomy Conjecture* of Feder and Vardi [12].) One special case when the dichotomy conjecture is known to hold is the case of undirected graphs (i.e., symmetric digraphs). In this case, HOM(H) is polynomial time solvable if H has a loop or is bipartite and is NP-complete otherwise [17]. For the list homomorphism problem L-HOM(H), it is shown in [6] that L-HOM(H) is polynomial time solvable if H is a so-called *bi-arc graph* (a simultaneous generalization of reflexive interval graphs and bipartite graphs whose complements are circular arc graphs), and is NP-complete otherwise. A more general result of Bulatov [1] handles all constraint satisfaction problems, implying, in particular, that for each digraph H, the list H-colouring problem L-HOM(H) is NP-complete or polynomial time solvable. By contrast, this is not known for trigraph list homomorphism problems, and in [3] it is only proved that for each trigraph H, the list H-colouring problem is NP-complete or *quasi-polynomial* (of complexity $n^{O(\log^k n)}$). All list H-colouring problems L-HOM(H) for trigraphs H with three or fewer vertices have been classified as NP-complete or polynomial time solvable in [10]. For symmetric trigraphs with four vertices, this has been accomplished in [2], with the exception of a single trigraph H; the corresponding problem remains open and has earned the name *the stubborn problem* (Figure 3a). The best known algorithm for this problem has complexity $n^{O(\log n/\log\log n)}$, implying it is unlikely to be NP-complete [9]. Thus (polynomial / NP-complete) dichotomy for trigraph list homomorphism problems seems less likely than for *digraph* list homomorphism problems.

In this paper we prove dichotomy for the class of *trigraph trees*, i.e., for trigraphs whose underlying graph is a tree. It turns out that if H is a trigraph tree, then the list H-colouring problem is polynomially equivalent to a list H^--colouring problem where H^- is a *digraph* obtained from H by removing all vertices with a strong loop and removing all other strong edges xy (and their converses yx if any).

We conduct the proof of this result in such a way that it in fact implies the dichotomy for a large class of trigraphs, which includes all digraphs and all trigraph trees. We think of these trigraphs as *tree-like*, although it is only the structure of the strong edges that is tree-like. For trigraphs that are not tree-like (in our definition), we illustrate the possible complications. We believe that our class of tree-like trigraphs covers an important portion of the class of trigraphs H in which the strong edges do not significantly impact the complexity of the list H-colouring problem.

2. Tools

Let H be a trigraph, and let G be a digraph with list $L(u) \subseteq V(H)$ for each $u \in V(G)$. We shall denote by n the number of vertices in G, and by k the number of vertices in H. We say that lists L' are *a reduction of* L, if we have $L'(u) \subseteq L(u)$ for each $u \in V(G)$.

In the following text, we shall say that *lists L can be reduced to satisfy property P* to mean the following statement: for every instance consisting of a graph G with lists L, there exists lists L' such that (i) the lists L' can be found in polynomial time, (ii) the lists L' are a reduction of L, (iii) the lists L' satisfy property P, and (iv) G has a list H-colouring with respect to L if and only if G has a list H-colouring with respect to L'.

We shall say that *lists L can be transformed to satisfy property P* to mean that for every instance G with lists L, there exists a set $\mathcal{L} = \{L_i\}_{i\in I}$ of lists such that (i) the set

$\mathcal{L}$ can be constructed in polynomial time, (ii) each L_i is a reduction of L, (iii) each L_i satisfies property P, and (iv) G has a list H-colouring with respect to L if and only if there exists $i \in I$ such that G has a list H-colouring with respect to L_i.

We illustrate the two concepts in Figure 1.

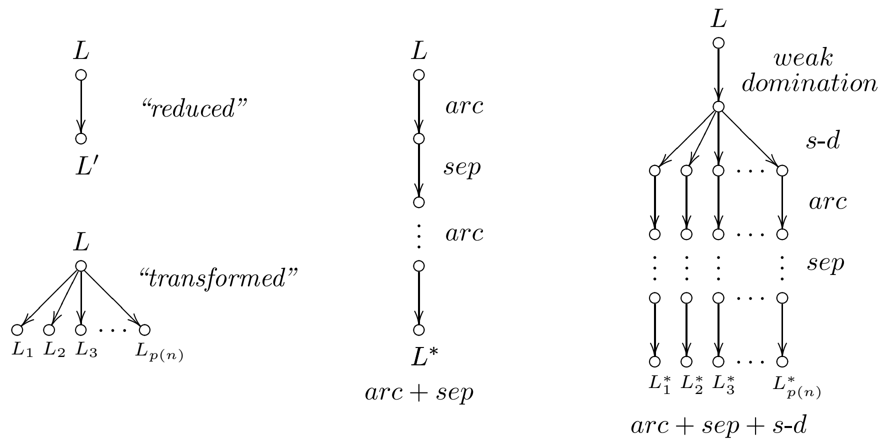

Figure 1: Illustrating the concepts of "reduced" and "transformed". The circles represent lists. An arrow from L to L' means that L' is a reduction of L. The middle and the right figure illustrate Lemma 6 and 9 respectively. Labels *arc, sep,* and *s-d* are shorthands for arc-consistency, separator-consistency, and sparse-dense-consistency.

We say that lists L are *non-empty* if the list $L(u)$ for each vertex $u \in V(G)$ is not empty. Clearly, if G admits a list homomorphism to H with respect to L, then the lists L are non-empty. Hence, to avoid trivial cases in what follows, we shall always assume that lists L are non-empty.

We say that lists L are *arc-consistent* if for each $u, v \in V(G)$ and each $x \in L(u)$, there exists $y \in L(v)$ such that (i) $xy \in W(H) \cup S(H)$ if $uv \in E(G)$, (ii) $yx \in W(H) \cup S(H)$ if $vu \in E(G)$, (iii) $xy \notin S(H)$ if $uv \notin E(G)$, and (iv) $yx \notin S(H)$ if $vu \notin E(G)$.

Lemma 1. *Lists L can be reduced to be arc-consistent.*

Proof. If some $u, v \in V(G)$ violate the above condition for $x \in L(u)$, no H-colouring of G with respect to L maps u to x. Hence, x can be removed from $L(u)$ without changing the solution. We repeatedly test for such violations and reduce the lists if a violation is found. After at most $n \times k$ such steps either we obtain arc-consistent lists, or some list $L(u)$ becomes empty, in which case there is no solution. $\square$

We say that lists L contain *representatives*, if there is a set $X \subseteq V(H)$ such that

(i) for each $v \in V(G)$, the list $L(v) \subseteq X$, and
(ii) for each $x \in X$, there is a vertex $v \in V(G)$ with $L(v) = \{x\}$,

Lemma 2. *Lists L can be transformed to contain representatives.*

Proof. We choose a subset X of the vertices of H. For each $x \notin X$, we remove x from all lists. For each $x \in X$, we find some vertex v with $x \in L(v)$ and change its list to $\{x\}$. If this is possible, we obtain lists containing representatives. Otherwise, we make a different choice of vertices v or set X. It can be seen that we have at most $(n+1)^k$ different lists we can construct this way. The claim now follows. $\square$

The *symmetric trigraph H' associated with H* is the trigraph on the vertices of H with strong edges xy iff $xy, yx \in S(H)$, and with weak edges xy iff $xy \notin S(H)$ or $yx \notin S(H)$ but $xy, yx \in W(H) \cup S(H)$.

A graph G is *chordal* if G contains no induced cycle of length four or longer. A *chordal completion* G' of G is a chordal graph on the vertices of G with $E(G') \supseteq E(G)$. A chordal completion G' is *minimal* if no chordal completion G'' of G satisfies $E(G'') \subsetneqq E(G')$.

