Unique perfect phylogeny is intractable

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Abstract

A *phylogeny* is a tree capturing evolution and ancestral relationships of a set of taxa (e.g., species). Reconstructing phylogenies from molecular data plays an important role in many areas of contemporary biological research. A phylogeny is perfect if (in rough terms) it correctly captures all input data. Determining if a perfect phylogeny exists was shown to be intractable in 1992 by Mike Steel [32] and independently by Bodlaender et al. [4]. In light of this, a related problem was proposed in [32]: given a perfect phylogeny, determine if it is the unique perfect phylogeny for the given dataset, where the dataset is provided as a set of quartet (4-leaf) trees. It was suggested that this problem may be more tractable [32], and determining its complexity became known as the Quartet Challenge [33].

In this paper, we resolve this question by showing that the problem is CoNP-complete. We prove this by relating perfect phylogenies to satisfying assignments of Boolean formulas. To this end, we cast the question as a chordal sandwich problem. As a particular consequence of our method, we show that the unique minimal chordal sandwich problem is CoNP-complete, and counting minimal chordal sandwiches is #P-complete.

Key words: perfect phylogeny, chordal graph, triangulation, chordal sandwich, intractability, unique solution

1. Introduction

One of the major efforts in molecular biology has been the computation of phylogenetic trees, or *phylogenies*, which describe the evolution of a set of species from a common ancestor. A phylogenetic tree for a set of species is a tree in which the leaves represent the species from the set and the internal nodes represent the (hypothetical) ancestral species. One standard model for describing the species is in terms of *characters*, where a character is an equivalence relation on the species set, partitioning it into different *character states*. In this model, we also assign character states to the (hypothetical) ancestral species. The desired property is that for each state of each character, the set of nodes in the tree having that character state forms a connected subgraph. When a phylogeny has this property, we say it is *perfect*. The Perfect Phylogeny problem [20] then asks *for a given set of characters defining a species set, does there exist a perfect phylogeny?* Note that we allow that states of some characters are unknown for some species; we call such characters *partial*, otherwise we speak of *full* characters. This approach to constructing phylogenies has been studied since the 1960s [8, 25, 26, 27, 35] and was given a precise mathematical formulation in the 1970s [12, 13, 14, 15]. In particular, Buneman [7] showed that the Perfect Phylogeny problem reduces to a specific graph-theoretic problem, the problem of finding a chordal completion of a graph that respects a prescribed colouring. In fact, the two problems are polynomially equivalent [23]. Thus, using this formulation, it has been proved that the Perfect Phylogeny problem is NP-hard in [4] and independently in [32]. These two results rely on the fact that the input may contain partial characters. In fact, the characters in these constructions only have two states. If we insist on full characters, the situation is different as for any fixed number r of character states, the problem can be solved in time polynomial [1] in the size of the input (and exponential in r). In particular, for $r = 2$ (or $r = 3$), the solution exists if and only if it exists for every pair (or triple) of characters [15, 24]. Also, when the number of characters is k (even if there are partial characters), the complexity [28] is polynomial in the number of species (and exponential in k).

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Another common formulation of this problem is the problem of a *consensus tree* [10, 19, 32], where a collection of subtrees with labelled leaves is given (for instance, the leaves correspond to species of a partial character). Here, we ask for a (phylogenetic) tree such that each of the input subtrees can be obtained by contracting edges of the tree (we say that the tree *displays* the subtree). The problem does not change [31] if we only allow particular input subtrees, the so-called *quartet trees*, which have exactly six vertices and four leaves. This follows from the fact that every *ternary phylogenetic tree* (all internal nodes have degree 3) can be uniquely described by a collection of quartet trees [31]. However, a collection of quartet trees does not necessarily uniquely describe a ternary phylogenetic tree. (Note that some authors use the term *binary tree* [5, 31] or *subcubic tree* for what we call here a *ternary tree* as defined in [30].)

This leads to a natural question (first posed in [32]): *What is the complexity of deciding whether or not a collection of quartet trees uniquely describes a (ternary) phylogenetic tree?* Initially, it was suggested [32] that this problem may be more tractable. Indeed, *a priori* it is possible that unique solutions only exist for special collections of quartet trees and thus have special structure which could be easy to test. However, as the problem was open for a number of years, and perhaps from experience with real datasets, it became more clear that this probably is not the case. This was reflected in the problem being conjectured to be intractable by Mike Steel who named it Quartet Challenge and listed it on his personal webpage [33] alongside with other challenging research problems from the area of phylogenetics. In particular, to emphasize the importance of the problem, a price of \$100 was offered for the first proof of intractability.

In this paper, we resolve the problem by showing that it is indeed intractable. Namely, we show the following.

Theorem 1. *It is CoNP-complete to determine, given a ternary phylogenetic X-tree $\mathcal{T}$ and a collection $\mathcal{Q}$ of quartet subtrees on X, whether or not $\mathcal{T}$ is the only phylogenetic tree that displays $\mathcal{Q}$.*

To prove this theorem, we investigate the graph-theoretical formulation of the problem [7] and view it through the notion of chordal sandwich [17]. In contrast, an alternative proof of the theorem, which recently appeared as [5], is based on the betweenness property, extending the hardness result of [32]; our proof extends the hardness from [4].

In light of this, we note that there are special cases of the problem that are known to be solvable in polynomial time. For instance, this is so if the collection $\mathcal{Q}$ contains a subcollection $\mathcal{Q}'$ with the same set $\mathcal{L}$ of labels of leaves and with $|\mathcal{Q}'| = |\mathcal{L}| - 3$. However, finding such a subcollection is known to be NP-complete. For these and similar results, we refer the reader to [3].

We prove Theorem 1 by describing a polynomial-time reduction from the uniqueness problem for ONE-IN-THREE-3SAT, which is CoNP-complete by [22].

Theorem 2. [22] *It is CoNP-complete to decide, given an instance I of* ONE-IN-THREE-3SAT, *and a truth assignment σ that satisfies I, whether or not σ is the unique satisfying truth assignment for I.*

We extract this from [22] by encoding the problem as the ternary relation $\{(0,0,1),(0,1,0),(1,0,0)\}$. We check that this relation is not: 0-valid, 1-valid, Horn, anti-Horn, affine, 2SAT, or complementive. Thus the uniqueness of the satisfiability problem corresponding to this relation is CoNP-complete by [22].

Our construction in the reduction is essentially a modification of the construction of [4] which proves NP-hardness of the Perfect Phylogeny problem. Recall that the construction of [4] produces instances $\mathcal{Q}$ that have a perfect phylogeny if and only if a particular boolean formula Φ is satisfiable. While studying this construction, we immediately observed that these instances $\mathcal{Q}$ have, in addition, the property that Φ has a unique satisfying assignment if and only if there is a unique minimal restricted chordal completion of the partial partition intersection graph of $\mathcal{Q}$ (for definitions see §2). This is precisely one of the two necessary conditions for uniqueness of perfect phylogeny as proved by Semple and Steel in [30] (see Theorem 5). Thus by modifying the construction of [4] to also satisfy the other condition of uniqueness of [30], we obtained the construction that we present in this paper. Note that, however, unlike [4] which uses 3SAT, we had to use a different problem in order for the construction to work correctly. Also, to prove that the construction is correct, we employ a variant of the characterization of [30] that uses the more general chordal sandwich problem [17] instead of the restricted chordal completion problem (see Theorem 8). In fact, by way of Theorems 6 and 7, we establish a direct connection between the problem of perfect phylogeny and the chordal sandwich problem, which apparently has not been yet observed. (Note that the connection to the (restricted) chordal completion problem of coloured graphs as mentioned above [7, 23] is a special case of this.)

Finally, as a corollary, we obtain the following result which is very interesting by itself.

Corollary 3. *The unique minimal chordal sandwich problem is CoNP-complete. The problem of counting the number of minimal chordal sandwiches is #P-complete.*

The first part follows directly from Theorems 2 and 9, while the second part follows from Theorem 9 and [9].

The paper is structured as follows. In §2, we describe some preliminary definitions and results needed for the construction in our reduction. In particular, we describe, based on [30], necessary and sufficient conditions for the existence of a unique perfect phylogeny in terms of the minimal chordal sandwich problem (cf. [16, 17]). The proof of this characterization is postponed until §5.

In §3 and §4, we present our hardness reduction, first informally and then formally. We state the two uniqueness conditions (Theorems 9 and 10) relating satisfying assignments of an instance I of ONE-IN-THREE-3SAT to minimal chordal sandwiches and phylogenetic trees uniquely determined by these assignments. The proofs are presented later in §6 and §7. In §8, we put these results together to prove Theorem 1.

We conclude in §9 with some other consequences and open questions related to this work.

2. Preliminaries

We mostly follow the terminology of [30, 31] and the graph-theoretical notions of [34].

In this paper, a graph is always simple, undirected, with no loops or parallel edges. For a graph $G = (V, E)$, we write $V(G)$ to denote its vertex set and $E(G)$ to denote its edge set. We write uv for the edge $(u, v) \in E(G)$, and say that u, v are *neighbours* or *adjacent* in G. For a vertex $v \in V(G)$, we denote by $N_G(v)$ the *neighbourhood* of v in G, i.e, the set of neighbours of v in G. We write $N_G[v]$ for $N_G(v) \cup \{v\}$. When appropriate, we drop the index G and simply write $N(v)$ and $N[v]$. For a set $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G *induced* by X, i.e., the graph with vertex set X and edges uv such that $u, v \in X$ and $uv \in E(G)$. We write $G - X$ for the graph $G[V(G) \setminus X]$. Similarly, for a set of edges $F \subseteq E(G)$, we write $G - F$ for the graph with vertex set $V(G)$ and edge set $E(G) \setminus F$. We write $G - x$ as a shorthand for $G - \{x\}$. We say that X is a *clique* of G if $G[X]$ is a *complete graph* (i.e., has all possible edges). A vertex $v \in V(G)$ is a *simplicial vertex* of G if all its neighbours are pairwise adjacent.

A graph is a *chordal graph* if it does not contains an induced cycle of length four or more. A *perfect elimination ordering* of a graph G is an ordering $v_1, v_2, \ldots, v_n$ of the vertices of G such that for every $i \in \{1 \ldots n\}$, the vertex v_i is a simplicial vertex of $G\big[\{v_1, \ldots, v_i\}\big]$, i.e., all its neighbours among $\{v_1 \ldots, v_{i-1}\}$ are pairwise adjacent. It is well-known [11] that a graph is chordal if and only if it admits a perfect elimination ordering.

Let X be a non-empty set. An *X-tree* is a pair (T, ϕ) where T is tree and $\phi : X \to V(T)$ is a mapping such that $\phi^{-1}(v) \neq \emptyset$ for all vertices $v \in V(T)$ of degree at most two. An X-tree (T, ϕ) is *ternary* if all internal vertices of T have degree three. Two X-trees $(T_1, \phi_1), (T_2, \phi_2)$ are *isomorphic* if there exists an isomorphism $\psi : V(T_1) \to V(T_2)$ between T_1 and T_2 that satisfies $\phi_2 = \psi \circ \phi_1$.

An X-tree (T, ϕ) is a *phylogenetic X-tree* (or a *free X-tree* in [30]) if ϕ is a bijection between X and the set of leaves of T. A *partial partition* of X is a partition of a non-empty subset of X into at least two sets. If $A_1, A_2, \ldots, A_t$ are these sets, we call them *cells* of this partition, and denote the partition $A_1 \big| A_2 \big| \ldots \big| A_t$. If $t = 2$, we call the partition a *partial split*. A partial split $A_1 \big| A_2$ is trivial if $|A_1| = 1$ or $|A_2| = 1$.

A *quartet tree* is a ternary phylogenetic tree with a label set of size four, that is, a ternary tree $\mathcal{T}$ with 6 vertices, 4 leaves labelled a, b, c, d, and with only one non-trivial partial split $\{a, b\} \big| \{c, d\}$ that it displays. Note that such a tree is unambiguously defined by this partial split. Thus, in the subsequent text, we identify the quartet tree $\mathcal{T}$ with the partial split $\{a, b\} \big| \{c, d\}$, that is, we say that $\{a, b\} \big| \{c, d\}$ is both a quartet tree and a partial split.

Let $\mathcal{T} = (T, \phi)$ be an X-tree, and let $\pi = A_1 \big| A_2 \big| \ldots \big| A_t$ be a partial partition of X. Let $F \subseteq E(T)$ be a set of edges of T. We say that F *displays* π in $\mathcal{T}$ if for all distinct $i, j \in \{1 \ldots t\}$, the sets $\phi(A_i)$ and $\phi(A_j)$ are subsets of the vertex sets of different connected components of $T - F$. We say that $\mathcal{T}$ *displays* π if there is a set of edges that displays π in $\mathcal{T}$. Further, an edge e of T is *distinguished* by π if every set of edges that displays π in $\mathcal{T}$ contains e.

Let $\mathcal{Q}$ be a collection of partial partitions of X. An X-tree $\mathcal{T}$ *displays* $\mathcal{Q}$ if it displays every partial partition in $\mathcal{Q}$. An X-tree $\mathcal{T} = (T, \phi)$ is *distinguished* by $\mathcal{Q}$ if every internal edge of T is distinguished by some partial partition in $\mathcal{Q}$; we also say that $\mathcal{Q}$ *distinguishes* $\mathcal{T}$. The set $\mathcal{Q}$ *defines* $\mathcal{T}$ if $\mathcal{T}$ displays $\mathcal{Q}$, and all other X-trees that display $\mathcal{Q}$ are isomorphic to $\mathcal{T}$. Note that if $\mathcal{Q}$ defines $\mathcal{T}$, then $\mathcal{T}$ is necessarily a ternary phylogenetic X-tree, since otherwise

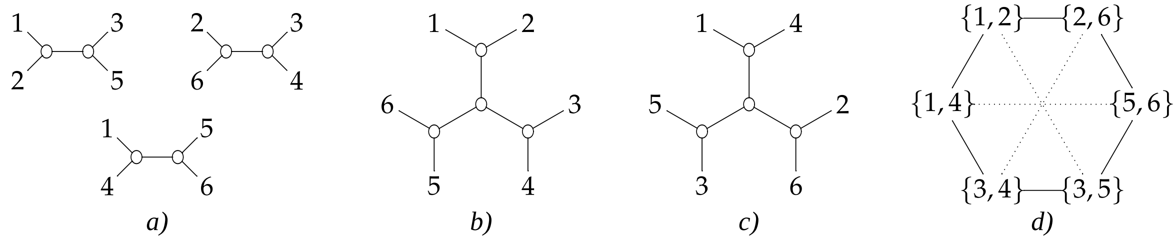

Figure 1: *a)* quartet trees $\mathcal{Q}$, *b), c)* two X-trees displaying $\mathcal{Q}$ and distinguished by $\mathcal{Q}$, *d)* $\text{int}^*(\mathcal{Q})$; dotted lines represent the edges in $\text{forb}(\mathcal{Q})$.

"resolving" any vertex either of degree four or more, or with multiple labels results in a non-isomorphic X-tree that also displays $\mathcal{Q}$ (also, see [30, Proposition 2.6]). See Fig. 1 for an illustration of these concepts.

The *partial partition intersection graph* of $\mathcal{Q}$, denoted by $\text{int}(\mathcal{Q})$, is a graph whose vertex set is $\{(A, \pi) \mid \text{where } A \text{ is a cell of } \pi \in \mathcal{Q}\}$ and two vertices (A, π), (A', π') are adjacent just if the intersection of A and A' is non-empty.

A *chordal completion* of a graph $G = (V, E)$ is a chordal graph $G' = (V, E')$ with $E \subseteq E'$. A *restricted chordal completion* of $\text{int}(\mathcal{Q})$ is a chordal completion G' of $\text{int}(\mathcal{Q})$ with the property that if A_1,A_2 are cells of $\pi \in \mathcal{Q}$, then (A_1, π) is not adjacent to (A_2, π) in G'. A restricted chordal completion G' of $\text{int}(\mathcal{Q})$ is *minimal* if no proper subgraph of G' is a restricted chordal completion of $\text{int}(\mathcal{Q})$.

The problem of perfect phylogeny is equivalent to the problem of determining the existence of an X-tree that displays the given collection $\mathcal{Q}$ of partial partitions. In [7], it was given the following graph-theoretical characterization.

Theorem 4. [7, 31, 32] *Let $\mathcal{Q}$ be a set of partial partitions of a set X. Then there exists an X-tree that displays $\mathcal{Q}$ if and only if there exists a restricted chordal completion of* $\text{int}(\mathcal{Q})$.

Of course, the X-tree in the above theorem might not be unique. For the problem of uniqueness, Semple and Steel [30, 31] describe necessary and sufficient conditions for when a collection of partial partitions defines an X-tree.

Theorem 5. [30] *Let $\mathcal{Q}$ be a collection of partial partitions of a set X. Let $\mathcal{T}$ be a ternary phylogenetic X-tree. Then $\mathcal{Q}$ defines $\mathcal{T}$ if and only if:*

(i) *$\mathcal{T}$ displays $\mathcal{Q}$ and is distinguished by $\mathcal{Q}$, and*

(ii) *there is a unique minimal restricted chordal completion of* $\text{int}(\mathcal{Q})$.

In order to simplify our proof of Theorem 1, we now describe a variant of the above theorem that, instead, deals with the notion of chordal sandwich [17].

Let $G = (V, E)$ and $H = (V, F)$ be two graphs on the same set of vertices with $E \cap F = \emptyset$. A *chordal sandwich* of (G,H) is a chordal graph $G' = (V, E')$ with $E \subseteq E'$ and $E' \cap F = \emptyset$. We say that E are the *forced* edges and F are the *forbidden* edges. (For other possible formulations of this notion, see [17].) A chordal sandwich G' of (G,H) is *minimal* if no proper subgraph of G' is a chordal sandwich of (G,H).

The *cell intersection graph* of $\mathcal{Q}$, denoted by $\text{int}^*(\mathcal{Q})$, is the graph whose vertex set is $\{A \mid \text{where } A \text{ is a cell of } \pi \in \mathcal{Q}\}$ and two vertices A, A' are adjacent just if the intersection of A and A' is non-empty. Let $\text{forb}(\mathcal{Q})$ denote the graph whose vertex set is that of $\text{int}^*(\mathcal{Q})$ in which there is an edge between A and A' just if A,A' are cells of some $\pi \in \mathcal{Q}$. See Fig. 1d for an example.

The relationship between the notion of partial partition intersection graph and the cell intersection graph is captured by the following theorem.

Theorem 6. *Let $\mathcal{Q}$ be a collection of partial partitions of a set X. Then there exists a bijective mapping between the minimal restricted chordal completions of* $\text{int}(\mathcal{Q})$ *and the minimal chordal sandwiches of* $(\text{int}^*(\mathcal{Q}), \text{forb}(\mathcal{Q}))$.

(The proof of this theorem is rather technical and it is presented as §5.)

This combined with Theorem 4 yields that there exists a phylogenetic X-tree that displays $\mathcal{Q}$ if and only if there exists a chordal sandwich of $(\text{int}^*(\mathcal{Q}), \text{forb}(\mathcal{Q}))$. Conversely, we can express every instance of the chordal sandwich problem as a corresponding instance of the problem of perfect phylogeny as follows.

Theorem 7. *Let (G, H) be an instance of the chordal sandwich problem. Then there exists a collection $\mathcal{Q}$ of partial splits such that there is a bijective mapping between the minimal chordal sandwiches of (G, H) and the minimal restricted chordal completions of* $\mathrm{int}(\mathcal{Q})$*. In particular, there exists a chordal sandwich for (G, H) if and only if there exists a phylogenetic tree that displays $\mathcal{Q}$.*

PROOF. Consider the instance (G, H) where $G = (V, E)$ and $H = (V, F)$ are two graphs with $E \cap F = \emptyset$.

Without loss of generality, we may assume that each connected component of G has at least three vertices. (We can safely remove any component with two or fewer vertices without changing the number of minimal chordal completions, since every such component is already chordal.)

We define the collection $\mathcal{Q}$ of partial splits (of the set E) as follows: for every edge $xy \in F$, we construct the partial split $D_x \Big| D_y$, where D_x are the edges of E incident to x, and D_y are the edges of E incident to y. By definition, the vertex set of the graph $\mathrm{int}^*(\mathcal{Q})$ is precisely $\{D_v \mid v \in V\}$. Further, it can be easily seen that the mapping ψ that, for each $v \in V$, maps v to D_v is an isomorphism between G and $\mathrm{int}^*(\mathcal{Q})$. (Here, one only needs to verify that $D_u = D_v$ implies $u = v$; for this we use that each component of G has at least three vertices.) Moreover, $\mathrm{forb}(\mathcal{Q})$ is precisely $\{\psi(x)\psi(y) \mid xy \in F\}$ by definition. Therefore, by Theorem 6, there is a one-to-one correspondence between the minimal chordal sandwiches of (G, H) and the minimal restricted chordal completions of $\mathrm{int}(\mathcal{Q})$. This proves the first part of the claim; the second part follows directly from Theorem 4. □

As an immediate corollary, we obtain the following desired characterization.

Theorem 8. *Let $\mathcal{Q}$ be a collection of partial partitions of a set X. Let $\mathcal{T}$ be a ternary phylogenetic X-tree. Then $\mathcal{Q}$ defines $\mathcal{T}$ if and only if:*

(i) *$\mathcal{T}$ displays $\mathcal{Q}$ and is distinguished by $\mathcal{Q}$, and*

(ii) *there is a unique minimal chordal sandwich of* $\Big(\mathrm{int}^*(\mathcal{Q}), \mathrm{forb}(\mathcal{Q})\Big)$.

We remark that the main technical advantage of this theorem over Theorem 5 is that it is less restrictive; it allows us to construct instances with arbitrary sets of forbidden edges rather than just with forbidden edges between vertices of the same colour. This makes our proof of Theorem 1 much simpler and more manageable.

3. Overview of the proof

Consider an instance I of ONE-IN-THREE-3SAT. The instance I consists of n *variables* $v_1, \ldots, v_n$ and m *clauses* $\mathcal{C}_1, \ldots, \mathcal{C}_m$ each of which is a disjunction of exactly three *literals* (i.e., variables v_i or their negations $\overline{v_i}$).

To simplify the presentation, we shall denote literals by capital letters X, Y, etc., and indicate their negations by $\overline{X}$, $\overline{Y}$, etc. (For instance, if $X = v_i$ then $\overline{X} = \overline{v_i}$, and if $X = \overline{v_i}$ then $\overline{X} = v_i$.)

A *truth assignment* for the instance I is a mapping $\sigma : \{v_1, \ldots, v_n\} \to \{0, 1\}$ where 0 and 1 represent *false* and *true*, respectively. To simplify the notation, we write $v_i = 0$ and $v_i = 1$ in place of $\sigma(v_i) = 0$ and $\sigma(v_i) = 1$, respectively, and extend this notation to literals X,Y, etc., i.e., write $X = 0$ and $X = 1$ in place of $\sigma(X) = 0$ and $\sigma(X) = 1$, respectively. A truth assignment σ is a *satisfying assignment for I* if in each clause $\mathcal{C}_j$ exactly one of the three literals evalues to true. That is, for each clause $\mathcal{C}_j = X \vee Y \vee Z$, either $X = 1$, $Y = 0$, $Z = 0$, or $X = 0$, $Y = 1$, $Z = 0$, or $X = 0$, $Y = 0$, $Z = 1$.

By standard arguments, we may assume that no variable appears twice in the same clause, since otherwise we can replace the instance I by an equivalent instance with this property. In particular, we can replace each clause of the form $v_i \vee \overline{v_i} \vee v_j$ by clauses $v_i \vee x \vee v_j$ and $\overline{v_i} \vee \overline{x} \vee v_j$ where x is a new variable, and replace each clause of the form $v_i \vee v_i \vee v_j$ by clauses $v_i \vee v_j \vee x$, $v_i \vee \overline{v_j} \vee \overline{x}$, and $\overline{v_i} \vee \overline{v_j} \vee x$ where x is again a new variable. Note that these two transformations preserve the number of satisfying assignments, since in the former the new variable x has always the truth value of $\overline{v_i}$ while in the latter x is always false in any satisfying assignment of this modified instance.

In what follows, we discuss the following objects arising from the instance I:

– the set of labels $\mathcal{X}_I$,

– the collection $\mathcal{Q}_I$ of quartet trees whose leaves are labelled by elements of $\mathcal{X}_I$,

– the ternary tree T_I, and

– the labelling $\phi_\sigma : \mathcal{X}_I \to V(T_I)$ of the leaves of T_I, where σ is a satisfying assigment for I,

which together yield

– the phylogenetic $\mathcal{X}_I$-tree $\mathcal{T}_\sigma = (T_I, \phi_\sigma)$.

The formal definitions of these objects is given as §4.

We then prove that the satisfying assignments to I are in bijection with the minimal chordal sandwiches of $\text{int}^*(\mathcal{Q}_I)$, the cell intersection graph of $\mathcal{Q}_I$, and $\text{forb}(\mathcal{Q}_I)$. Further, we show that every satisfying assignment σ for I defines a perfect phylogeny for $\mathcal{Q}_I$, namely the tree $\mathcal{T}_\sigma = (T_I, \phi_\sigma)$, that is distinguished by $\mathcal{Q}_I$. These together will imply Theorem 1, the main result of this paper. We summarize this as the following two theorems.

Theorem 9. *There is a bijective mapping between the satisfying assignments of the instance I and the minimal chordal sandwiches of* $(\text{int}^*(\mathcal{Q}_I), \text{forb}(\mathcal{Q}_I))$.

Theorem 10. *If σ is a satisfying assignment for I, then $\mathcal{T}_\sigma = (T_I, \phi_\sigma)$ is a ternary phylogenetic $\mathcal{X}_I$-tree that displays $\mathcal{Q}_I$ and is distinguished by $\mathcal{Q}_I$.*

We present the proofs of these theorems as §6 and §7, respectively. In the rest of this section, we informally discuss the constructions involved to prepare the reader for the technical nature of the proofs that will follow.

Before describing the collection $\mathcal{Q}_I$, let us briefly review the construction from [4] that proves NP-hardness of the Perfect Phylogeny problem. For convenience, we describe it in terms of the chordal sandwich problem whose input is a graph with (forced) edges and forbidden edges. In [4], one similarly considers a collection $\mathcal{C}_1, \ldots, \mathcal{C}_m$ of 3-literal clauses, and treats it as an instance I of 3-SATISFIABILITY. From this instance, one constructs a graph where each variable v_i corresponds to two *shoulders* S_{v_i} and $S_{\overline{v_i}}$, and where each literal W in a clause $\mathcal{C}_j$ corresponds to a pair of *knees* K^j_W and $K^j_{\overline{W}}$. In addition, there are two special vertices *the head* H and *the foot* F. All shoulders are adjacent to the head while all knees are adjacent to the foot. Further, if v_i occurs in the clause $\mathcal{C}_j$ (positively or negatively), then the vertices H, S_{v_i}, $K^j_{\overline{v_i}}$, F, $K^j_{v_i}$, $S_{\overline{v_i}}$ form an induced 6-cycle (see Fig. 2a). Also, if $\mathcal{C}_j = X \vee Y \vee Z$, then the vertices K^j_X, K^j_Y, K^j_Z induce a triangle with pendant edges $K^j_X K^j_{\overline{Y}}$, $K^j_Y K^j_{\overline{Z}}$, and $K^j_Z K^j_{\overline{X}}$ (the *clause gadget*, see Fig. 2b).

Finally, the edge between H and F is forbidden in the desired chordal sandwich, and so is the edge between S_{v_i} and $S_{\overline{v_i}}$, and between $K^j_{v_i}$ and $K^j_{\overline{v_i}}$ (the dotted edges in Fig. 2) for all indices i and j for which these vertices exist.

The main idea of this construction is that each of the 6-cycles allows only two possible chordal sandwiches: either the path $H, K^j_{v_i}, S_{v_i}, F$ is added, or the path $H, K^j_{\overline{v_i}}, S_{\overline{v_i}}, F$ is added (the authors of [4] call this path the "Mark of Zorro"). These two choices correspond to assigning v_i the value *true* or *false*, respectively, and the construction ensures that this choice is consistent over all clauses. This only produces satisfying assignments to 3-SATISFIABILITY, since we notice that no chordal sandwich adds a triangle on $K^j_{\overline{X}}, K^j_{\overline{Y}}, K^j_{\overline{Z}}$.

One can try to use this construction to prove Theorem 1 (we explain later why this fails). Indeed, it can be observed that the truth assignments satisfying the clauses $\mathcal{C}_1, \ldots, \mathcal{C}_m$ are in one-to-one correspondence with the minimal chordal sandwiches of the above graph G. To see this, one describes all edges that we are forced to have in the sandwich after the marks of Zorro are added according to a satisfying assignment. It turns out that these edges yield a chordal sandwich, and thus a minimal chordal sandwich.