Proposition 3. *[19] Let G' be a minimal chordal completion of G, and let C be a clique of G. Then there are no edges in G' between vertices of different components of $G - C$.*

We say that lists L contain *strong representatives*, if for each strong loop x of H, there is a set $S_x \subseteq V(H)$ such that

(i) $L(v) \subseteq S_x$ whenever $x \in L(v)$,
(ii) each vertex of $S_x \setminus \{x\}$ is a symmetric neighbour of x, and
(iii) $S_x \setminus \{x\}$ is connected in the symmetric graph of H.

Lemma 4. *Lists L can be transformed to contain strong representatives.*

Proof. Let G be a digraph with lists L, and let f be a list H-colouring of G with respect to L. By Lemmata 1 and 2, we may assume that the lists L are arc-consistent and contain representatives.

Let x be a strong loop of H. Since f is a homomorphism, if $f(u) = f(v) = x$ for $u, v \in V(G)$, we must have both uv and vu in $E(G)$. Hence, the set $C = f^{-1}(x)$ induces a symmetric clique in G.

Let G' be the symmetric graph of G, and H' be the symmetric trigraph associated with H. Clearly, C also induces a clique in G'. Moreover, it is easy to show that f is also a list H'-colouring of G' with respect to L.

Now, suppose that x appears on some list. Then, since the lists L contain representatives, there is a vertex $v_x \in V(G)$ with $L(v_x) = \{x\}$.

Let N denote the subset of $V(H)$ containing x and its symmetric neighbours. Let M denote the subset of $V(G)$ containing v_x and its symmetric neighbours. Let B denote the subset of $V(G)$ containing all vertices $u \in V(G)$ with $x \in L(u)$. Since the lists L are arc-consistent, we must have $B \subseteq M$. Also, for each $u \in M$, we have $L(u) \subseteq N$.

Now, since f respects lists L, we have $x \in L(u)$, for each $u \in C$. Hence, $C \subseteq B$, and therefore, C induces a clique in $G'[B]$. Furthermore, since $B \subseteq M$, we have $L(u) \subseteq N$ for each $u \in B$. Hence, f restricted to B is a homomorphism of $G'[B]$ to $H'[N]$. In particular, each component of $G'[B] - C$ maps by f to a unique component of $H'[N] - x$.

Now, let G'' be a minimal chordal completion of $G'[B]$. Let C'' be the vertices of a maximal clique of G'' that completely contains C. By Proposition 3, all vertices of $C'' \setminus C$ belong to one component of $G'[B] - C$. Hence, by the above remark, there is a unique component K of $H'[N] - x$ such that $f(u) \in V(K)$ for each $u \in C'' \setminus C$.

It now follows that G admits a list H-colouring with respect to L if and only if for some maximal clique C'' of a minimal chordal completion G'' of $G'[B]$, and some component K of $H'[N]-x$, the graph G admits a list H-colouring with respect to L such that the vertices outside of C'' do not map to x, and the vertices in C'' map to x or the vertices of K. Hence, we can modify the lists L by removing x from the vertices outside C'' and by reducing the lists of the vertices in C'' to $V(K) \cup \{x\}$. Such lists, clearly, contain a strong representative for x. (Take S_x to be $V(K) \cup \{x\}$.)

To conclude, we remark that the graph G' and the sets B and N, as well as, a minimal chordal completion G'' of $G'[B]$ by the result [19] can be found in polynomial time. Since G'' is chordal, it has at most n maximal cliques [14]. Also, there are at most k different components K of $H'[N]-x$. Hence, we can reduce the problem to at most $(nk)^k$ different instances with strong representatives. The proof is now complete. $\square$

Note that if lists L contain representatives, respectively strong representatives, then any non-empty reduction L' of L also contains representatives, respectively strong representatives.

Let F be a set of edges of H. Let $G \mathbin{\backslash\!\!\backslash} F$ denote the graph obtained from G by removing all edges $uv \in E(G)$ such that $xy \in F$ for some $x \in L(u)$ and $y \in L(v)$.

We say that lists L are *separator-consistent on* F, if for each component C of $G \mathbin{\backslash\!\!\backslash} F$ and each component K of $H \setminus F$,

(i) either there exists a list K-colouring of C with respect to L,
(ii) or no vertex of K appears on the list of any vertex of C.

Lemma 5. *If L-HOM($H \setminus F$) is polynomial time solvable, then lists L can be reduced to be separator-consistent on F.*

Proof. We obtain a separator-concistent lists L' from L as follows. For each component C of $G \mathbin{\backslash\!\!\backslash} F$, and each component K of $H \setminus F$, we test if C admits a list K-colouring with respect to L. If not, then we remove all vertices of K from the lists of the vertices of C. Since homomorphisms map connected graphs only to connected graphs, the claim follows. $\square$

In particular, we have the following property.

Lemma 6. *If L-HOM($H \setminus F$) is polynomial time solvable, then lists L can be reduced to be arc-consistent and separator-consistent on F.*

Proof. We apply Lemmata 1 and 5 to L until the lists no longer change. Since at each step the lists are reduced, the claim follows. $\square$

We remark that for $X \subseteq V(H)$, we say that lists L are *separator-consistent on* X, if they are separator-consistent on F, where F is the set of edges of H with at least one endpoint in X.

Let X and Y be two sets of vertices of H such that each vertex of X has a strong loop, and no vertex of Y has a loop.

We say that lists L are *sparse-dense-consistent on* X *and* Y if for each $v \in V(G)$ such that $L(v) \subseteq X \cup Y$, we have $L(v) \subseteq X$ or $L(v) \subseteq Y$.

Lemma 7. *If L-HOM($H[X]$) and L-HOM($H[Y]$) are polynomial time solvable, then lists L can be transformed to be sparse-dense-consistent on X and Y.*

Proof. Let $\mathcal{S}$ and $\mathcal{D}$ be classes of digraphs closed under taking induced subgraphs. Suppose that there is a constant $c = c(\mathcal{S}, \mathcal{D})$ such that each digraph in $\mathcal{S} \cap \mathcal{D}$ has at most c vertices. Then by [7], for any n-vertex digraph G, there are at most n^{2c} partitions $V(G) = V_1 \cup V_2$ such that $G[V_1] \in \mathcal{S}$ and $G[V_2] \in \mathcal{D}$. (We call each such partition an $(\mathcal{S}, \mathcal{D})$-*partition* of G.) All these partitions can be enumerated in time $n^{2c+2}T(n)$ where $T(n)$ is the complexity of recognizing digraphs in $\mathcal{S}$ and in $\mathcal{D}$.

Now, let $\mathcal{S}$ be the set of all digraphs admitting an $H[X]$-colouring and $\mathcal{D}$ be the set of all digraphs admitting an $H[Y]$-colouring; we have $c(\mathcal{S}, \mathcal{D}) \leq |X| \cdot |Y|$. Let Z denote the vertices u of G with $L(u) \subseteq X \cup Y$. We observe that any list H-colouring of G with respect to L induces an $(\mathcal{S}, \mathcal{D})$-partition of $G[Z]$. Hence, for each $(\mathcal{S}, \mathcal{D})$-partition $Z = A \cup B$ of $G[Z]$, we construct lists L' on H as follows: $L'(u) = L(u) \cap X$ if $u \in A$, $L'(u) = L(u) \cap Y$ if $u \in B$, and $L'(u) = L(u)$ otherwise. Clearly, L' is sparse-dense consistent on X and Y. Now, since there are at most n^{k^2} such partitions, the claim follows. $\square$

Let D be a set of vertices of H, and let x, y be two vertices of H. We say that y *weakly dominates* x *on* D, if for each $z \in D$, we have that zy is a weak edge in H whenever zx is a weak edge in H, and yz is a weak edge in H whenever xz is a weak edge in H.

We say that y *weakly dominates* x if y weakly dominates x on $V(H)$.

Lemma 8. *If y weakly dominates x, then lists L can be reduced to satisfy $x \notin L(u)$ whenever $y \in L(u)$.*

Proof. By Lemmata 1 and 2, we may assume that the lists L are arc-consistent and contain representatives. Let f be a list H-colouring of G with respect to L. Let f' be a mapping such that $f'(u) = y$ if $f(u) = x$ and $y \in L(u)$, and $f'(u) = f(u)$ otherwise. We show that f' is also a homomorphism and the claim will follow.