From G, using Theorems 6 and 7, one can further construct a collection $\mathcal{Q}$ of partial splits (phylogenetic trees) such that the satisfying assignments of the clauses $\mathcal{C}_1, \ldots, \mathcal{C}_m$ are in a one-to-one correspondence with the minimal chordal sandwiches of $(\text{int}^*(\mathcal{Q}), \text{forb}(\mathcal{Q}))$. In particular, this collection $\mathcal{Q}$ satisfies the condition (ii) of Theorem 8 if and only if the clauses $\mathcal{C}_1, \ldots, \mathcal{C}_m$ have a unique satisfying assignment. Since this is CoNP-complete to determine [22], it would seem like we almost have a proof of Theorem 1. Unfortunately, we are missing a crucial piece which is the phylogenetic tree $\mathcal{T}$ satisfying the condition (i) of Theorem 8 for the collection $\mathcal{Q}$. A straightforward construction of such a tree based on [30] yields a phylogenetic tree that is distinguished by $\mathcal{Q}$, but whose internal nodes may have

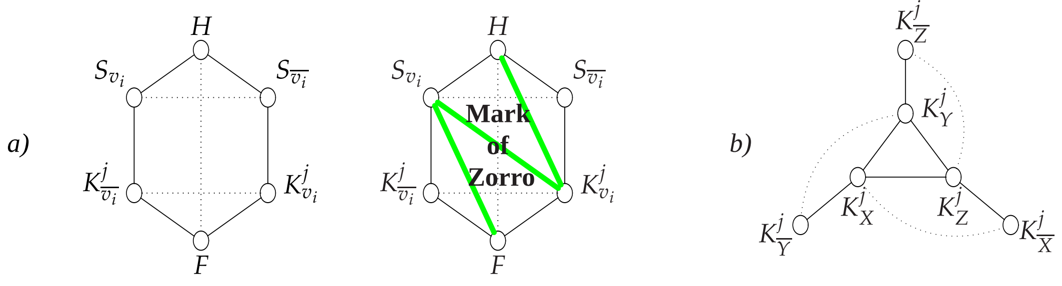

Figure 2: Configurations from [4].

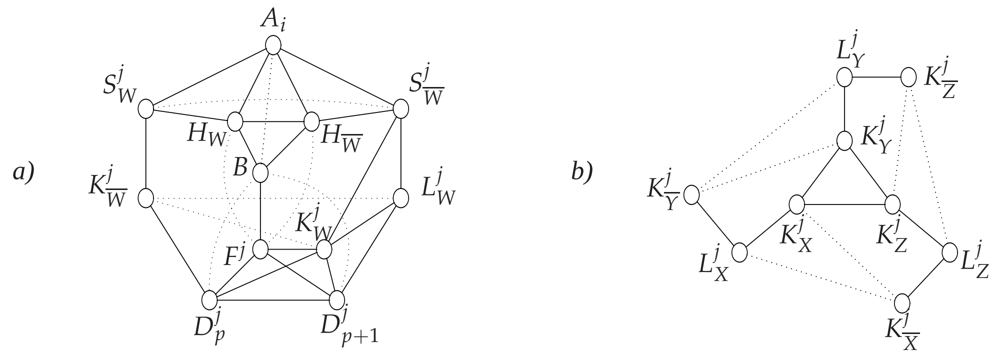

Figure 3: Configurations from our construction (note that, on the left, W is a literal, either v_i or $\overline{v_i}$, and is the p-th literal of the clause $\mathcal{C}_j$)

degree higher than three. If we try to fix this (by "resolving" the high-degree nodes in order to get a ternary tree), the resulting tree may no longer be distinguished by $\mathcal{Q}$. Moreover, the collection $\mathcal{Q}$ may not consist of quartet trees only. For all these reasons, we need to modify the construction of G.

First, we discuss how to modify G so that it corresponds to a collection of quartet trees. To do this, we must ensure that the neighbourhood of each vertex consists of two cliques (with possibly edges between them).

We construct a new graph G_I by modifying G as follows. Instead of one head H, we introduce, for each variable v_i, two *heads* H_{v_i}, $H_{\overline{v_i}}$, and an *auxiliary head* A_i. For a literal W in the clause $\mathcal{C}_j$, we introduce two *shoulders* S^j_W and $S^j_{\overline{W}}$, and, as before, two *knees* K^j_W and $K^j_{\overline{W}}$, but also an additional *auxiliary knee* L^j_W. Further, for each clause $\mathcal{C}_j$, we introduce a *foot* F^j and three *auxiliary feet* D^j_1, D^j_2, and D^j_3. Finally, we add one additional vertex B known as *the backbone*. The resulting modifications to the 6-cycles and the clause gadgets can be seen in Fig. 3a and 3b. (The forbidden edges are again indicated by dotted lines.) Note that, unlike in the case of G, this is not a complete description of G_I as we need to add some additional (forced) edges and forbidden edges not shown in these diagrams in order to make the reduction work. This is rather technical and we omit this for brevity.

From the construction, we conclude that, just like in G, the "6-cycles" of G_I (Fig. 3a) admit only two possible kinds of sandwiches, and this is consistent over different clauses. However, unlike in G, the chordal sandwiches of G_I no longer correspond to satisfying assignments of 3-SATISFIABILITY but rather to satisfying assignments of ONE-IN-THREE-3-SAT. Fortunately, the uniqueness variant of this problem is CoNP-complete (see Theorem 2).

Now, from G_I, we construct a collection $\mathcal{Q}_I$ of quartet trees. To do this, we cannot simply use Theorem 7 as before, since this may create partial partitions that do not correspond to quartet trees. Moreover, even if we use [31] to replace these partitions by an equivalent collection of quartet trees, this process may not preserve the number of solutions. We need a more careful construction.

We recall that each vertex v of G_I belongs to two cliques that completely cover its neighbourhood; we assign greek letters to these two cliques (to distinguish them from vertices), and associate them with v.

In particular, we use the following symbols: α_W, β^j_W, $\gamma^j_1, \gamma^j_2, \gamma^j_3$, λ^j, δ, μ where W is a literal and $j \in \{1 \dots m\}$. They define specific cliques of G_I as follows. The letter α_W defines the clique of G_I consisting of all heads and shoulders of W. The letter β_W^j corresponds to the clique formed by the shoulder S_W^j and the knees $K_{\overline{W}}^j$, $L_{\overline{W}}^j$ (if exists). Further, λ^j is the clique on F^j, D_1^j, D_2^j, D_3^j, K_X^j, K_Y^j, K_Z^j where $\mathcal{C}_j = X \vee Y \vee Z$, while the clique for γ_p^j where $p \in \{1,2,3\}$ is formed by D_p^j, $K_{\overline{W}}^j$, L_U^j where W and U are the p-th and $(p-1)$-th (modulo 3) literals of $\mathcal{C}_j$. Finally, δ corresponds to the clique containing B and all heads H_W, while μ corresponds to the clique with B and all feet F^j.

From this, we construct the collection $\mathcal{Q}_I$ by considering every forbidden edge uv of G_I and by constructing a partial partition with two cells in which one cell is the set of cliques assigned to u and the other is the set of cliques assigned to v. Since we assign to each vertex of G_I exactly two cliques, this yields partitions corresponding to quartet trees. For instance, in Fig. 3b, we have a forbidden edge $K_X^j K_{\overline{X}}^j$ where K_X^j is assigned cliques $\beta_{\overline{X}}^j, \lambda^j$, and $K_{\overline{X}}^j$ is assigned β_X^j, γ_1^j. This yields a quartet tree $\{\beta_{\overline{X}}^j, \lambda^j\} \,\big|\, \{\beta_X^j, \gamma_1^j\}$. Finally, since by construction every vertex of G_I is incident to at least one forbidden edge, we conclude that $G_I = \text{int}^*(\mathcal{Q}_I)$.

This completes the overview of the construction. From this, the proof of Theorem 9 follows, essentially along the same lines as the uniqueness property we discussed for G. That is, we describe the edges that are forced in the sandwich by a satisfying assignment for I, treated as an instance of ONE-IN-THREE-3SAT, and prove that this yields a chordal sandwich, i.e., a minimal chordal sandwich.

To complete the result, we need to explain how to construct a phylogenetic tree corresponding to a satisfying assignment σ for I, namely the tree $\mathcal{T}_\sigma = (T_I, \phi_\sigma)$, and show that it displays and is distinguished by the trees in $\mathcal{Q}_I$, as stated in Theorem 10. As this is rather technical, we instead discuss a small example here.

The example instance I^+ consists of four variables v_1, v_2, v_3, v_4 and three clauses $\mathcal{C}_1 = v_1 \vee v_2 \vee v_3$, $\mathcal{C}_2 = \overline{v_1} \vee v_2 \vee v_4$, and $\mathcal{C}_3 = v_3 \vee \overline{v_2} \vee \overline{v_4}$. The unique satisfying assignment assigns true to v_1, v_4 and false to v_2, v_3. The corresponding phylogenetic tree $\mathcal{T} = (T, \phi)$ is shown in Fig. 4.

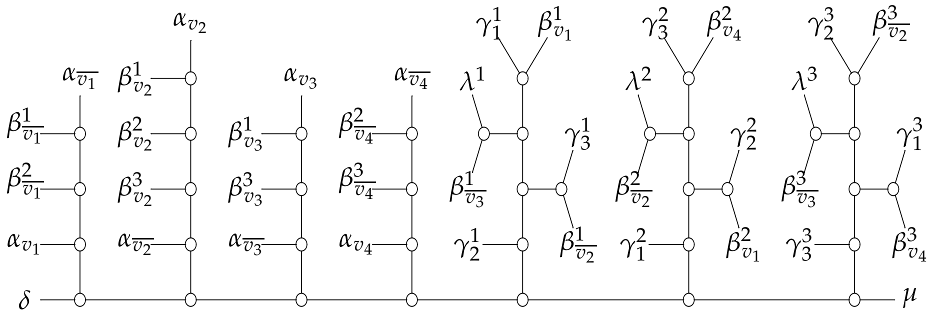

Figure 4: The phylogenetic tree for the example instance I^+.

For instance, one of the quartet trees in $\mathcal{Q}_{I^+}$ is $\pi = \{\alpha_{v_1}, \beta_{v_1}^1\} \,\big|\, \{\alpha_{\overline{v_1}}, \beta_{\overline{v_1}}^1\}$ representing the forbidden edge of G_{I^+} between $S_{v_1}^1$ and $S_{\overline{v_1}}^1$. It is easy to verify $\mathcal{T}$ displays π. Another example from $\mathcal{Q}_{I^+}$ is $\pi' = \{\beta_{\overline{v_1}}^1, \lambda^1\} \,\big|\, \{\beta_{v_1}^1, \gamma_1^1\}$ representing the forbidden edge $K_{v_1}^1 K_{\overline{v_1}}^1$. Again, it is displayed by $\mathcal{T}$, but this time one internal edge of T is contained in every set of edges of T that displays π' in $\mathcal{T}$; hence, this edge is distinguished by π'. This way we can verify all other quartet trees in $\mathcal{Q}_{I^+}$ and conclude that they are displayed by $\mathcal{T}$ and they distinguish $\mathcal{T}$.

Now, with the help of Theorem 8, this allows us to prove that given an instance I of ONE-IN-THREE-3SAT and a satisfying assignment σ for I, one can in polynomial time construct a phylogenetic tree $\mathcal{T}$ and a collection of quartet trees $\mathcal{Q}$ such that $\mathcal{T}$ is the unique tree defined by $\mathcal{Q}$ if and only if σ is the unique satisfying assignment for I. Combined with Theorem 2, this proves Theorem 1.

That concludes this section.

4. Formal Construction

Let I be an instance of ONE-IN-THREE-3SAT consisting of n *variables* $v_1, \ldots, v_n$ and m *clauses* $\mathcal{C}_1, \ldots, \mathcal{C}_m$ each of which is a disjunction of exactly three *literals*. Assume that no variable appears twice in the same clause.

For each $i \in \{1 \dots n\}$, we let Δ_i denote all indices j such that v_i or $\overline{v_i}$ appears in the clause $\mathcal{C}_j$. In the following, we define the set $\mathcal{X}_I$, introduce notation for some of its 2-element subsets, and using these define the collection $\mathcal{Q}_I$.

4.1. Definition of $\mathcal{X}_I$

The set $\mathcal{X}_I$ consists of the following elements:

- $\alpha_{v_i}, \alpha_{\overline{v_i}}$ for each $i \in \{1 \dots n\}$,
- $\beta^j_{v_i}, \beta^j_{\overline{v_i}}$ for each $i \in \{1 \dots n\}$ and $j \in \Delta_i$,
- $\gamma^j_1, \gamma^j_2, \gamma^j_3, \lambda^j$ for each $j \in \{1 \dots m\}$,
- δ and μ.

4.2. Selected subsets of $\mathcal{X}_I$

$$B = \left\{\mu, \delta\right\}$$

For each $i \in \{1, \dots, n\}$:

$$H_{v_i} = \{\alpha_{v_i}, \delta\}, H_{\overline{v_i}} = \{\alpha_{\overline{v_i}}, \delta\}, A_i = \left\{\alpha_{v_i}, \alpha_{\overline{v_i}}\right\},$$
$$S^j_{v_i} = \left\{\alpha_{v_i}, \beta^j_{v_i}\right\}, S^j_{\overline{v_i}} = \left\{\alpha_{\overline{v_i}}, \beta^j_{\overline{v_i}}\right\} \text{ for all } j \in \Delta_i$$

For each $j \in \{1 \dots m\}$ where $C_j = X \vee Y \vee Z$:

$$F^j = \left\{\lambda^j, \mu\right\},$$
$$K^j_{\overline{X}} = \left\{\beta^j_X, \gamma^j_1\right\},\ K^j_{\overline{Y}} = \left\{\beta^j_Y, \gamma^j_2\right\},\ K^j_{\overline{Z}} = \left\{\beta^j_Z, \gamma^j_3\right\},$$
$$K^j_X = \left\{\beta^j_{\overline{X}}, \lambda^j\right\},\ K^j_Y = \left\{\beta^j_{\overline{Y}}, \lambda^j\right\},\ K^j_Z = \left\{\beta^j_{\overline{Z}}, \lambda^j\right\},$$
$$L^j_X = \left\{\beta^j_{\overline{X}}, \gamma^j_2\right\},\ L^j_Y = \left\{\beta^j_{\overline{Y}}, \gamma^j_3\right\},\ L^j_Z = \left\{\beta^j_{\overline{Z}}, \gamma^j_1\right\},$$
$$D^j_1 = \left\{\gamma^j_1, \lambda^j\right\},\quad D^j_2 = \left\{\gamma^j_2, \lambda^j\right\},\quad D^j_3 = \left\{\gamma^j_3, \lambda^j\right\}$$

4.3. Definition of $\mathcal{Q}_I$

The collection $\mathcal{Q}_I$ of quartet trees is defined as the union of the following sets:

- $\displaystyle\bigcup_{i \in \{1..n\}} \left\{A_i \middle| B\right\}$
- $\displaystyle\bigcup_{j \in \{1..m\}} \left\{D^j_1 \middle| B\ ,\ D^j_2 \middle| B\ ,\ D^j_3 \middle| B\right\}$
- $\displaystyle\bigcup_{\substack{i \in \{1..n\} \\ j,j' \in \Delta_i}} \left\{S^j_{v_i} \middle| S^{j'}_{\overline{v_i}}\right\}$
- $\displaystyle\bigcup_{\substack{1 \le i' < i \le n \\ j \in \Delta_i}} \left\{H_{v_{i'}} \middle| S^j_{v_i}\ ,\ H_{\overline{v_{i'}}} \middle| S^j_{v_i}\ ,\ H_{v_{i'}} \middle| S^j_{\overline{v_i}}\ ,\ H_{\overline{v_{i'}}} \middle| S^j_{\overline{v_i}}\right\}$
- $\displaystyle\bigcup_{\substack{j \in \{1...m\} \\ \text{where } \mathcal{C}_j = X \vee Y \vee Z}} \left\{ \begin{array}{l} K^j_{\overline{X}} \middle| K^j_X\ ,\ K^j_{\overline{Y}} \middle| K^j_Y\ ,\ K^j_{\overline{Z}} \middle| K^j_Z\ ,\ K^j_{\overline{X}} \middle| L^j_X\ ,\ K^j_{\overline{Y}} \middle| L^j_Y\ ,\ K^j_{\overline{Z}} \middle| L^j_Z \\ S^j_Y \middle| K^j_X\ ,\ S^j_Z \middle| K^j_Y\ ,\ S^j_X \middle| K^j_Z\ ,\ S^j_Z \middle| L^j_X\ ,\ S^j_X \middle| L^j_Y\ ,\ S^j_Y \middle| L^j_Z \end{array} \right\}$
- $\displaystyle\bigcup_{\substack{i \in \{1..n\} \\ j,j' \in \Delta_i \text{ and } j < j'}} \left\{S^j_{v_i} \middle| K^{j'}_{\overline{v_i}}\ ,\ S^j_{\overline{v_i}} \middle| K^{j'}_{v_i}\right\}$
- $\displaystyle\bigcup_{\substack{i \in \{1..n\} \\ j \in \Delta_i \text{ and } j < j' \le m}} \left\{K^j_{\overline{v_i}} \middle| F^{j'}\ ,\ K^j_{v_i} \middle| F^{j'}\right\}$
- $\displaystyle\bigcup_{\substack{i \in \{1..n\} \\ j \in \{1..m\}}} \left\{H_{\overline{v_i}} \middle| F^j\ ,\ H_{v_i} \middle| F^j\right\}$

Note that in each clause $\mathcal{C}_j = X \vee Y \vee Z$ there is a particular type of symmetry between the literals X, Y, and Z. In particular, if we replace, in the above, the indices X, Y, Z and 1, 2, 3 as follows: $X \to Y \to Z \to X$ and $1 \to 2 \to 3 \to 1$, we obtain precisely the same definition of $\mathcal{Q}_I$ as the above. We shall refer to this as the *rotational symmetry* between X, Y, Z.

Now, we formally define the tree T_I corresponding to the instance I. For satisfying assignments σ, we also define the labelling ϕ_σ of the leaves of T_I by the elements of $\mathcal{X}_I$. This (as we prove later in Theorem 10) will constitute a perfect phylogeny, an $\mathcal{X}_I$-tree $\mathcal{T}_\sigma = (T_I, \phi_\sigma)$, for the collection $\mathcal{Q}_I$.

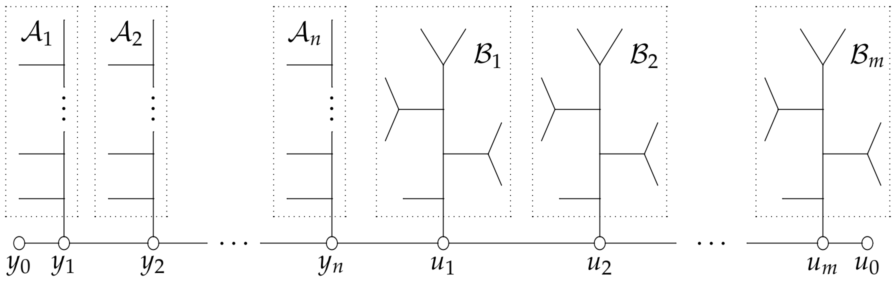

Figure 5: The tree T_I.

4.4. Definition of the tree T_I

$$V(T_I) = \left\{y_0, y_1, y'_1, \ldots, y_n, y'_n\right\} \cup \left\{a_1, a'_1, \ldots, a_n, a'_n\right\} \cup \left\{c_i^j, z_i^j \;\middle|\; i \in \{1 \ldots n\} \text{ and } j \in \Delta_i\right\}$$
$$\cup \left\{u_0, u_1, \ldots, u_m\right\} \cup \left\{x_1^j, x_2^j, x_3^j, x_4^j, x_5^j, x_6^j, b_1^j, b_2^j, b_3^j, g_1^j, g_2^j, g_3^j, \ell^j \;\middle|\; j \in \{1 \ldots m\}\right\}$$

$$E(T_I) = \left\{y_1y'_1, y_2y'_2, \ldots, y_ny'_n\right\} \cup \left\{a_1y'_1, a_2y'_2, \ldots a_ny'_n\right\} \cup \left\{c_i^jz_i^j \;\middle|\; j \in \Delta_i\right\}_{i=1}^{n}$$
$$\cup \left\{y_0y_1, y_1y_2, y_2y_3, \ldots, y_{n-1}y_n\right\} \cup \left\{y_nu_1, u_1u_2, u_2u_3, \ldots, u_{m-1}u_m, u_mu_0\right\}$$
$$\cup \left\{u_jx_1^j, x_1^jx_2^j, x_2^jx_3^j, x_2^jx_4^j, x_4^jx_5^j, x_4^jx_6^j, b_1^jx_6^j, b_2^jx_3^j, b_3^jx_5^j, g_1^jx_6^j, g_2^jx_1^j, g_3^jx_3^j, \ell^jx_5^j \;\middle|\; j \in \{1 \ldots m\}\right\}$$
$$\cup \left\{a'_iz_i^{j_1}, z_i^{j_1}z_i^{j_2}, \ldots, z_i^{j_{t-1}}z_i^{j_t}, z_i^{j_t}y'_i \;\middle|\; i \in \{1 \ldots n\} \text{ and } j_1 < j_2 < \ldots < j_t \text{ are elements of } \Delta_i\right\}$$

4.5. Definition of the labelling ϕ_σ

Let σ be a satisfying assignment for the instance I. The mapping $\phi_\sigma : \mathcal{X}_I \to V(T_I)$ is defined as follows:

- $\phi_\sigma(\delta) = y_0$ and $\phi_\sigma(\mu) = u_0$,
- for each $i \in \{1 \ldots n\}$:

  if $v_i = 1$, then $\phi_\sigma(\alpha_{v_i}) = a_i$, $\phi_\sigma(\alpha_{\overline{v_i}}) = a'_i$, and $\phi_\sigma(\beta^j_{\overline{v_i}}) = c_i^j$ for all $j \in \Delta_i$,

  if $v_i = 0$, then $\phi_\sigma(\alpha_{\overline{v_i}}) = a_i$, $\phi_\sigma(\alpha_{v_i}) = a'_i$, and $\phi_\sigma(\beta^j_{v_i}) = c_i^j$ for all $j \in \Delta_i$,
- for each $j \in \{1 \ldots m\}$ where $\mathcal{C}_j = \mathrm{X} \vee \mathrm{Y} \vee \mathrm{Z}$:

  if $\mathrm{X} = 1$, then $\phi_\sigma(\beta^j_{\mathrm{X}}) = b_1^j$, $\phi_\sigma(\beta^j_{\overline{\mathrm{Y}}}) = b_2^j$, $\phi_\sigma(\beta^j_{\overline{\mathrm{Z}}}) = b_3^j$,
  $\phi_\sigma(\gamma_1^j) = g_1^j$, $\phi_\sigma(\gamma_2^j) = g_2^j$, $\phi_\sigma(\gamma_3^j) = g_3^j$, $\phi_\sigma(\lambda^j) = \ell^j$,

  if $\mathrm{Y} = 1$, then $\phi_\sigma(\beta^j_{\mathrm{Y}}) = b_1^j$, $\phi_\sigma(\beta^j_{\overline{\mathrm{Z}}}) = b_2^j$, $\phi_\sigma(\beta^j_{\overline{\mathrm{X}}}) = b_3^j$,
  $\phi_\sigma(\gamma_2^j) = g_1^j$, $\phi_\sigma(\gamma_3^j) = g_2^j$, $\phi_\sigma(\gamma_1^j) = g_3^j$, $\phi_\sigma(\lambda^j) = \ell^j$,

  if $\mathrm{Z} = 1$, then $\phi_\sigma(\beta^j_{\mathrm{Z}}) = b_1^j$, $\phi_\sigma(\beta^j_{\overline{\mathrm{X}}}) = b_2^j$, $\phi_\sigma(\beta^j_{\overline{\mathrm{Y}}}) = b_3^j$,
  $\phi_\sigma(\gamma_3^j) = g_1^j$, $\phi_\sigma(\gamma_1^j) = g_2^j$, $\phi_\sigma(\gamma_2^j) = g_3^j$, $\phi_\sigma(\lambda^j) = \ell^j$,

For illustration of the construction of T_I and ϕ_σ, see Fig. 5 and 6.

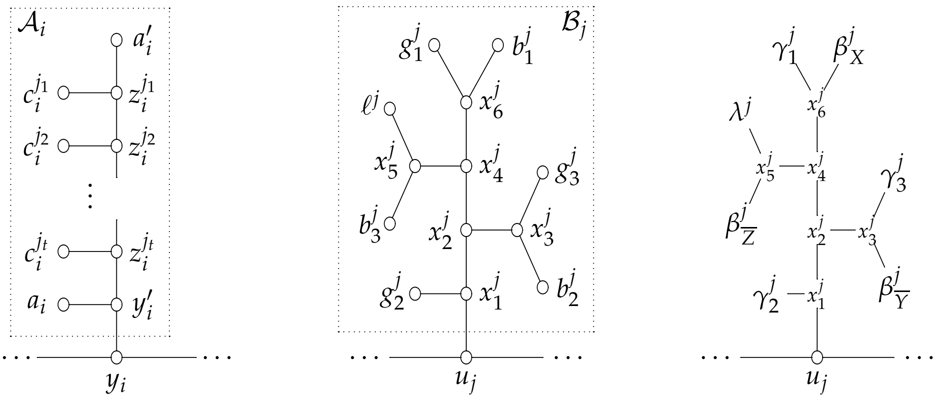

Figure 6: *a)* subtree $\mathcal{A}_i$ for the variable v_i, *b)* subtree $\mathcal{B}_j$ for the clause $\mathcal{C}_j$, *c)* labelling of leaves of $\mathcal{B}_j$ when $\sigma(X) = 1$, $\sigma(Y) = \sigma(Z) = 0$.

5. Perfect Phylogenies and Minimal Chordal Sandwiches

In this section, we prove Theorem 6. As a particular consequence of this theorem, we obtain Theorem 8, which allows us to cast the problem of uniqueness of perfect phylogenies as a minimal chordal sandwich problem.

We need to introduce some additional tools. The following is a standard property of minimal chordal completions.

Lemma 11. *Let G' be a chordal completion of G. Then G' is a minimal chordal completion of G if and only if for all $uv \in E(G') \setminus E(G)$, the vertices u, v have at least two non-adjacent common neighbours in G'.*

PROOF. Suppose that G' is a minimal chordal completion. Let $uv \in E(G') \setminus E(G)$, and let $G'' = G' - uv$. Since G' is a minimal chordal completion and $uv \notin E(G)$, we conclude that G'' is not chordal. Thus, there exists a set $C \subseteq V(G')$ that induces a cycle in G''. Since G' is chordal, C does not induce a cycle in G'. This implies $u, v \in C$, and hence, uv is the unique chord of $G'[C]$. So, we conclude $|C| = 4$, because otherwise $G'[C]$ contains an induced cycle. Let x, y be the two vertices of $C \setminus \{u, v\}$. Clearly, $xy \notin E(G')$ and both x and y are common neighbours of u and v in G', as required.

Conversely, suppose that G' is not a minimal chordal completion. Then by [29], there exists an edge $uv \in E(G') \setminus E(G)$ such that $G' - uv$ is a chordal graph. If the vertices u, v have non-adjacent common neighbours x, y in G', then $\{u, x, v, y\}$ induces a 4-cycle in $G' - uv$. This is impossible as we assume that $G' - uv$ is chordal.

That concludes the proof. □

Using this tool, we prove the following two important lemmas.

Lemma 12. *Let G be a graph and G' be a minimal chordal completion of G. If G contains vertices u, v with $N_G(u) \subseteq N_G(v)$, then also $N_{G'}(u) \subseteq N_{G'}(v)$.*

PROOF. Let u, v be vertices of G with $N_G(u) \subseteq N_G(v)$. Let $B = N_{G'}(u) \setminus N_{G'}(v)$ and $A = N_{G'}(u) \cap N_{G'}(v)$. Assume for contradiction that $B \neq \emptyset$, and let A_1 denote the vertices of A with at least one neighbour in B. Look at the graph $G_1 = G'[A_1 \cup B \cup \{v\}]$.

By the definition of A_1 and B, the vertex v is adjacent to each vertex in A_1 and non-adjacent to each vertex in B. Hence, no vertex in A_1 is a simplicial vertex of G_1, since it is adjacent to v and at least one vertex in B.

Now, consider $w \in B$. By the definition of B, we have that w is adjacent in G' to u but not v. Thus, uw is not an edge of G, since $N_G(u) \subseteq N_G(v)$ and $N_G(v) \subseteq N_{G'}(v)$. So, by Lemma 11, the vertices u, w have non-adjacent common neighbours x, y in G'. Since x, y are adjacent to u, we have $x, y \in A \cup B$. In fact, since w has no neighbours in $A \setminus A_1$, we conclude $x, y \in A_1 \cup B$. Thus, w is not a simplicial vertex of G_1.

This proves that no vertex of G_1, except possibly for v, is simplicial in G_1. Also, G_1 is not a complete graph, since $B \neq \emptyset$, and v has no neighbour in B. Recall that G_1 is chordal because G' is. Thus, by the result of Dirac [11], it follows that G_1 must contain at least two non-adjacent simplicial vertices, but this is clearly impossible.

Hence, we must conclude $B = \emptyset$. In other words, $N_{G'}(u) \subseteq N_{G'}(v)$ as promised. □

Lemma 13. *Let G be a graph, and let H be a graph obtained from G by substituting complete graphs for the vertices of G. Then there is a one-to-one correspondence between minimal chordal completions of G and H.*

PROOF. Let $v_1, v_2, \ldots, v_n$ be the vertices of G. Since H is obtained from G by substituting complete graphs, there is a partition $C_1 \cup \ldots \cup C_n$ of $V(H)$ where each C_i induces a complete graph in H, and for all distinct $i, j \in \{1 \ldots n\}$:

($\star$) each $x \in C_i$, $y \in C_j$ satisfy $v_i v_j \in E(G)$ if and only if $xy \in E(H)$.

We define the following mapping Ψ: if G' is a graph with vertex set $V(G)$, then $H' = \Psi(G')$ denotes be the graph constructed from G' by considering each $i \in \{1 \ldots n\}$, substituting the set C_i for the vertex v_i, and making all vertices in C_i pairwise adjacent. Thus, for all distinct $i, j \in \{1 \ldots n\}$:

($\star\star$) each $x \in C_i$, $y \in C_j$ satisfy $v_i v_j \in E(G')$ if and only if $xy \in E(H')$.