Let t be any vertex (including x and y) of H. Suppose that $xt \in S(H)$, yt is not an edge, and t appears on some list. We claim that no list $L(u)$ contains both x and y. To prove this, let v_t be a vertex with $L(v_t) = \{t\}$. Suppose that $x, y \in L(u)$. If $uv_t \in E(G)$, then by arc-consistency of L, we obtain $y \notin L(u)$. If $uv_t \notin E(G)$, similarly $x \notin L(u)$, a contradiction.

Moreover, by symmetry, if $tx \in S(H)$ and ty is not an edge, or if x and y exchange places, we also have that no list contains both x and y. In addition, since y weakly dominates x, we obtain that either both x and y have strong loops, or both have no loops, or y has a weak loop.

Now, it is not difficult to directly verify that f' is a homomorphism using the above observations and the fact that y weakly dominates x.

Hence, G has a list H-colouring with respect to L if and only if G has a list H-colouring with respect to lists L' obtained from L by removing x from each list $L(v)$ that also contains y. That concludes the proof. $\square$

Let X, Y, and Z be three sets of vertices of H such that each vertex of X has a strong loop, no vertex of Y has a loop, and each vertex of Z has a weak loop. Suppose that each vertex of Z weakly dominates each vertex of Y, and that we have $L(u) \subseteq X \cup Y \cup Z$ whenever $L(u) \cap X \neq \emptyset$.

Lemma 9. *Let X, Y, Z be as above. If L-HOM($H[X]$) and L-HOM($H - X$) are polynomial time solvable, then lists L can be transformed to be arc-consistent, separator-consistent on X, and sparse-dense-consistent on X and Y.*

Proof. We observe that by Lemma 8, we may assume that no list $L(u)$ contains both a vertex of Y and a vertex of Z. Hence, either $L(u) \subseteq X \cup Z$, or $L(u) \subseteq X \cup Y$, or $L(u) \cap X = \emptyset$ for each $u \in V(G)$.

Now, let L_i be one of the lists we obtain from L by applying sparse-dense-consistency on X and Y (Lemma 7). Clearly, we have either $L_i(u) \subseteq X$, or $L_i(u) \subseteq Y$, or $L_i(u) \cap X = \emptyset$, or $L_i(u) \cap Y = \emptyset$ for each $u \in V(G)$. In particular, any reduction of L_i must also satisfy this condition. Hence, we can apply Lemma 6 to L_i, and the claim follows. □

Let $A \cup B$ be a partition of the vertices of H. Let F be the edges of H that have exactly one endpoint in A. Let X respectively Y be the vertices of A respectively B with at least one incident edge of F.

We say that G is *separable* on F, if for each vertex $v \in V(G)$,

(i) $L(v)$ contains at most one vertex of X and at most one vertex of Y,
(ii) if $L(v)$ contains a vertex of X, it contains no vertex of $A \setminus X$, and
(iii) if $L(v)$ contains a vertex of Y, it contains no vertex of $B \setminus Y$.

Lemma 10. *If L-HOM($H - X - Y$) is polynomial time solvable, then L-HOM(H) is polynomial time solvable on the class of all digraphs G separable on F.*

Proof. First, we observe if G with lists L is separable on F, then for any reduction L' of L, the graph G with lists L' is separable on F. Hence, by Lemmata 1 and 2, we may assume that lists L are arc-consistent and contain representatives.

Next, we show that L-HOM($H[A]$) can be solved in polymial time on any $G \setminus\!\!\setminus F$ given that G is separable on F. Indeed, consider a component C of $G \setminus\!\!\setminus F$, and let L' be lists such that $L'(u) = L(u) \cap A$ for each $u \in V(C)$. Since G is separable on F, we have either $L'(u) \subseteq A \setminus X$, or $|L'(u)| = 1$. Hence, if B denotes the vertices with $|L'(u)| = 1$, then arc-consistency of L implies that C has a list $H[A]$-colouring with respect to L if and only if $C - B$ has a list $(H - X - Y)$-colouring with respect to L'. Similarly, for L-HOM($H[B]$). Hence, L-HOM($H \setminus F$) can be solved in polynomial time for $G \setminus\!\!\setminus F$ given G is separable on F.

It now follows by Lemma 6 that we may assume that the lists L are arc-consistent and separator-consistent on F. We also assume that lists L are non-empty, since otherwise there is no solution.

Let H_0 be the trigraph constructed from $H[X \cup Y]$ by adding vertices a and b with weak loops such that a has a weak symmetric edge to each vertex of X, and b has a weak symmetric edge to each vertex of Y.

Let L_0 be lists obtained from L by replacing by a each $z \in L(u)$ such that $z \in A \setminus X$, and replacing by b each $z \in L(u)$ such that $z \in B \setminus Y$. Also, let $A_0 = X \cup \{a\}$ and $B_0 = Y \cup \{b\}$.

Let f_0 be a list H_0-colouring of G with respect to L_0. Let f be a list $(H \setminus F)$-colouring of $G \setminus\!\!\setminus F$ with respect to L such that for each $u \in V(G)$, we have $f(u) \in A$ if $f_0(u) \in A_0$, and $f(u) \in B$ if $f_0(u) \in B_0$. Since the lists L are separator-consistent on F, such colouring can be found.

We make some observations about f. First, note that $f(u) = x$ if $f_0(u) = x$ where $x \in X$. Indeed, this follows since G is separable on F, and f_0 respects lists L_0. Similarly, $f(u) = y$ if $f_0(u) = y \in Y$. Moreover, $f(u) \notin X$ if $f_0(u) = a$, and $f(u) \notin Y$ if $f_0(u) = b$. It now follows that f is a homomorphism of G to H with respect to L.

Finally, we observe that the lists L_0 are all of size at most two. Hence, the mapping f_0 can be found in polynomial time by the standard reduction to $2SAT$. That concludes the proof. $\square$

3. Dichotomy for trigraph trees

In this section, we prove the dichotomy for L-HOM(H) for trigraphs trees H, i.e., for trigraphs H whose underlying graph is a tree. We remark that some partial results along these lines are included in [18]; specifically, the case when the underlying graph of H is a path is discussed there.

Let H^- be the digraph obtained from a trigraph H by removing all edges xy such that at least one of xy, yx is a strong edge of H, and by removing all vertices x such that xx is a strong loop in H.

Theorem 11. *If H is a trigraph tree, then L-HOM(H) is polynomially equivalent to L-HOM(H^-).*

Corollary 12. *If H is a trigraph tree, then L-HOM(H) is polynomial time solvable or NP-complete.*

Proof. Suppose that L-HOM(H) is polynomial time solvable. Observe that, since the underlying graph of H is a tree, H^- is an induced subgraph of H. Hence, any instace of L-HOM(H^-) is also an instance of L-HOM(H). Therefore, L-HOM(H^-) is also polynomial time solvable.

Suppose that L-HOM(H^-) is polynomial time solvable. We prove the Theorem by induction on the size of $V(H)$. Hence, we shall assume that for each vertex x of H, L-HOM($H - x$) is polynomial time solvable.

Let G with lists L be an instance of L-HOM(H). If H contains no strong loops or strong edges, then $H = H^-$ and there is nothing to prove.

Suppose that H contains a strong loop at x. By Lemmata 1, 2, and 4 we may assume that the lists L are arc-consistent and contain representatives and strong representatives.

Consider the strong representative S_x of x. Let B denote the vertices $u \in V(G)$ with $x \in L(u)$. If x has no symmetric neighbours, $S_x = \{x\}$. Hence, $L(v) = \{x\}$ whenever $x \in L(v)$. Therefore, since the lists L are arc-consistent, G admits a list H-colouring with respect to L if and only if $G - B$ admits a list $(H - x)$-colouring with respect to L. Since L-HOM($H - x$) is polynomial time solvable, the claim follows.