We prove that Ψ is a bijection between the minimal chordal completions of G and H which will yield the lemma.

Let G' be a minimal chordal completion of G, and let $H' = \Psi(G')$. Clearly, H' is chordal, since G' is chordal, and chordal graphs are closed under the operation of substituting a complete graph for a vertex. Also, observe that $V(H) = V(H')$. If $xy \in E(H)$ where $x, y \in C_i$ for some $i \in \{1 \ldots n\}$, then also $xy \in E(H')$, since C_i induces a complete graph in H'. If $xy \in E(H)$ and $x \in C_i$, $y \in C_j$ for distinct $i, j \in \{1 \ldots n\}$, then $v_i v_j \in E(G)$ by ($\star$), implying $v_i v_j \in E(G')$, since $E(G) \subseteq E(G')$. Hence, $xy \in E(H')$ by ($\star\star$). This proves that $E(H) \subseteq E(H')$, and thus, H' is a chordal completion of H.

To prove that H' is a minimal chordal completion of H, it suffices, by Lemma 11, to show that for all $xy \in E(H') \setminus E(H)$, the vertices x, y have at least two non-adjacent common neighbours in H'. Consider $xy \in E(H') \setminus E(H)$, and let $i, j \in \{1 \ldots n\}$ be such that $x \in C_i$ and $y \in C_j$. Since $xy \notin E(H)$ and C_i induces a complete graph in H, we conclude $i \neq j$. Thus, by ($\star\star$), we have $v_i v_j \in E(G')$, and so, $v_i v_j \in E(G') \setminus E(G)$ by ($\star$). Now, recall that G' is a minimal chordal completion of G. Thus, by Lemma 11, the vertices v_i, v_j have non-adjacent common neighbours v_k, v_ℓ in G'. So, we let $w \in C_k$ and $z \in C_\ell$. By ($\star\star$), we conclude $wz \notin E(H')$, since $v_k v_\ell \notin E(G')$. Moreover, ($\star\star$) also implies that z, w are common neighbours of x, y, since v_k, v_ℓ are common neighbours of v_i, v_j. This proves that x, y have non-adjacent common neighbours, and thus shows that H' is a minimal chordal completion of H.

Conversely, let H' be a minimal chordal completion of H. Let G' be the graph with $V(G') = V(G)$ such that $v_i v_j \in E(G')$ if and only if there exists $x \in C_i$, $y \in C_j$ with $xy \in E(H')$. Let $i \in \{1 \ldots n\}$ and consider vertices $x, y \in C_i$ in the graph H. Recall that C_i induces a complete graph in H. This implies that $xy \in E(H)$ and both x and y are adjacent in H to every $z \in C_i \setminus \{x, y\}$. Further, by ($\star$), if $z \in C_j$ where $j \neq i$, then x, y are both adjacent to z if $v_i v_j \in E(G)$, and x, y are both non-adjacent to z if $v_i v_j \notin E(G)$. This shows that $N_H(x) = N_H(y)$, and hence, $N_{H'}(x) = N_{H'}(y)$ by Lemma 12 and the fact that H' is a minimal chordal completion of H. This proves that $H' = \Psi(G')$, and hence, G' is chordal. In fact, $E(G) \subseteq E(G')$ by ($\star$) and ($\star\star$). Thus G' is a chordal completion of G.

It remains to show that G' is a minimal chordal completion of G. Again, it suffices to show that for each $v_i v_j \in E(G') \setminus E(G)$, the vertices v_i, v_j have non-adjacent common neighbours in G'. Consider $v_i v_j \in E(G') \setminus E(G)$, and let $x \in C_i$ and $y \in C_j$. So, $i \neq j$ and $xy \in E(H')$ by ($\star\star$). Further, $xy \in E(H') \setminus E(H)$ by ($\star$) and the fact that $v_i v_j \notin E(G)$. So, the vertices x, y have non-adjacent common neighbours w, z in H' by Lemma 12 and the fact that H' is a minimal chordal completion of H. Let $k, \ell \in \{1 \ldots n\}$ be such that $w \in C_k$ and $z \in C_\ell$. Since $xz \in E(H')$ but $wx \notin E(H')$, we conclude by ($\star\star$) that $i \neq k$. By symmetry, also $i \neq \ell$, $j \neq k$, and $j \neq \ell$. Further, $k \neq \ell$, since $wx \notin E(H')$ and C_k induces a complete graph in H'. Thus, ($\star\star$) implies that v_k, v_ℓ are non-adjacent common neighbours of v_i, v_j in G', since w, z are non-adjacent common neighbours of x, y in H'. This proves that G' is indeed a minimal chordal completion of G.

That concludes the proof. □

Now, we are finally ready to prove Theorem 6.

PROOF OF THEOREM 6. We observe that the graph $\text{int}(\mathcal{Q})$ can be obtained by substituting complete graphs for the vertices of $\text{int}^*(\mathcal{Q})$. Namely, for each vertex A of $\text{int}^*(\mathcal{Q})$, we substitute A by the complete graph on vertices $C_A = \{(A, \pi) \mid \pi \in \mathcal{Q} \text{ and } A \text{ is a cell of } \pi\}$. Thus, by Lemma 13, there is a bijection Ψ between the minimal chordal completions of $\text{int}(\mathcal{Q})$ and $\text{int}^*(\mathcal{Q})$. By translating the condition ($\star\star$) from the proof of Lemma 13, we conclude that if G' is a minimal chordal completion of $\text{int}^*(\mathcal{Q})$, then $H' = \Psi(G')$ is the graph whose vertex set is that of $\text{int}(\mathcal{Q})$ and in which for all $A, A' \in V(G')$:

($\star\star$) all meaningful $\pi, \pi' \in \mathcal{Q}$ satisfy $AA' \in V(G') \iff (A,\pi)(A',\pi') \in V(H')$.

We show that Ψ is a bijection between the minimal restricted chordal completions of $\text{int}(\mathcal{Q})$ and the minimal chordal sandwiches of $(\text{int}^*(\mathcal{Q}), \text{forb}(\mathcal{Q}))$.

First, let H' be a minimal restricted chordal completion of $\text{int}(\mathcal{Q})$. Then $G' = \Psi^{-1}(H')$ is a minimal chordal completion of $\text{int}^*(\mathcal{Q})$. Consider two cells A_1, A_2 of $\pi \in \mathcal{Q}$. Since H' is a restricted chordal completion of $\text{int}(\mathcal{Q})$, we have that (A_1,π) is not adjacent to (A_2,π) in H'. Thus, $A_1A_2 \notin E(G')$ by ($\star\star$). This shows that G' contains no edge from $\text{forb}(\mathcal{Q})$. Thus G' is a minimal chordal sandwich of $(\text{int}^*(\mathcal{Q}), \text{forb}(\mathcal{Q}))$, since it is also a minimal chordal completion of $\text{int}^*(\mathcal{Q})$.

Conversely, let G' be a minimal chordal sandwich of $(\text{int}^*(\mathcal{Q}), \text{forb}(\mathcal{Q}))$. Then $H' = \Psi(G')$ is a minimal chordal completion of $\text{int}(\mathcal{Q})$. Consider two cells A_1, A_2 of $\pi \in \mathcal{Q}$. Since A_1A_2 is an edge of $\text{forb}(\mathcal{Q})$, and G' is a minimal chordal sandwich of $(\text{int}^*(\mathcal{Q}),\text{forb}(\mathcal{Q}))$, we have $A_1A_2 \notin E(G')$. Thus, $(A_1,\pi)(A_2,\pi) \notin E(H')$ by ($\star\star$). This shows that H' is a minimal restricted chordal completion of $\text{int}(\mathcal{Q})$.

That concludes the proof. □

6. Minimal Chordal Sandwiches and Boolean Satisfiability

In this section, we prove Theorem 9. We consider an instance I of ONE-IN-THREE-3SAT, and carefully analyze chordal sandwiches of $(\text{int}^*(\mathcal{Q}_I), \text{forb}(\mathcal{Q}_I))$. For a truth assignment σ for the instance I, we construct graphs G_σ, G'_σ, and G^*_σ, starting from $\text{int}^*(\mathcal{Q}_I)$. We show that if σ is a satisfying assignment for I, then G^*_σ is a minimal chordal sandwich of $(\text{int}^*(\mathcal{Q}_I), \text{forb}(\mathcal{Q}_I))$. Conversely, for every minimal chordal sandwich G' of $(\text{int}^*(\mathcal{Q}_I), \text{forb}(\mathcal{Q}_I))$, we describe a satisfying assignment σ for I such that $G' = G^*_\sigma$. From this the theorem will follow.

For later, we need the following simple properties. The proofs are straightforward and left to the reader.

Lemma 14. *Let G be a chordal graph, and let a, b be non-adjacent vertices of G. Then every two common neighbours of a and b are adjacent.*

Lemma 15. *Let G be a chordal graph, and $C = \{a,b,c,d,e\}$ be a 5-cycle in G with edges ab, bc, cd, de, ae.*

(a) *If $bd, ce \notin E(G)$, then $ac, ad \in E(G)$, and*
(b) *if $bd, be \notin E(G)$, then $ac \in E(G)$.*

Lemma 16. *Let G be a chordal graph, and $C = \{a,b,c,d,e,f\}$ be a 6-cycle in G with edges ab, bc, cd, de, ef, af.*

(a) *If $bd, ce, df \notin E(G)$, then $ac, ad, ae \in E(G)$,*
(b) *if $bd, ce, cf \notin E(G)$, then $ac, ad \in E(G)$, and*
(c) *if $be, bf, ce, cf \notin E(G)$, then $ad \in E(G)$.*

To assist the reader in following the subsequent arguments, we now list here the cliques of $\text{int}^*(\mathcal{Q}_I)$ according to the elements from which they arise:

δ: B, $H_{v_1}, \ldots, H_{v_n}$, $H_{\overline{v_1}}, \ldots, H_{\overline{v_n}}$

μ: B, $F^1, \ldots, F^m$

For each $i \in \{1 \ldots n\}$ where $j_1, j_2, \ldots, j_k$ are the elements of Δ_i:

α_{v_i}: H_{v_i}, A_i, $S^{j_1}_{v_i}$, $S^{j_2}_{v_i}, \ldots, S^{j_t}_{v_i}$, $\qquad$ $\alpha_{\overline{v_i}}$: $H_{\overline{v_i}}$, A_i, $S^{j_1}_{\overline{v_i}}$, $S^{j_2}_{\overline{v_i}}, \ldots, S^{j_t}_{\overline{v_i}}$,

For each $j \in \{1 \ldots m\}$ where $\mathcal{C}_j = X \vee Y \vee Z$:

λ^j: K^j_X, K^j_Y, K^j_Z, D^j_1, D^j_2, D^j_3, F^j

γ^j_1: $K^j_{\overline{X}}$, L^j_Z, D^j_1 $\qquad$ γ^j_2: $K^j_{\overline{Y}}$, L^j_X, D^j_2 $\qquad$ γ^j_3: $K^j_{\overline{Z}}$, L^j_Y, D^j_3

β^j_X: S^j_X, $K^j_{\overline{X}}$ $\qquad$ β^j_Y: S^j_Y, $K^j_{\overline{Y}}$ $\qquad$ β^j_Z: S^j_Z, $K^j_{\overline{Z}}$

$\beta^j_{\overline{X}}$: $S^j_{\overline{X}}$, K^j_X, L^j_X $\qquad$ $\beta^j_{\overline{Y}}$: $S^j_{\overline{Y}}$, K^j_Y, L^j_Y $\qquad$ $\beta^j_{\overline{Z}}$: $S^j_{\overline{Z}}$, K^j_Z, L^j_Z

We start with a useful lemma describing an important property of $\text{int}^*(\mathcal{Q}_I)$.

Lemma 17. *Let G' be a chordal sandwich of $(\text{int}^*(\mathcal{Q}_I), \text{forb}(\mathcal{Q}_I))$, and let $i \in \{1 \ldots n\}$. Then*

(a) *there exists $W \in \{v_i, \overline{v_i}\}$ such that for all $j \in \Delta_i$, the vertex K^j_W is adjacent to B, and*

(b) *for each $j \in \Delta_i$, and each $W \in \{v_i, \overline{v_i}\}$, if K^j_W is adjacent to B, then the vertices S^j_W, K^j_W, L^j_W (if exists) are adjacent to B, A_i, H_W, $H_{\overline{W}}$, F^j. (See Fig. 7a)*

PROOF. Let $i \in \{1 \ldots n\}$. First, we observe the following.

($\star$) *for each $j \in \Delta_i$, each $W \in \{v_i, \overline{v_i}\}$, at least one of $S^j_{\overline{W}}$, K^j_W is adjacent to B.*

We may assume that $S^j_{\overline{W}}$ is not adjacent to B, otherwise we are done. Observe that $S^j_{\overline{W}}$ is adjacent to K^j_W, since $\beta^j_{\overline{W}} \in K^j_W \cap S^j_{\overline{W}}$. Moreover, there exists $p \in \{1,2,3\}$ such that $K^j_W \cap D^j_p$ contains λ^j or γ^j_p, implying that K^j_W is adjacent to D^j_p. Also, F^j is adjacent to D^j_p and B, since $\lambda^j \in D^j_p \cap F^j$ and $\mu \in B \cap F^j$, respectively. Further, $H_{\overline{W}}$ is adjacent to $S^j_{\overline{W}}$ and B, since $\alpha_{\overline{W}} \in H_{\overline{W}} \cap S^j_{\overline{W}}$ and $\delta \in H_{\overline{W}} \cap B$. Finally, $H_{\overline{W}}$ is not adjacent to F^j, and B is not adjacent to D^j_p, since $H_{\overline{W}} \,\big|\, F^j$ and $D^j_p \,\big|\, B$ are in $\mathcal{Q}_I$. So, by Lemma 16 applied to the cycle $\{K^j_W$, $S^j_{\overline{W}}$, $H_{\overline{W}}$, B, F^j, $D^j_p\}$, we conclude that K^j_W is adjacent to B. This proves ($\star$).

Now, to prove (a), suppose for contradiction that there are $j, j' \in \Delta_i$ such that both $K^j_{\overline{v_i}}$ and $K^{j'}_{v_i}$ are not adjacent to B. Then by ($\star$), both $S^j_{v_i}$ and $S^{j'}_{\overline{v_i}}$ are adjacent to B. Note also that A_i is adjacent to both $S^j_{v_i}$, $S^{j'}_{\overline{v_i}}$, since $\alpha_{v_i} \in A_i \cap S^j_{v_i}$ and $\alpha_{\overline{v_i}} \in A_i \cap S^{j'}_{\overline{v_i}}$. Further, note that $A_i B$ and $S^j_{v_i} S^{j'}_{\overline{v_i}}$ are not edges of G', since $A_i \,\big|\, B$ and $S^j_{v_i} \,\big|\, S^{j'}_{\overline{v_i}}$ are in $\mathcal{Q}_I$. But then G' contains an induced 4-cycle on $\{S^j_{v_i}$, A_i, $S^{j'}_{\overline{v_i}}$, $B\}$, which is impossible, since G' is chordal. This proves (a).