Now, suppose that x has symmetric neighbours. Since the underlying graph of H is a tree, it follows that $S_x = \{x, y\}$ where y is a symmetric neighbour of x.

First, suppose that y has no loop. We apply Lemma 9 for $X = \{x\}$, $Y = \{y\}$, and $Z = \emptyset$, and we see that we may assume that the lists L are arc-consistent and sparse-dense-consistent on $\{x\}$ and $\{y\}$. That is, we have $L(u) = \{x\}$ or $L(u) = \{y\}$ or $L(u) \setminus \{x, y\} \neq \emptyset$ for each $u \in V(G)$. In particular, because $S_x = \{x, y\}$, we have $L(u) = \{x\}$ whenever $x \in L(u)$. Since the lists L are arc-consistent, the claim follows exactly as above.

Next, suppose that y has a weak loop. Let K be the component of $H - x$ to which y belongs. Observe that y weakly dominates x on $V(K)$. We apply Lemma 9 for $X = \{x\}$, $Y = \emptyset$, and $Z = \{y\}$. This implies that we may assume that the lists L are arc-consistent and separator-consistent on X. Moreover, $G - B$ admits a list $(H - x)$-colouring with respect to L, since otherwise LHOM(H) has no solution for G with lists L.

Hence, we let f_0 be a list $(H - x)$-colouring of $G - B$ with respect to L such that f_0 reduced to any component C of $G - B$ is a K-colouring as long as a vertex of K appears on the list of some vertex of C. Since the lists L are separator-consistent on X, such mapping must exist. In fact, f_0 can be constructed in polynomial time, since L-HOM($H - x$) is polynomial time solvable.

We extend this mapping to G and show that this yields a homomorphism. Let f be a mapping defined as follows.

$$f(u) = \begin{cases} x & \text{if } u \in B \text{ and } L(u) = \{x\} \\ y & \text{if } u \in B \text{ and } L(u) = \{x, y\} \\ f_0(u) & \text{otherwise} \end{cases}$$

Note that since $S_x = \{x, y\}$, we have for $u \in B$ either $L(u) = \{x\}$ or $L(u) = \{x, y\}$. Hence, the mapping f is well-defined. Moreover, f clearly respects lists L. We show that f is also a homomorphism.

Suppose that $f(u)f(v) \notin W(H) \cup S(H)$ for some $uv \in E(G)$. Clearly, at least one of u, v must belong to B, since f_0 is a homomorphism. Also, if $L(u) = \{x\}$, or $L(v) = \{x\}$, or both $u, v \in B$, we obtain a contradiction by arc-consistency of L. Hence, we can assume, by symmetry, that $u \in B$, $v \notin B$, and $f(u) = y$. Since the list L are arc-consistent, there exists $s \in L(v)$ such that $ys \in W(H) \cup S(H)$. In particular, $s \neq x$ since $v \notin B$, and hence, s belongs to K. Therefore, if C is the component of $G - B$ to which v belongs, f_0 restricted to C is a K-colouring. In particular, $f(v) = f_0(v) \in V(K)$.

Now, if $xf(v) \in S(H)$, both x and y cannot belong to $L(u)$ by arc-consistency. If $xf(v) \in W(H)$, then $yf(v) \in W(H)$ since y weakly dominates x on $V(K)$. Therefore, $xf(v) \notin W(H) \cup S(H)$. But, since the lists L are arc-consistent, and $L(u) = \{x, y\}$, we obtain a contradiction.

Suppose now that $f(u)f(v) \in S(H)$ for $uv \notin E(G)$. Clearly, $u \in B$ or $v \in B$. Again, if $L(u) = \{x\}$, or $L(v) = \{x\}$, or $u, v \in B$, we have a contradiction. Hence, we can assume, by symmetry, that $u \in B$, $v \notin B$, and $f(u) = y$. Also, since $yf(v)$ is an edge and $v \notin B$, $f(v)$ belongs to K. Now, similarly as above, $xf(v)$ can neither be a non-edge nor a weak edge by arc-consistency of L, respectively the fact that y weakly dominates x on $V(K)$. Therefore, $xf(v) \in S(H)$, and again, since the lists L are arc-consistent and $L(u) = \{x, y\}$, we have a contradiction.

This proves that f is indeed a list H-colouring of G with respect to L, and clearly, f can be constructed in polynomial time.

Next, suppose that y has a strong loop. Consider the strong representative S_y.

If $S_y = \{y\}$ or $S_y = \{y, x'\}$ where $x' \neq x$, it follows that $L(u) = \{x\}$ whenever $x \in L(u)$. In this case, the claim follows, as above, from arc-consistency of L. Hence, we may assume $S_x = S_y = \{x, y\}$. We observe that $H \setminus \{xy, yx\}$ contains two components, a component A which contains x, and a component B which contains y. Therefore, if we let $X = \{x\}$ and $Y = \{y\}$, then the list $L(v)$ of any vertex $v \in V(G)$ either does not contain both x and y, or $L(v) \subseteq \{x, y\}$. This shows that G is separable on $F = \{xy\}$. In addition, L-HOM($H - \{x, y\}$) is polynomial time solvable. Therefore, by Lemma 10, we can solve L-HOM(H) for G with lists L in polynomial time.

Finally, suppose that H contains a strong edge xy, and yx is an edge of H. Let A be the component of $H \setminus \{xy, yx\}$ which contains x, and B be the component which contains y. Let $X = \{x\}$ and $Y = \{y\}$. If x does not appear on the list of any vertex in G, then G with lists L is an instance of L-HOM($H - x$) which is polynomial time solvable. Similarly for y. Hence, since the lists L contain representatives, there exist vertices $v_x, v_y \in V(G)$ such that $L(v_x) = \{x\}$ and $L(v_y) = \{y\}$. Now, suppose that the list $L(u)$ contains x. By arc-consistency, we must have $uv_y \in E(G)$. Since y is not adjacent to any vertex of $A \setminus X$ and $uv_y \in E(G)$, no vertex of $A \setminus X$ can appear on the list $L(u)$. Similarly, if $y \in L(u)$, no vertex of $B \setminus Y$ appears on the list $L(u)$. This shows that G is separable on $F = \{xy, yx\}$. If yx is not an edge of H, the same argument shows that G is separable on $F = \{xy\}$. In both cases, the claim follows by Lemma 10.

That concludes the proof. $\square$

4. Extensions

In this section, we extend the dichotomy from the previous section to a larger class of trigraphs which we refer to as *tree-like*. In fact, we prove the dichotomy for this class using the same tools we used in the previous section. We have stated these tools in a sufficiently general way to easily allow for this extension.

Domination property

We say that an ordering $x_1, \ldots, x_t$ of vertices is a *domination ordering*, if x_i weakly dominates x_j whenever $i < j$.

Let H be a trigraph, and suppose that x is strong loop of H. For a component K of $H - x$, let R_K denote the set of all symmetric neighbours of x in K. Recall that according to our conventions, y is a symmetric neighbour of x just if both xy and yx are in $W(H) \cup S(H)$.

We say that a component K satisfies *the domination property* if

(D1) no vertex of R_K has a strong loop, and
(D2) the vertices of R_K with weak loops admit a domination ordering, each weakly dominates x on $V(K)$, and each weakly dominates every vertex of R_K without a loop.

Matching property

Let F be the edges of H. We say that F *separates* H if each edge $xy \in F$ has its endpoints x, y in different components of $H \setminus F$. Thus F separates H if and only if there is a partition of $V(H)$ such that F is the set of all edges between different parts.

Let F^* consist of all edges $xy \in F$ such that neither xy nor yx is a strong edge of H. We say that F satisfies *the matching property* if

(M1) F separates H,
(M2) for any $xy \in F^*$, both xx and yy are strong loops,
(M3) if $xy, zy \in F$, and neither xy nor zy is a bridge of H, then xz is not a symmetric edge, and xx or zz is a strong loop,
(M4) if $xy, xz \in F$, and neither xy nor xz is a bridge of H, then yz is not a symmetric edge, and yy or zz is a strong loop.
(M5) if $xy \in F^*$, and xz and yw are symmetric edges with $xz \notin F$, then $zw \notin F^*$ and $xw \notin F^*$.
(M6) if $xy \in F^*$, and xz and yw are symmetric edges with $yw \notin F$, then $zw \notin F^*$ and $zy \notin F^*$.

4.1. Tree-like trigraphs

We are now in a position to define the class $\mathcal{T}$ of *tree-like trigraphs.*

(T1) If H has no strong edges (or loops), then $H \in \mathcal{T}$
(T2) If H contains a strong loop at x such that $H - x \in \mathcal{T}$ and each component of $H - x$ satisfies the domination property, then $H \in \mathcal{T}$.
(T3) If H contains a set F of edges such that $H \setminus F \in \mathcal{T}$ and the set F satisfies the matching property, then $H \in \mathcal{T}$.

Theorem 13. *If H is a tree-like trigraph, i.e., if $H \in \mathcal{T}$, then L-HOM(H) is polynomially equivalent to L-HOM(H^-). In particular, L-HOM(H) is polynomial time solvable or NP-complete.*

Proof. We prove the Theorem by structural induction on H. We shall assume that H is connected, otherwise we treat each component of H separately. If H is a digraph, there is nothing to prove.

Suppose that H contains a strong loop at x such that $H - x \in \mathcal{T}$ and each component of $H - x$ satisfies the domination property. If L-HOM(H) is polynomial time solvable, then so is L-HOM($H - x$). On the other hand, since $H - x \in \mathcal{T}$, we shall assume, by induction, that L-HOM($H - x$) is polynomial time solvable.

Let G with lists L be an instance of L-HOM(H). By Lemmata 1, 2, and 4, we may assume that the lists L contain representatives and strong representatives.

Consider the strong representative S_x. Let B be the vertices $u \in V(G)$ with $x \in L(u)$. It follows that there is a unique connected component K of $H - x$ such that $S_x \setminus \{x\} \subseteq K$. Let $X = \{x\}$, let Y be the vertices of S_x with no loops, and Z be the vertices of S_x with weak loops. Since K satisfies the domination property, we have $S_x = X \cup Y \cup Z$. Also, Z admits a domination ordering $z_1, \ldots, z_{|Z|}$, and each vertex of Z dominates every vertex of Y. For $u \in V(G)$, let $\min(u)$ denote the vertex $z_i \in L(u)$ with the smallest index i. Now, by Lemma 7, we may assume that for each $u \in V(G)$, either $L(u) = \{x\}$, or $L(u) = \{x, \min(u)\}$, or $x \notin L(u)$.

Moreover, L-HOM($H[X]$) and L-HOM($H - X$) are polynomial time solvable since $X = \{x\}$, and by the inductive hypothesis, respectively. Hence, we can apply Lemma 9 to X, Y, Z to obtain that we may assume that the lists L are arc-consistent, separator-consistent on X, and sparse-dense-consistent on X and Y.

Now, let f_0 be a list $(H-x)$-colouring of $G-B$ with respect to L, such that f_0 reduced to any component C of $G-B$ is a K-colouring as long as a vertex of K appears on the list of some vertex of C. Since the lists L are separator-consistent on X, such mapping must exist and can be found in polynomial time.

We extend the mapping f_0 to a mapping f as follows.

$$f(u) = \begin{cases} x & \text{if } u \in B \text{ and } L(u) = \{x\} \\ \min(u) & \text{if } u \in B \text{ and } L(u) = \{x, \min(u)\} \\ f_0(u) & \text{otherwise} \end{cases}$$

By the above remark, the mapping f is well-defined. It also clearly respects the lists L.

It remains to show that f is a homomorphism. The proof of this follows exactly as in the proof of Theorem 11, and hence, we skip further details.

Now, suppose that H contains edges F satisfying the matching property and such that $H \setminus F \in \mathcal{T}$. Again, we assume that L-HOM$(H \setminus F)$ is polynomial time solvable, and that the lists L are arc-consistent, contain representatives and strong representatives.

First, we observe that any subset F_0 of F satisfies the matching property in $H' = H \setminus (F \setminus F_0)$. In particular, $H' \setminus F_0 = H \setminus F \in \mathcal{T}$.

Therefore, we can proceed by induction on F. We denote by V_F the vertices of H with at least one incident edge from F.

We observe that if a vertex $x \in V_F$ does not appear on any list, we can remove from F all edges incident to x. Since there is at least one such edge, the claim follows by induction. Hence, we shall assume that each vertex of V_F appears on some list.

Let $x \in V_F$ be a strong loop. Consider the strong representative S_x. We claim that either $S_x = \{x, y\}$ where $xy \in F$, or S_x belongs to a component of $H \setminus F$. To prove this, suppose that $y, y' \in S_x \setminus \{x\}$ and $xy \in F$. Clearly, both xy, xy', and yy' are symmetric edges. Since F separates H, at least one of xy', yy' must be in F. But that contradicts the matching property.

Now, suppose that S_x belongs to a component of $H \setminus F$. Let Q_x denote the union of lists of vertices $u \in V(G)$ with $x \in L(u)$. We have $Q_x \subseteq S_x$. Since $x \in V_F$, there exists y with $xy \in F$ or $yx \in F$.

By symmetry, suppose that $xy \in F$, and suppose also that xy is a weak edge. Hence, y is a strong loop. Consider the strong representative S_y. Suppose first that S_y also belongs to a component of $H \setminus F$. Clearly, $S_x \cap S_y = \emptyset$. In particular, $Q_x \cap Q_y = \emptyset$ where Q_y is the union of lists $L(u)$ with $y \in L(u)$. Furthermore, for $z \in Q_x \setminus \{x\}$ and $w \in Q_y \setminus \{y\}$, we have by the matching property that zy, xw are not edges of H, and either $zw \in S(H)$ or zw is not an edge of H. In particular, if $zw \in S(H)$, then arc-consistency of L implies $z \notin Q_x$. Hence, it follows that the only edge in H from Q_x to Q_y is the edge xy. This implies that for any $uv \in E(G)$ with $x \in L(u)$ and $y \in L(v)$, we have $L(u) = \{x\}$ and $L(v) = \{y\}$ by arc-consistency of L. Therefore, we can remove all such edges uv, and after that, we can remove xy from F. The claim now follows by induction.

Next, suppose that $Q_y = \{y, w\}$ where $wy \in F$. Let $z \in Q_x \setminus \{x\}$. By the matching property, we have $zy \notin W(H) \cup S(H)$. Also, either $zw \in S(H)$ or zw is not an edge, and either $xw \in S(H)$ or xw is not an edge. If $zw \in S(H)$, then arc-consistency of L implies

$w \notin Q_y$. If $xw \in S(H)$ and zw is not an edge, then arc-consistency of L implies $z \notin Q_x$. Again, the only edge from Q_x to Q_y is the edge xy, and the claim follows by induction.

Therefore, we may assume that any strong loop $x \in V_F$ is either incident to a strong edge of F, or we have $S_x = \{x, y\}$ where $xy \in F$.

Recall that F separates H. First, suppose that $H \setminus F$ contains exactly two components. We prove that G is separable on F. Suppose that we have $x \in L(u)$ where $x \in V_F$. Let K be the component of $H \setminus F$ which contains x. Suppose that there exists y such that $xy \in F$ or $yx \in F$ and one of xy, yx is strong. Then, by the matching property and arc-consistency of L, the list $L(u)$ contains no vertex of K other than x. If no such y exists, then xx must be a strong loop. Hence, by the above assumption, we have $S_x = \{x, y\}$ where $xy \in F$. Therefore, $L(u)$ again contains no vertex of K other than x. The claim now follows by Lemma 10.