For (b), suppose that K^j_W is adjacent to B for $j \in \Delta_i$ and $W \in \{v_i, \overline{v_i}\}$. First observe that K^j_W is adjacent to $S^j_{\overline{W}}$, and the vertex $K^j_{\overline{W}}$ is adjacent to S^j_W, since $\beta^j_{\overline{W}} \in K^j_W \cap S^j_{\overline{W}}$ and $\beta^j_W \in K^j_{\overline{W}} \cap S^j_W$. Moreover, there exists $p \in \{1,2,3\}$ such that $K^j_W \cap D^j_p$ and $K^j_{\overline{W}} \cap D^j_p$ contain γ^j_p and λ^j, respectively, or λ^j and γ^j_p, respectively. This implies that K^j_W and $K^j_{\overline{W}}$ are adjacent to D^j_p. Also, A_i is adjacent to S^j_W and $S^j_{\overline{W}}$, since $\alpha_W \in A_i \cap S^j_W$ and $\alpha_{\overline{W}} \in A_i \cap S^j_{\overline{W}}$. Further,

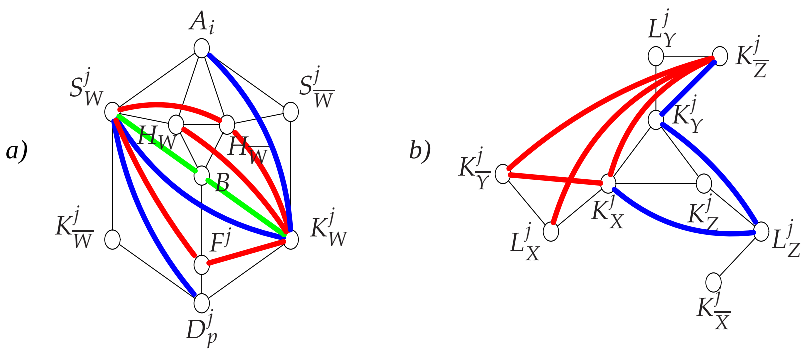

Figure 7: Chordal completion edges for *a)* $W = 1$, *b)* $X = 1$, $Y = 0$, $Z = 0$.

note that $D^j_p B$, $A_i B$, $K^j_W K^j_{\overline{W}}$, and $S^j_W S^j_{\overline{W}}$ are not edges of G', since $D^j_p \,\big|\, B$, $A_i \,\big|\, B$, $K^j_W \,\big|\, K^j_{\overline{W}}$, and $S^j_W \,\big|\, S^j_{\overline{W}}$ are in $\mathcal{Q}_I$. This implies that $K^j_{\overline{W}}$ is not adjacent to B, since otherwise G' contains an induced 4-cycle on $\{K^j_W, B, K^j_{\overline{W}}, D^j_p\}$. So, by ($\star$), we have that S^j_W is adjacent to B. Thus, Lemma 15 applied to $\{K^j_W, S^j_{\overline{W}}, A_i, S^j_W, B\}$ yields that K^j_W is adjacent to A_i and S^j_W. So, by Lemma 14 applied to $\{S^j_W, K^j_W, D^j_p, K^j_{\overline{W}}\}$, we have that S^j_W is adjacent to D^j_p.

Now, observe that H_W, $H_{\overline{W}}$ are adjacent to both A_i and B, since $\alpha_W \in H_W \cap A_i$, $\alpha_{\overline{W}} \in H_{\overline{W}} \cap A_i$, and $\delta \in B \cap H_W \cap H_{\overline{W}}$. Thus, by Lemma 14 applied to $\{u, A_i, u', B\}$ where $u \in \{S^j_W, K^j_W\}$ and $u' \in \{H_W, H_{\overline{W}}\}$, we conclude that S^j_W and K^j_W are adjacent to both H_W and $H_{\overline{W}}$. Similarly, we observe that F^j is adjacent to B and D^j_p, since $\mu \in F^j \cap B$ and $\lambda^j \in D^j_p \cap F^j$. Thus, Lemma 14 applied to $\{u, B, F^j, D^j_p\}$ yields that S^j_W and K^j_W are also adjacent to F^j.

Lastly, suppose that L^j_W exists. Then there is $q \in \{1, 2, 3\}$ such that $\gamma^j_q \in D^j_q \cap L^j_W$, implying that L^j_W is adjacent to D^j_q. Moreover, F^j is adjacent to D^j_q and B, since $\lambda^j \in D^j_q \cap F^j$ and $\mu \in F^j \cap B$. Also, $H_{\overline{W}}$ is adjacent to B, $S^j_{\overline{W}}$, and the vertex $S^j_{\overline{W}}$ is adjacent to L^j_W, since $\delta \in B \cap H_{\overline{W}}$, $\alpha_{\overline{W}} \in H_{\overline{W}} \cap S^j_{\overline{W}}$, and $\beta^j_{\overline{W}} \in S^j_{\overline{W}} \cap L^j_W$. Further, $H_{\overline{W}} F^j$ and $D^j_q B$ are not edges of G', since $H_{\overline{W}} \,\big|\, F^j$ and $D^j_q \,\big|\, B$ are in $\mathcal{Q}_I$. Also, $S^j_{\overline{W}} B$ is not an edge of G', since otherwise G' contains an induced 4-cycle on $\{S^j_W, B, S^j_{\overline{W}}, A_i\}$. Thus, by Lemma 15 applied to $\{L^j_W, S^j_{\overline{W}}, H_{\overline{W}}, B, F^j, D^j_q\}$, we conclude that L^j_W is adjacent to $H_{\overline{W}}$, B, and F^j. Moreover, by Lemma 15 applied to $\{L^j_W, B, S^j_W, A_i, S^j_{\overline{W}}\}$, we conclude that L^j_W is adjacent to A_i. Finally, recall that H_W is adjacent to both A_i and B. Thus, Lemma 14 applied to $\{L^j_W, A_i, H_W, B\}$ yields that L^j_W is also adjacent to H_W.

That concludes the proof. □

Now, let σ be a truth assignment for the instance I. Recall that, for simplicity, we write $X = 0$ and $X = 1$ in place of $\sigma(X) = 0$ and $\sigma(X) = 1$, respectively. To facilitate the arguments in the subsequent proofs, we introduce a naming convention for the vertices in $\text{int}^*(\mathcal{Q}_I)$ similar to that of [4], as we already indicated in §3.

The vertices S^j_W for all meaningful choices of j and W are called *shoulders*. For a fixed j, we call them *shoulders of the clause* $\mathcal{C}_j$, and for a fixed W, we call them *shoulders of the literal* W. A shoulder is a *a true shoulder* if $W = 1$. Otherwise, it is a *false shoulder*. The vertices K^j_W, L^j_W for all meaningful choices of j and W are called *knees*. For a fixed j, we call them *knees of the clause* $\mathcal{C}_j$, and for a fixed W, we call them *knees of the literal* W. A knee is a *true knee* if $W = 1$. Otherwise, it is a *false knee*. The vertices A_i, D^j_p, H_W, F^j for all meaningful choices of indices are called *A-vertices*, *D-vertices*, *H-vertices*, and *F-vertices*, respectively.

Based on σ, we define the following three graphs: G_σ, G'_σ, and G^*_σ.

6.1. Definition of G_σ

The graph G_σ is constructed from $\text{int}^*(\mathcal{Q}_I)$ by performing the following steps:

(i) make B adjacent to all true knees and true shoulders

6.2. Definition of G'_σ

The graph G'_σ is constructed from G_σ by performing the following steps:

(ii) make $\{$true knees, true shoulders$\}$ pairwise adjacent,

(iii) for all $i \in \{1 \ldots n\}$, make A_i adjacent to all true knees of the literals v_i and $\overline{v_i}$,

(iv) for all $1 \le i' \le i \le n$, make H_{v_i}, $H_{\overline{v_i}}$ adjacent to all true knees and true shoulders of the literals $v_{i'}$ and $\overline{v_{i'}}$,

(v) for all $1 \le j \le j' \le m$, make F^j adjacent to all true knees and true shoulders of the clause $\mathcal{C}_{j'}$,

(vi) for all $i \in \{1 \ldots n\}$ and all $j, j' \in \Delta_i$ such that $j \le j'$:

a) if $v_i = 1$, make $S^{j'}_{\overline{v_i}}$ adjacent to $K^j_{v_i}$, $L^j_{v_i}$ (if exists),

b) if $v_i = 0$, make $S^{j'}_{v_i}$ adjacent to $K^j_{\overline{v_i}}$, $L^j_{\overline{v_i}}$ (if exists).

6.3. Definition of G^*_σ

The graph G^*_σ is constructed from G'_σ by adding the following edges:

(vii) for all $j \in \{1 \ldots m\}$ where $\mathcal{C}_j = X \vee Y \vee Z$:

a) if $X = 1$, then add edges $F^j L^j_Z$, $K^j_X L^j_Z$, $K^j_Y K^j_{\overline{Z}}$, $D^j_2 K^j_{\overline{Z}}$, $D^j_2 S^j_{\overline{Y}}$, $D^j_3 S^j_{\overline{Y}}$ and also add all possible edges between the vertices D^j_1, D^j_2, D^j_3, S^j_X, $S^j_{\overline{Z}}$, L^j_Z, K^j_Y,

b) if $Y = 1$, then add edges $F^j L^j_X$, $K^j_Y L^j_X$, $K^j_Z K^j_{\overline{X}}$, $D^j_3 K^j_{\overline{X}}$, $D^j_3 S^j_{\overline{Z}}$, $D^j_1 S^j_{\overline{Z}}$ and also add all possible edges between the vertices D^j_1, D^j_2, D^j_3, S^j_Y, $S^j_{\overline{X}}$, L^j_X, K^j_Z,

c) if $Z = 1$, then add edges $F^j L^j_Y$, $K^j_Z L^j_Y$, $K^j_X K^j_{\overline{Y}}$, $D^j_1 K^j_{\overline{Y}}$, $D^j_1 S^j_{\overline{X}}$, $D^j_2 S^j_{\overline{X}}$ and also add all possible edges between the vertices D^j_1, D^j_2, D^j_3, S^j_Z, $S^j_{\overline{Y}}$, L^j_Y, K^j_X.

Lemma 18. *G'_σ is a subgraph of every chordal sandwich of $(G_\sigma, \mathrm{forb}(\mathcal{Q}_I))$.*

PROOF. Let G' be a chordal sandwich of $(G_\sigma, \mathrm{forb}(\mathcal{Q}_I))$. We prove the claim by showing that G' contains all edges defined in steps (ii)-(vi). We consider these steps one by one.

- for (ii), consider true shoulders S^j_W, $S^{j'}_{W'}$ and true knees K^j_W, $K^{j'}_{W'}$ and L^j_W, $L^{j'}_{W'}$ (if they exist). We allow that W is possibly equal to W' and possibly $j = j'$. First, we observe that, by (i), the true knees K^j_W and $K^{j'}_{W'}$ are adjacent to B. Therefore, by Lemma 17, the vertices S^j_W, K^j_W, L^j_W are adjacent to H_W and F^j, whereas $S^{j'}_{W'}$, $K^{j'}_{W'}$, $L^{j'}_{W'}$ are adjacent to $H_{W'}$ and $F^{j'}$. Also, H_W is adjacent to $H_{W'}$, and F^j is adjacent to $F^{j'}$, since $\delta \in H_W \cap H_{W'}$ and $\mu \in F^j \cap F^{j'}$, respectively. Further, $H_W F^j$, $H_W F^{j'}$, $H_{W'} F^j$, $H_{W'} F^{j'}$ are not edges of G', since $H_W \big| F^j$, $H_W \big| F^{j'}$, $H_{W'} \big| F^j$, $H_{W'} \big| F^{j'}$ are in $\mathcal{Q}_I$. Thus, if $j = j'$ and W is equal to W', then, by Lemma 14 applied to cycles $\{u, H_W, u', F^j\}$ where $u, u' \in \{S^j_W, S^{j'}_{W'}, K^j_W, K^{j'}_{W'}, L^j_W, L^{j'}_{W'}\}$, we conclude that $\{S^j_W, S^{j'}_{W'}, K^j_W, K^{j'}_{W'}, L^j_W, L^{j'}_{W'}\}$ are pairwise adjacent in G'. If $j \ne j'$ and W is not equal to W', we reach the same conclusion by Lemma 16 applied to the cycles $\{u, H_W, H_{W'}, u', F^{j'}, F^j\}$. Otherwise, we obtain the conclusion by applying Lemma 15 either to cycles $\{u, H_W, u', F^{j'}, F^j\}$ or cycles $\{u, F^j, u', H_{W'}, H_W\}$. This proves (ii).
- for (iii), consider the vertex A_i for $i \in \{1 \ldots n\}$. Let $W \in \{v_i, \overline{v_i}\}$ be such that $W = 1$. Then, for each $j \in \Delta_i$, the vertex K^j_W is adjacent to B by (i). Thus, by Lemma 17, both K^j_W and L^j_W (if exists) are adjacent to A_i. This proves (iii).
- for (iv), we consider $1 \le i' \le i \le n$. Let $W' \in \{v_{i'}, \overline{v_{i'}}\}$ be such that $W' = 1$. Then, for all $j \in \Delta_{i'}$, the vertex $K^j_{W'}$ is adjacent to B by (i), and hence, the vertices $S^j_{W'}$, $K^j_{W'}$ and $L^j_{W'}$ (if exists) are adjacent by

Lemma 17 to $H_{v_{i'}}$, $H_{\overline{v_{i'}}}$. In other words, the vertices $H_{v_{i'}}$, $H_{\overline{v_{i'}}}$ are adjacent to all true knees and true shoulders of the literals $v_{i'}$, $\overline{v_{i'}}$. Thus, we may assume that $i' < i$. Now, the vertex $H_{v_{i'}}$ is adjacent to H_{v_i},$H_{\overline{v_i}}$, since $\delta \in H_{v_i} \cap H_{\overline{v_i}} \cap H_{v_{i'}}$. Let $W \in \{v_i, \overline{v_i}\}$ be such that $W = 1$. Then K^j_W is adjacent to B by (i), and hence, S^j_W is adjacent to H_{v_i}, $H_{\overline{v_i}}$ by Lemma 17. Also, S^j_W is adjacent to all true knees and true shoulders of the literals $v_{i'}$, $\overline{v_{i'}}$, by (ii), and the same is true for $H_{v_{i'}}$ as proved earlier in this paragraph. Further, S^j_W is not adjacent to $H_{v_{i'}}$, since $H_{v_{i'}} \,\big|\, S^j_W$ is in $\mathcal{Q}_I$. Thus, by Lemma 14, both H_{v_i} and $H_{\overline{v_i}}$ are adjacent to all true knees and true shoulders of the literals $v_{i'}$, $\overline{v_{i'}}$. This proves (iv).