Finally, suppose that $H \setminus F$ contains more than two components. Let F_0 be a smallest subset of F such that $H \setminus F_0$ is disconnected. Since H is connected and $H \setminus F$ is disconnected, the set F_0 must exist. As remarked above, $F \setminus F_0$ satisfies the matching property in $H \setminus F_0$. Also, it can be seen that F_0 satisfies the matching property in H. Since F_0 contains at least one edge and $H \setminus F_0$ contains exactly two components because of minimality, the claim follows by induction.

That concludes the proof. □

The class $\mathcal{T}$ admits a few natural extensions which we shall only mention tangentially. For instance, we shall observe the following simple fact.

Theorem 14. *If each vertex of H has a strong loop, and the symmetric graph of H contains no triangles, then L-HOM(H) is polynomial time solvable.*

Proof. Let G with lists L be an instance of L-HOM(H). By Lemma 4, we may assume that the lists L contain strong representatives. Since the symmetric graph of H has no triangles, it follows that for each $x \in V(H)$, the strong representative S_x contains at most two elements. The problem now can be reduced to $2SAT$ which is polynomial time solvable. □

As a consequence, we can extend the class $\mathcal{T}$ by adding another basis clause to its recursive description:

(T1′) If each vertex of H has a strong loop, and the symmetric graph of H contains no triangles, then $H \in \mathcal{T}$.

Similarly, if H has no weak edges, i.e., if $W(H) = \emptyset$, then L-HOM(H) is polynomially solvable [3, 7]. Hence, we can also add the following clause:

(T2′) If H has no weak edges, then $H \in \mathcal{T}$.

4.2. Trigraph trees and special tree-like trigraphs

Since the recursive description of the class $\mathcal{T}$ is complex, we shall identify a subclass of $\mathcal{T}$ which can be defined directly.

Let $F(H)$ denote all edges xy of H such that either

(i) xy or yx is a strong edge of H, or
(ii) xy and yx are weak edges of H, and x and y are strong loops of H.

We say that H is a *special tree-like trigraph* if there is a set $F' \supseteq F(H)$ of edges of H such that

(S1) F' satisfies the matching property, and
(S2) for every strong loop x of $H \setminus F'$, each component K of $H \setminus F' - x$ satisfies the domination property in $H \setminus F'$.

Let $\mathcal{S}$ denote the class of all special tree-like trigraphs. Also, let $\mathcal{T}_0$ denote the class of all trigraph trees.

Theorem 15. $\mathcal{T}_0 \subseteq \mathcal{S} \subseteq \mathcal{T}$.

Proof. First, let H be in $\mathcal{T}_0$. Consider the set $F' = F(H)$. Since each edge in F' is a bridge of H, conditions (M1), (M3) and (M4) are satisfied for F'. Also, the edges of F' do not form cycles in H since H is a trigraph tree, and hence, (M5) and (M6) are satisfied. Therefore, F' satisfies the matching property. On the other hand, for every vertex x of H with a strong loop, each component of $H - x$ satisfies the domination property, since x is adjacent to at most one vertex of this component. It follows that $H \in \mathcal{S}$.

Now, for $H \in \mathcal{S}$, it suffices to observe that edges xy of $F' \setminus F(H)$ have both x and y strong loops. Hence, after removing F' using (T3) and then removing all vertices with strong loops using (T2), we obtain precisely H^- which is a digraph. Hence, by (T1), we conclude that $H \in \mathcal{T}$. $\square$

Corollary 16. *If H is a special tree-like trigraph, i.e., if $H \in \mathcal{S}$, then L-HOM(H) is polynomial time solvable or NP-complete.*

Although some simple extensions of tree-like, or special tree-like trigraphs are possible (such as, say, Theorem 14), we shall mention in the conclusions some example trigraphs outside $\mathcal{T}$ for which the first part of Theorem 13 fails.

4.3. Representatives of strong edges

In this section, we describe an extension of the notion of strong representatives to strong edges of trigraphs.

The *underlying trigraph H' of H* is the trigraph on the vertices of H with strong edges xy such that $xy \in S(H)$ or $yx \in S(H)$, and with weak edges xy such that $xy, yx \notin S(H)$ and $xy \in W(H)$ or $yx \in W(H)$.

We denote by G^2 the digraph on the vertices of G with edges xy such that $xy \in E(G)$ or $xz, zy \in E(G)$ for some $z \in V(G)$.

We denote by H^2 the trigraph on the vertices of H with strong edges xy such that $xy \in S(H)$ or $xz, zy \in S(H)$ for some $z \in V(G)$, and with weak edges xy such that $xy \notin S(H^2)$, and $xy \in W(H)$ or $xz, zy \in W(H) \cup S(H)$ for some $z \in V(G)$.

We say that an edge xy of H is *admissible*, if there exist vertices $u, v \in V(G)$ such that $x \in L(u)$ and $y \in L(v)$.

We say that lists L contain *representatives for strong edges*, if for each admissible strong edge xy in H, there is a set S_{xy} such that

(i) $L(v) \subseteq S_{xy}$ whenever $x \in L(v)$ or $y \in L(v)$,
(ii) each vertex of $S_{xy} \setminus \{x, y\}$ is a neighbour of x or a neighbour of y,
(iii) if x and y have no common neighbours, then $S_{xy} \backslash \{x, y\}$ contains only neighbours of x or only neighbours of y,
(iv) if yx is a strong edge, then each vertex of $S_{xy} \backslash \{x, y\}$ is a symmetric neighbours of x or a symmetric neighbour of y,
(v) if yx is a strong edge, and x and y have no common symmetric neighbours, then $S_{xy} \backslash \{x, y\}$ contains only symmetric neighbours of x or only symmetric neighbours of y.

Lemma 17. *Lists L can be transformed to contain representatives for strong edges.*

Proof. By Lemmata 1 and 2, we may assume that the lists L are arc-consistent and contain representatives.

Let f be a list H-colouring of G with respect to L, and let xy be an admissible strong edge of H. Let $C = f^{-1}(x) \cup f^{-1}(y)$, let N denote all vertices of $V(H)$ that are neighbours of x or y, and let B denote all vertices $u \in V(G)$ with $x \in L(u)$ or $y \in L(u)$. We have $C \subseteq B$. Also, since lists L are arc-consistent, we have $L(u) \subseteq N$ for each $u \in B$.

Let G' be the underlying graph of G and H' be the underlying trigraph of H. It is easy to verify that f is a homomorphism of G' to H'. In particular, f is a surjective mapping from of B to N_0, where $N_0 = f(B)$. Clearly, $N_0 \subseteq N$.

Now, it is not difficult to prove that f is a homomorphism of $(G'[B])^2$ to $(H'[N_0])^2$. (For this, we need the above remark about surjectivity.) We observe that C induces a clique in $(G'[B])^2$. Hence, using the same argument as in the proof of Lemma 4, there is a maximal clique C'' of a minimal chordal completion G'' of $(G'[B])^2$ such that for each $u \in C'' \setminus C$, we have $f(u) \in V(K)$ where K is a unique component of $(H'[N_0])^2 - \{x, y\}$. In particular, if x and y have no common neighbours, then $V(K)$ either contains only neighbours of x or it contains only neighbours of y. Hence, we let $S_{xy} = V(K) \cup \{x, y\}$.

Now, if yx is also a strong edge, we obtain S_{xy} by replacing G with the symmetric graph of G, and replacing H with the symmetric trigraph of H. The remainder of the proof follows exactly as in Lemma 4. □

4.4. Trigraph cycles

A trigraph H is a *trigraph cycle* if the underlying graph of H is a cycle. Let H be a trigraph cycle. We say that H is a *good cycle* if H has at least one of the following:

(i) two strong edges, or
(ii) three consecutive strong loops, or
(iii) two pairs of consecutive strong loops, or
(iv) a strong edge and a distinct pair of consecutive strong loops, or
(v) two strong loops joined by a nonsymmetric edge, or
(vi) a strong loop whose neighbours have no loops, or
(vii) a strong loop with non-symmetric edges to neighbours, or
(viii) a strong edge with at least one endpoint having no loop.

Theorem 18. *If H is a good cycle, then the problem L-HOM(H) is polynomial time solvable or NP-complete.*

Figure 2 illustrates example trigraph cycles whose complexity is not determined by our theorem. These are cycles H that contain vertices x, y such that either $xy \in S(H)$ and $xx, yy \in W(H)$, or all of xx, xy, yx, and yy are edges (weak or strong) but at least one is strong. (Only three typical cases of this are shown in the figure).

We prove the theorem by reducing the problem to an induced subgraph of H. Unfortunately, if H is a trigraph cycle that is not a good cycle, such reduction may not be possible at all. The complement of the stable cutset problem (Figure 3c) is a good example illustrating this difficulty.

Proof. We assume that H has at least five vertices, since otherwise the claim follows from [2]. Let G with lists L be an instance of L-HOM(H). By Lemmata 1, 2, 4, and 17, we assume that the lists L are arc-consistent, contain representatives, strong representatives, and representatives for strong edges. Also, we assume that each vertex x of H appears on some list, since otherwise we can reduce the problem $H - x$, which is a trigraph tree, and the claim follows by Theorem 11.

Suppose that H contains two strong edges e, e'. If $F = \{e, e'\}$ satisfies the matching property, then we are done by Theorem 13. Otherwise, we must have $e = xy$ and $e' = zy$. (The case $e = yx$ and $e' = yz$ is symmetric.) Let t be any vertex (including x, z) of H other than y. By arc-consistency of L, no list $L(u)$ contains both y and t since xy and zy are strong, but at least one of xt, zt is not an edge. Hence, we have $L(u) = \{y\}$ whenever $y \in L(u)$. By arc-consistency of L, we can reduce the problem to $H - y$ and we are done.

Next, if H contains two pairs of consecutive strong loops x, y and z, w with possibly $y = z$, or a strong edge xy and a pair of strong loops z, w where $\{z, w\} \neq \{x, y\}$, then $F = \{xy, zw\}$ satisfies the matching property, and again we are done.

Hence, suppose that H contains strong loops x, y where xy is an edge, but yx is not. Consider the strong representatives S_x and S_y. Clearly, S_x and S_y are disjoint. Also, the only edge from S_x to S_y is the edge xy. Hence, by arc-consistency of L, if $uv \in E(G)$ with $x \in L(u)$ and $y \in L(v)$, we have $L(u) = \{x\}$ and $L(v) = \{y\}$. In particular, we can remove all such edges uv, and afterwards, we remove the edge xy from H. Now, since $H \setminus xy$ is a trigraph tree, by Theorem 11, we conclude that L-HOM($H \setminus xy$) is polynomially equivalent to L-HOM($H - x$). Hence, we are done.

Next, suppose that H contains a strong loop x with neighbours y, y' having no loops. Consider the strong representative S_x. If $S_x = \{x, y\}$, then we apply Lemma 9 for $X = \{x\}$, $Y = \{y\}$, and $Z = \emptyset$. This yields that the lists L are arc-consistent and sparse-dense-consistent on X and Y. In particular, $L(u) = \{x\}$ whenver $x \in L(u)$. Similarly, if $S_x = \{x, y'\}$. Hence, we can reduce the problem to $H - x$ and we are done.

Now, suppose that H contains a strong loop x with non-symmetric edges between x and its neighbours y, y'. Clearly, we have $S_x = \{x\}$. Hence, we can reduce the problem to $H - x$ and we are done.

Finally, suppose that H contains a strong edge xy whose one endpoint has no loop. By symmetry suppose that x has no loop. Let z be the other neighbour of x and let w be the other neighbour of y. Consider the representative S_{xy}. We have that either $S_{xy} = \{x, y, w\}$ or $S_{xy} = \{z, x, y\}$. By arc-consistency of L, no list $L(u)$ contains both y and w since xy is strong, but xw is not an edge. Similarly, no list $L(u)$ contains both x and y, or both x and z. Hence, if $S_{xy} = \{z, x, y\}$, we have $L(u) = \{x\}$ whenever $x \in L(u)$, and if $S_{xy} = \{x, y, w\}$, we have $L(u) = \{y\}$ whenever $y \in L(u)$. Therefore, we can reduce the problem to $H - x$ or to $H - y$, and we are done. $\square$

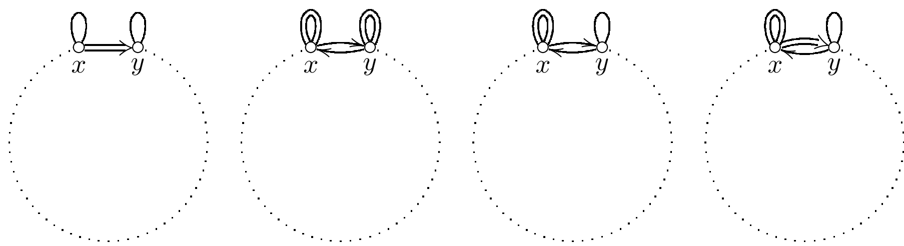

Figure 2: Unresolved trigraph cycles.

4.5. Surjective list homomorphism

Finally, we describe how we can use our results to classify the complexity of finding surjective list homomorphisms for some trigraphs.

We say that a homomorphism f of G to H is a *a (vertex) surjective homomorphism* if f is a surjective mapping of $V(G)$ onto $V(H)$.

The *surjective list H-colouring problem* SL-HOM(H) takes as input a digraph G with lists L, and asks whether or not G admits a surjective list homomorphism to H with respect to L.

Let H^{--} be the digraph obtained from a trigraph H by removing all vertices x with a strong loop at x or a strong edge xy or yx for some y.

Theorem 19. *If H is a special tree-like trigraph, i.e., if $H \in \mathcal{S}$, then SL-HOM(H) is polynomially equivalent to SL-HOM(H^{--}). In particular, SL-HOM(H) is polynomial time solvable or NP-complete.*

Proof. Let H be any trigraph (not necessarily a special tree-like). First, we observe that SL-HOM(H) is polynomially reducible to L-HOM(H), since SL-HOM(H) is a special case of L-HOM(H).

Now, suppose that H contains a digraph H_0 as an induced subgraph. Let G_0 with lists L_0 be an instance of L-HOM(H_0). Let H' be the digraph associated with H, and let G be the disjoint union of G_0 and H'. Define $L(x) = \{x\}$ for $x \in V(H')$, and $L(x) = L_0(x)$ for $x \in V(G_0)$. It now follows that G_0 admits a list H_0-colouring with respect to L_0 if and only if G admits a surjective list H-colouring with respect to L.

This yields that L-HOM(H^{--}) is polynomially equivalent to SL-HOM(H^{--}). In fact, we can also conclude that SL-HOM(H^{--}) is polynomially reducible to SL-HOM(H).