- for (v), consider $1 \le j \le j' \le m$. Again, we observe that if $K^{j'}_{W'}$ is a true knee, then $K^{j'}_{W'}$ is adjacent to B by (i), and hence, $S^{j'}_{W'}$, $K^{j'}_{W'}$, and $L^{j'}_{W'}$ (if exists) are adjacent to $F^{j'}$ by Lemma 17. In other words, the vertex $F^{j'}$ is adjacent to all true knees and true shoulders of the clause $\mathcal{C}_{j'}$. So, we may assume that $j < j'$. Now, let K^j_W be any true knee of the clause $\mathcal{C}_j$. Then K^j_W is adjacent to B, and hence, to F^j by (i) and Lemma 17, respectively. Also, K^j_W is adjacent to all true shoulders and true knees of $\mathcal{C}_{j'}$ by (ii). Further, F^j is adjacent to $F^{j'}$, since $\mu \in F^j \cap F^{j'}$, and the vertex K^j_W is not adjacent to $F^{j'}$, since $K^j_W \,\big|\, F^{j'}$ is in $\mathcal{Q}_I$. Thus, by Lemma 14, the vertex F^j is adjacent to all true knees and true shoulders of the clause $\mathcal{C}_{j'}$. This proves (v).

- for (vi), let $i \in \{1 \ldots n\}$ and consider $j, j' \in \Delta_i$ with $j \le j'$. Let $W \in \{v_i, \overline{v_i}\}$ be such that $W = 1$. Observe that K^j_W is adjacent to $S^j_{\overline{W}}$, since $\beta^j_{\overline{W}} \in S^j_{\overline{W}} \cap K^j_W$. If L^j_W exists, also L^j_W is adjacent to $S^j_{\overline{W}}$, since then $\beta^j_{\overline{W}} \in S^j_{\overline{W}} \cap L^j_W$. Thus, we may assume that $j < j'$. Now, $S^{j'}_{\overline{W}}$ is adjacent to $S^j_{\overline{W}}$ and $K^{j'}_W$, since $\alpha_{\overline{W}} \in S^j_{\overline{W}} \cap S^{j'}_{\overline{W}}$, and $\beta^{j'}_{\overline{W}} \in S^{j'}_{\overline{W}} \cap K^{j'}_W$. Also, K^j_W and L^j_W (if exists) are adjacent to $K^{j'}_W$ by (ii). Further, $S^j_{\overline{W}} K^{j'}_W$ is not an edge of G', since $S^j_{\overline{W}} \,\big|\, K^{j'}_W$ is in $\mathcal{Q}_I$. Thus, by Lemma 14, the vertices K^j_W, L^j_W (if exists) are adjacent to $S^{j'}_{\overline{W}}$. This proves (vi).

The proof is now complete. □

Lemma 19. *If σ is a satisfying assignment for I, then G^*_σ is a subgraph of every chordal sandwich of $(G_\sigma, \mathrm{forb}(\mathcal{Q}_I))$.*

PROOF. Let G' be a chordal sandwich of $(G_\sigma, \mathrm{forb}(\mathcal{Q}_I))$, and assume that σ is a satisfying assignment for I. That is, for each clause $\mathcal{C}_j = X \vee Y \vee Z$, either $X = 1, Y = Z = 0$, or $Y = 1, X = Z = 0$, or $Z = 1, X = Y = 0$.

By Lemma 18, the graph G' contain all edges defined in (ii)-(vi). Thus it remains to prove that it also contains the edges defined in (vii).

Consider $j \in \{1 \ldots m\}$ where $\mathcal{C}_j = X \vee Y \vee Z$. By the rotational symmetry between X, Y, and Z, we may assume that $X = 1$, $Y = 0$, and $Z = 0$. Observe that K^j_Z is adjacent to K^j_X and L^j_Z, since $\lambda^j \in K^j_Z \cap K^j_X$ and $\beta^j_{\overline{Z}} \in K^j_Z \cap L^j_Z$. Further, $K^j_{\overline{X}}$ is adjacent to L^j_Z and S^j_X, since $\gamma^j_1 \in L^j_Z \cap K^j_X$ and $\beta^j_X \in K^j_{\overline{X}} \cap S^j_X$. By (ii), also K^j_X is adjacent to S^j_X. Moreover, $S^j_X K^j_Z$ and $K^j_X K^j_{\overline{X}}$ are not edges of G', since $S^j_X \,\big|\, K^j_Z$, $K^j_X \,\big|\, K^j_{\overline{X}}$ are in $\mathcal{Q}_I$. Thus, by Lemma 15 applied to the cycle $\{L^j_Z, K^j_Z, K^j_X, S^j_X, K^j_{\overline{X}}\}$, we conclude that L^j_Z is adjacent to S^j_X and K^j_X. Now, observe that L^j_Y is adjacent to K^j_Y and $K^j_{\overline{Z}}$, since $\beta^j_{\overline{Y}} \in L^j_Y \cap K^j_Y$ and $\gamma^j_3 \in L^j_Y \cap K^j_{\overline{Z}}$. Recall that K^j_Z is adjacent to L^j_Z and also to K^j_Y, since $\lambda^j \in K^j_Z \cap K^j_Y$. Moreover, S^j_X is adjacent to $K^j_{\overline{Z}}$ and L^j_Z by (ii) and the above. Further, $K^j_{\overline{Z}} L^j_Z$, $S^j_X L^j_Y$, $S^j_X K^j_Z$ are not edges of G', since $K^j_{\overline{Z}} \,\big|\, L^j_Z$, $S^j_X \,\big|\, L^j_Y$, $S^j_X \,\big|\, K^j_Z$ are in $\mathcal{Q}_I$. Thus, by Lemma 16 applied to the cycle $\{K^j_Y, L^j_Y, K^j_{\overline{Z}}, S^j_X, L^j_Z, K^j_Z\}$, we conclude that K^j_Y is adjacent to $K^j_{\overline{Z}}$, S^j_X, and L^j_Z. Next, observe that $S^j_{\overline{Z}}$ is adjacent to $K^j_{\overline{Z}}$ and K^j_Z by (ii) and since $\beta^j_{\overline{Z}} \in S^j_{\overline{Z}} \cap K^j_Z$. Recall that K^j_Y is adjacent to $K^j_{\overline{Z}}$ and K^j_Z. Further, $K^j_Z K^j_{\overline{Z}}$ is not an edge of G', since $K^j_Z \,\big|\, K^j_{\overline{Z}}$ is in $\mathcal{Q}_I$. Thus, by Lemma 14, the vertex $S^j_{\overline{Z}}$ is adjacent to K^j_Y. Now, recall that L^j_Z is adjacent to S^j_X and K^j_Z, and $S^j_X K^j_Z$ is not an edge of G'. Also, F^j is adjacent to S^j_X and K^j_Z by (v) and since $\lambda^j \in F^j \cap K^j_Z$. Thus, by Lemma 14, the vertex L^j_Z is adjacent to F^j. Now, observe that D^j_1 is adjacent to K^j_X, $K^j_{\overline{X}}$, since $\lambda^j \in D^j_1 \cap K^j_X$ and $\gamma^j_1 \in D^j_1 \cap K^j_{\overline{X}}$. Recall that also S_X is adjacent to both K^j_X and $K^j_{\overline{X}}$, and that $K^j_X K^j_{\overline{X}}$ is not an edge of G'. Thus, by Lemma 14, we have that D^j_1 is adjacent to S^j_X. Next, observe that D^j_2 is adjacent to K^j_Y, $K^j_{\overline{Y}}$, since $\lambda^j \in D^j_2 \cap K^j_Y$ and $\gamma^j_2 \in D^j_2 \cap K^j_{\overline{Y}}$. Recall that K^j_Y is adjacent to $K^j_{\overline{Z}}$ and S^j_X. Also, $K^j_{\overline{Y}}$ is adjacent to S^j_X, $S^j_{\overline{Y}}$, $K^j_{\overline{Z}}$ by (ii), and K^j_Y is adjacent to $S^j_{\overline{Y}}$, since $\beta^j_{\overline{Y}} \in K^j_Y \cap S^j_{\overline{Y}}$. Further, $K^j_Y K^j_{\overline{Y}}$ is not an edge of G', since $K^j_Y \big| K^j_{\overline{Y}}$ is in $\mathcal{Q}_I$. Thus, by Lemma 14, the vertices S^j_X, $S^j_{\overline{Y}}$, $K^j_{\overline{Z}}$ are adjacent to D^j_2. Now, observe that D^j_1, D^j_2 are adjacent to K^j_Z, since $\lambda^j \in D^j_1 \cap D^j_2 \cap K^j_Z$. Also, recall that S^j_X is adjacent to D^j_1, D^j_2, L^j_Z, the vertex K^j_Z is adjacent to $S^j_{\overline{Z}}$, L^j_Z, and $S^j_X K^j_Z$ is not an edge of G'. Further, S^j_X is adjacent to $S^j_{\overline{Z}}$ by (ii). Thus, by Lemma 14, both D^j_1 and D^j_2 are adjacent to $S^j_{\overline{Z}}$ and L^j_Z. Next, observe that D^j_3 is adjacent to K^j_Z, $K^j_{\overline{Z}}$, since $\lambda^j \in D^j_3 \cap K^j_Z$ and $\gamma^j_3 \in D^j_3 \cap K^j_{\overline{Z}}$. Recall that also $S^j_{\overline{Z}}$ is adjacent to K^j_Z, $K^j_{\overline{Z}}$, and that $K^j_Z K^j_{\overline{Z}}$ is not an edge of G'. Thus, by Lemma 14, the vertex D^j_3 is adjacent to $S^j_{\overline{Z}}$. Further, recall that L^j_Z is adjacent to K^j_Z, S^j_X, the vertex $K^j_{\overline{Z}}$ is adjacent to S^j_X, and $S^j_X K^j_Z$ and $K^j_{\overline{Z}} L^j_Z$ are not edges of G'. Thus, Lemma 15 applied to $\{D^j_3, K^j_Z, L^j_Z, S^j_X, K^j_{\overline{Z}}\}$ yields that D^j_3 is adjacent to both L^j_Z and S^j_X. Moveover, $S^j_{\overline{Y}}$ is also adjacent to S^j_X by (ii), and L^j_Y is also adjacent to D^j_3, $S^j_{\overline{Y}}$, since $\gamma^j_3 \in D^j_3 \cap L^j_Y$ and $\beta^j_{\overline{Y}} \in S^j_{\overline{Y}} \cap L^j_Y$. Further, recall that $S^j_X L^j_Y$ is not an edge of G'. Thus, by Lemma 14 applied to $\{D^j_3, L^j_Y, S^j_{\overline{Y}}, S^j_X\}$, the vertex D^j_3 is adjacent to $S^j_{\overline{Y}}$.

To prove (vii), we observe that the above analysis yields that G' contains edges $F^j L^j_Z$, $K^j_X L^j_Z$, $K^j_Y K^j_{\overline{Z}}$, $D^j_2 K^j_{\overline{Z}}$, $D^j_2 S^j_{\overline{Y}}$, and $D^j_3 S^j_{\overline{Y}}$. It remains to show that $\{D^j_1, D^j_2, D^j_3, S^j_X, S^j_{\overline{Z}}, L^j_Z, K^j_Y\}$ are pairwise adjacent. By the above paragraph, we have that S^j_X, $S^j_{\overline{Z}}$, L^j_Z are adjacent to D^j_1, D^j_2, D^j_3. Also, D^j_1, D^j_2, D^j_3 and K^j_Y are pairwise adjacent, since $\lambda^j \in D^j_1 \cap D^j_2 \cap D^j_3 \cap K^j_Y$. Further, L^j_Z is adjacent to S^j_X, and K^j_Y is adjacent to S^j_X, $S^j_{\overline{Z}}$, L^j_Z, by the above paragraph. Finally, $S^j_{\overline{Z}}$ is adjacent to S^j_X and L^j_Z by (ii) and since $\beta^j_{\overline{Z}} \in S^j_{\overline{Z}} \cap L^j_Z$. This proves (vii).

The proof is now complete. □

Lemma 20. *If σ is a satisfying assignment for I, then G^*_σ is a chordal graph.*

PROOF. Assume that σ is a satisfying assignment for I, namely for each clause $\mathcal{C}_j = X \vee Y \vee Z$, we have either $X = 1, Y = Z = 0$, or $Y = 1, X = Z = 0$, or $Z = 1, X = Y = 0$.

Consider the partition $V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$ of $V(G^*_\sigma)$ defined as follows:

$V_1 = \{$false knees, D-vertices$\}$,

$V_2 = \{$false shoulders$\}$,

$V_3 = \{A$-vertices$\}$,

$V_4 = \{H$-vertices, F-vertices$\}$, and

$V_5 = \{$true knees, true shoulders, the vertex $B\}$.

Let π be an enumeration of $V(G^*_\sigma)$ constructed by listing the elements of V_1, V_2, V_3, V_4, V_5 in that order such that:

1. the elements of V_1 are listed by considering each clause $\mathcal{C}_j = X \vee Y \vee Z$ and listing vertices (based on the truth assignment) as follows:
   a) if $X = 1$, then list $K^j_{\overline{X}}$, K^j_Z, L^j_Y, L^j_Z, D^j_1, K^j_Y, D^j_3, D^j_2 in that order,
   b) if $Y = 1$, then list $K^j_{\overline{Y}}$, K^j_X, L^j_Z, L^j_X, D^j_2, K^j_Z, D^j_1, D^j_3 in that order,
   c) if $Z = 1$, then list $K^j_{\overline{Z}}$, K^j_Y, L^j_X, L^j_Y, D^j_3, K^j_X, D^j_2, D^j_1 in that order,
2. the elements of V_2 are listed by listing the false shoulders of the clauses $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ in that order,
3. the elements of V_3 are listed in any order,

4. the elements of V_4 are listed as follows: first the vertices $H_{v_1}, H_{\overline{v_1}}$, $H_{v_2}, H_{\overline{v_2}}$, $\ldots H_{v_n}, H_{\overline{v_n}}$ in that order, and then the vertices F^m, F^{m-1}, $\ldots$, F^1 in that order,
5. the elements of V_5 are listed in any order.

We show that π is a perfect elimination ordering of G^*_σ which will imply the claim.