Now, let H be a special tree-like digraph. Let $F' \supseteq F(H)$ be the set of edges from the definition of H. Let G with lists L be an instance of SL-HOM(H). We can assume that each vertex of H appears on some list, since otherwise there is no solution. Now, following the proof of Theorem 13, we conclude (depending on L) that there is a set $F'' \subseteq F'$ such that the instance G, L of L-HOM(H) is polynomially reducible to an instance of L-HOM($H \setminus F''$). In fact, the proof implies the instance is reducible to an instance of L-HOM($H - U$) where U are the vertices incident to the edges of F''. Moreover, since each vertex of H appears on some list, the edges of $F' \setminus F''$ are only between strong loops. Hence, U contains all vertices of H incident to strong edges. Therefore, $(H - U)^{-} = H^{--}$. Also, by the definition of H, for each strong loop x of $H - U$, each connected component of $H - U - x$ satisfies the domination property. Hence,

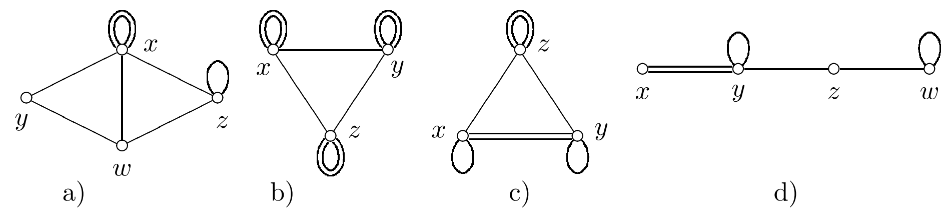

Figure 3: a) the stubborn problem, b) the complement of the 3-colouring problem, c) the complement of the stable cutset problem, d) a trigraph H with NP-complete L-HOM(H) but polynomial time solvable SL-HOM(H).

by Theorem 13, L-HOM($H - U$) is polynomially equivalent to L-HOM(H^{--}). Also, as remarked earlier, SL-HOM(H^{--}) is polynomially equivalent to L-HOM(H^{--}). Hence, SL-HOM(H) is polynomially reducible to SL-HOM(H^{--}).

That concludes the proof. □

The proof of this theorem in particular implies the following.

Corollary 20. *If H is a digraph, then L-HOM(H) and SL-HOM(H) are polynomially equivalent.*

In fact, similarly, one can prove a stronger statement.

Proposition 21. *If for no vertices x, y, z of H we have $xy \in S(H)$ and $xz \notin W(H) \cup S(H)$, then L-HOM(H) and SL-HOM(H) are polynomially equivalent.*

Note that if we have vertices x, y, z as described above, then an instance G in which x occurs on some list allows us to use arc-consistency to make sure that no vertex has both y and z on its list; while if x appears on no list, this no longer happens.

5. Conclusions

We investigated the list (and surjective list) homomorphism problems for trigraphs H. When H is a digraph, we know that each such problem is polynomial time solvable or NP-complete [1]. However, there are small trigraphs H for which such dichotomy is not known, for instance, the trigraph in Figure 3a (corresponding to the so-called "stubborn" problem from [2]).

Hence, we have tried to identify properties of trigraphs H which allow us to prove dichotomy. With this in mind, we have defined the class $\mathcal{T}$ of tree-like trigraphs, and we proved that these trigraphs enjoy dichotomy. Slightly easier to define is the class $\mathcal{S}$ of special tree-like trigraphs, included in $\mathcal{T}$. In particular, the class of trigraphs whose underlying graph is a tree, is included in $\mathcal{S}$ (and hence in $\mathcal{T}$).

We now offer some tangential evidence that our class $\mathcal{T}$ carves out a reasonable portion of trigraphs where L-HOM(H) is polynomially equivalent to L-HOM(H^{-}). For instance, one way to violate the matching property is by having a vertex with a strong loop adjacent to two other vertices with strong loops joined by a symmetric edge. The trigraph in Figure 3b illustrates this possibility, and also illustrates that a polynomial time solvable (in this case trivial) problem L-HOM(H^-) can arise from an NP-complete problem L-HOM(H). (In this case, G admits a homomorphism to H if and only if the complement of G is 3-colourable.) The trigraph in Figure 3c illustrates another way to have this take place. Here, the two other vertices have weak loops and are joined by a strong symmetric edge. In this case L-HOM(H) is also NP-complete, since it corresponds (in the complement) to the stable cutset problem [13]. Thus we again have an easy L-HOM(H^-) with a hard L-HOM(H).

Of course, it is possible that some class of trigraphs enjoys dichotomy for reasons different from polynomial equivalence of L-HOM(H) and L-HOM(H^-). However, the stubborn problem (Figure 3a) illustrates the fact that there are trigraphs $H \notin \mathcal{T}$ where even the dichotomy is not clear. We note that the trigraph H for the stubborn problem (the trigraph in Figure 3a) violates the domination property, since the statement (D2) does not hold for the strong loop at x.

Finally, we have also introduced the surjective list homomorphism problem SL-HOM(H) as an interesting variant of L-HOM(H). We were able to completely classify the complexity of SL-HOM(H) for trigraphs H in the class $\mathcal{S}$ by proving polynomial equivalence with SL-HOM(H^{--}). This result implies, in particular, that the two problems, L-HOM(H) and SL-HOM(H), may not necessarily have the same complexity for all trigraphs H. This difference was not noted before, since the two problems are polynomially equivalent for all digraphs H. In fact, we have given a general condition for trigraphs H under which the two problems SL-HOM(H) and L-HOM(H) are polynomially equivalent. Nonetheless, the trigraph H in Figure 3d illustrates a case where SL-HOM(H) is polynomial time solvable, because using representatives and arc-consistency reduces all lists to size at most two, while L-HOM(H) is NP-complete, because $H - x$ corresponds to the stable cutset problem [13].

References

[1] A. A. Bulatov, Tractable conservative constraint satisfaction problems, ACM Transactions on Computational Logic, to appear.

[2] K. Cameron, E. M. Eschen, C. T. Hoàng, R. Sritharan, The complexity of the list partition problem for graphs, SIAM Journal on Discrete Mathematics 21 (2007) 900–929.

[3] T. Feder, P. Hell, Full constraint satisfaction problems, SIAM Journal on Computing 36 (2006) 230–246.

[4] T. Feder, P. Hell, Matrix partitions of perfect graphs, Discrete Mathematics 306 (2006) 2450–2460.

[5] T. Feder, P. Hell, W. Hochstättler, Generalized colourings of cographs, in: Graph Theory in Paris, Birkhauser Verlag, 2006, pp. 149–167, invited chapter.

[6] T. Feder, P. Hell, J. Huang, Bi-arc graphs and the complexity of list homomorphisms, Journal of Graph Theory 42 (2003) 61–80.

[7] T. Feder, P. Hell, S. Klein, R. Motwani, List partitions, SIAM Journal on Discrete Mathematics 16 (2003) 449–478.

[8] T. Feder, P. Hell, S. Klein, L. T. Nogueira, F. Protti, List matrix partitions of chordal graphs, Theoretical Computer Science 349 (2005) 52–66.

[9] T. Feder, P. Hell, D. Král', J. Sgall, Two algorithms for general list matrix partitions, in: Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms (SODA), 2005, pp. 870–876.

[10] T. Feder, P. Hell, K. Tucker-Nally, Digraph matrix partitions and trigraph homomorphisms, Discrete Applied Mathematics 154 (2006) 2458–2469.

[11] T. Feder, P. Hell, W. Xie, Matrix partitions with finitely many obstructions, Electronic Journal of Combinatorics 14 (2007) R58.

[12] T. Feder, M. Y. Vardi, The computational structure of monotone monadic SNP and constraint satisfaction: A study through datalog and group theory, SIAM Journal on Computing 28 (1999) 57–104.

[13] C. M. H. de Figueiredo, S. Klein, The NP-completeness of multipartite cutset testing, Congressus Numerantium 119 (1996) 217–222.

[14] D. R. Fulkerson, O. A. Gross, Incidence matrices and interval graphs, Pacific Journal of Mathematics 15 (1965) 835–855.

[15] P. Hell, S. Klein, L. T. Nogueira, F. Protti, Partitioning chordal graphs into independent sets and cliques, Discrete Applied Mathematics 141 (2004) 185–194.

[16] P. Hell, J. Nešetřil, Graphs and Homomorphisms, Oxford University Press, 2004.

[17] P. Hell, J. Nešetřil, On the complexity of H-colouring, Journal of Combinatorial Theory B 48 (1990) 92–110.

[18] D. G. Schell, Matrix partitions of digraphs, Master's thesis, Simon Fraser University (2008).

[19] R. E. Tarjan, Decomposition by clique separators, Discrete Mathematics 55 (1985) 221–232.