- consider V_1. Let $j \in \{1 \ldots m\}$ and let $\mathcal{C}_j = X \vee Y \vee Z$. By the rotational symmetry of X, Y, Z, assume that $X = 1$ and $Y = Z = 0$. So, π lists the false knees and D-vertices of $\mathcal{C}_j$ as $K^j_{\overline{X}}$, K^j_Z, L^j_Y, L^j_Z, D^j_1, K^j_Y, D^j_3, D^j_2.
  - consider the vertex $K^j_{\overline{X}}$. Recall that $K^j_{\overline{X}} = \{\beta^j_X, \gamma^j_1\}$. Observe that S^j_X is the only other vertex containing β^j_X, and L^j_Z, D^j_1 are the only other vertices containing γ^j_1. Moreover, none of the rules (i)-(vii) adds edges incident to $K^j_{\overline{X}}$. Thus, S^j_X, L^j_Z, D^j_1 are the only neighbours of $K^j_{\overline{X}}$, and they are pairwise adjacent by (vii). This proves that $K^j_{\overline{X}}$ is indeed a simplicial vertex of G^*_σ.
  - consider K^j_Z. Since $K^j_Z = \{\beta^j_{\overline{Z}}, \lambda^j\}$, we conclude that K^j_Z is adjacent to $S^j_{\overline{Z}}$, L^j_Z, K^j_X, K^j_Y, D^1_j, D^2_j, D^3_j, and F^j. Moreover, K^j_Z has no other neighbours by observing the rules (i)-(vii). Now, by (vii), we conclude that $S^j_{\overline{Z}}$, L^j_Z, K^j_Y, D^j_1, D^j_2, D^j_3 are pairwise adjacent. Also, the vertices F^j, K^j_X, K^j_Y, D^j_1, D^j_2, D^j_3 are pairwise adjacent, since they all contain λ^j. Further, F^j is adjacent to $S^j_{\overline{Z}}$ and L^j_Z by (v) and (vii), respectively, and K^j_X is adjacent to $S^j_{\overline{Z}}$ and L^j_Z by (ii) and (vii), respectively. This proves that K^j_Z is a simplicial vertex of G^*_σ.
  - consider L^j_Y. The neighbours of L^j_Y are $S^j_{\overline{Y}}$, K^j_Y, $K^j_{\overline{Z}}$, and D^j_3. So, $S^j_{\overline{Y}}$ is adjacent to $K^j_{\overline{Z}}$, D^j_3, and K^j_Y by (ii), (vii), and since $\beta^j_{\overline{Y}} \in S^j_{\overline{Y}} \cap K^j_Y$. Similarly, K^j_Y is adjacent to $K^j_{\overline{Z}}$ and D^j_3 by (vii) and since $\lambda^j \in K^j_Y \cap D^j_3$. Finally, $K^j_{\overline{Z}}$ is adjacent to D^j_3, since $\gamma^j_3 \in K^j_{\overline{Z}} \cap D^j_3$. This proves that L^j_Y is a simplicial vertex of G^*_σ.
  - consider L^j_Z. The neighbours of L^j_Z are F^j, K^j_X, K^j_Y, K^j_Z, D^j_1, D^j_2, D^j_3, S^j_X, $S^j_{\overline{Z}}$, and $K^j_{\overline{X}}$. By (vii), the vertices D^j_1, D^j_2, D^j_3, S^j_X, $S^j_{\overline{Z}}$, K^j_Y are pairwise adjacent. Also, F^j, K^j_X, K^j_Y, D^j_1, D^j_2, D^j_3 are pairwise adjacent, since they all contain λ_j. Further, K^j_X and F^j are adjacent to S^j_X, $S^j_{\overline{Z}}$ by (ii) and (v), respectively. This proves that L^j_Z is a simplicial vertex of $G^*_\sigma - \{K^j_{\overline{X}}, K^j_Z\}$.
  - consider D^j_1. The neighbours of D^j_1 are F^j, K^j_X, K^j_Y, K^j_Z, D^j_2, D^j_3, S^j_X, $S^j_{\overline{Z}}$, L^j_Z, and $K^j_{\overline{X}}$. By (vii), the vertices D^j_2, D^j_3, S^j_X, $S^j_{\overline{Z}}$, K^j_Y are pairwise adjacent. Also, F^j, K^j_X, K^j_Y, D^j_2, D^j_3 are pairwise adjacent, since they all contain λ^j. Further, K^j_X and F^j are adjacent to S^j_X, $S^j_{\overline{Z}}$ by (ii) and (v), respectively. This proves that D^j_1 is a simplicial vertex of $G^*_\sigma - \{K^j_{\overline{X}}, K^j_Z, L^j_Z\}$.
  - consider K^j_Y. The neighbours of K^j_Y are F^j, K^j_X, K^j_Z, D^j_1, D^j_2, D^j_3, S^j_X, $S^j_{\overline{Y}}$, $S^j_{\overline{Z}}$, $K^j_{\overline{Z}}$, L^j_Y, and L^j_Z. By (vii), the vertices D^j_2, D^j_3, S^j_X, $S^j_{\overline{Z}}$ are pairwise adjacent. Also, F, K^j_X, D^j_2, D^j_3 are pairwise adjacent, since they all contain λ^j. Further, by (ii), the vertices S^j_X, $S^j_{\overline{Y}}$, $S^j_{\overline{Z}}$, K^j_X, and $K^j_{\overline{Z}}$ are pairwise adjacent, and are adjacent to F^j by (v). Moreover, by (vii), $S^j_{\overline{Y}}$ and $K^j_{\overline{Z}}$ are adjacent to D^j_2, and they are also adjacent to D^j_3 by (vii) and since $\gamma^j_3 \in K^j_{\overline{Z}} \cap D^j_3$, respectively. This proves that K^j_Y is a simplicial vertex of $G^*_\sigma - \{K^j_Z, L^j_Y, L^j_Z, D^j_1\}$.
  - consider D^3_j. The neighbours of D^3_j are F^j, K^j_X, K^j_Y, K^j_Z, D^j_1, D^j_2, S^j_X, $S^j_{\overline{Y}}$, $S^j_{\overline{Z}}$, $K^j_{\overline{Z}}$, L^j_Z, and L^j_Y. By (ii), the vertices S^j_X, $S^j_{\overline{Y}}$, $S^j_{\overline{Z}}$, K^j_X, $K^j_{\overline{Z}}$ are pairwise adjacent. Also, F^j, K^j_X, D^j_2 are pairwise adjacent, since they all contain λ^j. Further, F^j and D^j_2 are adjacent to S^j_X, $S^j_{\overline{Y}}$, $S^j_{\overline{Z}}$, $K^j_{\overline{Z}}$ by (v) and (vii), respectively. This proves that D^3_j is a simplicial vertex of $G^*_\sigma - \{K^j_Z, L^j_Y, L^j_Z, D^j_1, K^j_Y\}$.

- consider D^2_j. The neighbours of D^2_j are F^j, K^j_X, K^j_Y, K^j_Z, D^1_j, D^3_j, S^j_X, $S^j_{\overline{Y}}$, $S^j_{\overline{Z}}$, $K^j_{\overline{Z}}$, $K^j_{\overline{Y}}$, L^j_X and L^j_Z. By (ii), the vertices S^j_X, $S^j_{\overline{Y}}$, $S^j_{\overline{Z}}$, K^j_X, L^j_X, $K^j_{\overline{Y}}$, $K^j_{\overline{Z}}$ are pairwise adjacent, and are adjacent to F by (v). This proves that D^2_j is a simplicial vertex of $G^*_\sigma - \{K^j_Z, L^j_Z, D^j_1, K^j_Y, D^j_3\}$.

That concludes the vertices in V_1.

- consider the set V_2. Let $j \in \{1 \ldots m\}$ and consider a false shoulder S^j_W for some $W = 0$. Let i be such that $W = v_i$ or $W = \overline{v_i}$. The neighbours of S^j_W are the vertices H_W, A_i, and the elements of the following sets:

$$\mathcal{S}^- = \left\{ S^{j'}_W \;\middle|\; j' \in \Delta_i \text{ and } j' < j \right\} \qquad \mathcal{S}^+ = \left\{ S^{j''}_W \;\middle|\; j'' \in \Delta_i \text{ and } j'' > j \right\}$$

$$\mathcal{K}^- = \left\{ K^{j'}_{\overline{W}}, L^{j'}_{\overline{W}} \text{ (if exists)} \;\middle|\; j' \in \Delta_i \text{ and } j' \leq j \right\}$$

By (ii), the elements of $\mathcal{K}^-$ are pairwise adjacent. Similarly, the elements of $\{H_W, A_i\} \cup \mathcal{S}^+$ are pairwise adjacent, since they all contain α_W. Further, each element of $\mathcal{S}^+$ is adjacent to every element of $\mathcal{K}^-$ by (vi), and each element of $\mathcal{K}^-$ is adjacent to A_i and H_W by (iii) and (iv), respectively. This proves that S^j_W is a simplicial vertex of $G^*_\sigma - \mathcal{S}^-$, and note that the elements of $\mathcal{S}^-$ are false shoulders of the clauses $\mathcal{C}_1, \ldots, \mathcal{C}_{j-1}$.

- consider the set V_3. Let $i \in \{1 \ldots n\}$ and consider the vertex A_i. The neighbours of A_i are the vertices H_{v_i}, $H_{\overline{v_i}}$, all shoulders of the literals v_i, $\overline{v_i}$, and all true knees of $v_i, \overline{v_i}$. By (ii), the true knees and true shoulders of v_i, $\overline{v_i}$ are pairwise adjacent, and are adjacent to both H_{v_i} and $H_{\overline{v_i}}$ by (iv). Also, H_{v_i} is adjacent to $H_{\overline{v_i}}$, since $\delta \in H_{v_i} \cap H_{\overline{v_i}}$. Therefore A_i is a simplicial vertex of $G^*_\sigma - V_2$, since the false shoulders of v_i, $\overline{v_i}$ belong to V_2.

- consider the set V_4.
  - let $i \in \{1 \ldots n\}$ and consider H_{v_i}, $H_{\overline{v_i}}$. The vertices H_{v_i}, $H_{\overline{v_i}}$ are adjacent to the vertices B, A_i, the elements of the following sets

  $$\mathcal{H}^- = \left\{ H_{v_{i'}}, H_{\overline{v_{i'}}} \;\middle|\; i' < i \right\} \qquad \mathcal{H}^+ = \left\{ H_{v_{i''}}, H_{\overline{v_{i''}}} \;\middle|\; i'' > i \right\}$$

  and all true knees, true shoulders of $v_{i'}$, $\overline{v_{i'}}$ for all $i' \in \{1 \ldots i\}$. Further, H_{v_i} is adjacent to $H_{\overline{v_i}}$, to all shoulders of v_i and to no other vertices, whereas $H_{\overline{v_i}}$ is adjacent H_{v_i}, to all shoulders of $\overline{v_i}$ and to no other vertices. Now, by (ii), the true knees and true shoulders of $v_{i'}$, $\overline{v_{i'}}$ for all $i' \in \{1 \ldots i\}$, are pairwise adjacent, and are adjacent to B and each element of $\mathcal{H}^+$ by (i) and (iv), respectively. Also, the elements of $\{B\} \cup \mathcal{H}^+$ are pairwise adjacent, since they all contain δ. Finally, observe that A_i belongs to V_3, and the false shoulders of v_i, $\overline{v_i}$ belong to V_2. This proves that both H_{v_i} and $H_{\overline{v_i}}$ are simplicial vertices of $G^*_\sigma - (V_2 \cup V_3 \cup \mathcal{H}^-)$ as required.
  - let $j \in \{1 \ldots m\}$ and consider F^j. Let $\mathcal{C}_j = X \vee Y \vee Z$, and by the rotational symmetry, assume that $X = 1$ and $Y = Z = 0$. Then the neighbours of F^j are B, K^j_Y, K^j_Z, D^j_1, D^j_2, D^j_3, L^j_Z, the elements of the following sets

  $$\mathcal{F}^+ = \left\{ F^{j'} \;\middle|\; j' > j \right\} \qquad \mathcal{F}^- = \left\{ F^{j''} \;\middle|\; j'' < j \right\}$$

  and all true knees and true shoulders of the clause $\mathcal{C}_{j'}$ for all $j' \in \{j \ldots m\}$. By (ii), the true knees and true shoulders of the clause $\mathcal{C}_{j'}$ for all $j' \in \{j \ldots m\}$, are pairwise adjacent, and are adjacent to B and each elements of $\mathcal{F}^-$ by (i) and (v), respectively. Also, the vertices of $\{B\} \cup \mathcal{F}^-$ are pairwise adjacent, since they all contain μ. Thus F^j is a simplicial vertex of $G^*_\sigma - (V_1 \cup \mathcal{F}^+)$, since the vertices K^j_Y, K^j_Z, D^j_1, D^j_2, D^j_3, L^j_Z belong to V_1.

That concludes all vertices in V_4.

- consider the set V_5. Observe that all vertices of V_5 are pairwise adjacent by (i) and (ii).

That concludes the proof. □

Lemma 21. *For every chordal sandwich G' of $(\mathrm{int}^*(\mathcal{Q}_I), \mathrm{forb}(\mathcal{Q}_I))$, there exists σ such that G_σ is a subgraph of G', and such that σ is a satisfying assignment for I.*

PROOF. Let G' be a chordal sandwich of $(\mathrm{int}^*(\mathcal{Q}_I), \mathrm{forb}(\mathcal{Q}_I))$. By Lemma 17, for each $i \in \{1 \dots n\}$, there exists $W \in \{v_i, \overline{v_i}\}$ such that for all $j \in \Delta_i$, the vertices S^j_W, K^j_W, and L^j_W(if exists) are adjacent to B in G'. Set $\sigma(v_i) = 1$ if $W = v_i$, and otherwise set $\sigma(v_i) = 0$. It follows that for such a mapping σ, the graph G' contains all edges of G_σ. Thus, by Lemma 19, the graph G'_σ is a subgraph of G', that is, G' contains the edges defined in (ii)-(vi).

It remains to prove that σ is a satisfying assignment for I. Let $j \in \{1 \dots m\}$ and the clause $\mathcal{C}_j = X \vee Y \vee Z$. If $X = Y = 1$, then the vertex S^j_Y is a true shoulder, and K^j_X is a true knee. Thus, by (ii), we conclude that S^j_Y is adjacent K^j_X. However, this is impossible, since $S^j_Y \big| K^j_X$ is in $\mathcal{Q}_Y$. Similarly, if $X = Z = 1$, we have that S^j_X is adjacent to K^j_Z by (ii) while $S^j_X \big| K^j_Z$ is in $\mathcal{Q}_I$, and if $Y = Z = 1$, then S^j_Z is adjacent to K^j_Y by (ii) while $S^j_Z \big| K^j_Y$ is in $\mathcal{Q}_I$.

Now, suppose that $X = Y = Z = 0$. First, observe that K^j_X is adjacent to L^j_X, K^j_Z, and the vertex L^j_Z is adjacent to K^j_Z, $K^j_{\overline{X}}$, since $\beta^j_{\overline{X}} \in K^j_X \cap L^j_X$, $\lambda^j \in K^j_X \cap K^j_Z$, $\beta^j_{\overline{Z}} \in L^j_Z \cap K^j_Z$, and $\gamma^j_1 \in L^j_Z \cap K^j_{\overline{X}}$. Also, $K^j_{\overline{X}}$ is adjacent to $K^j_{\overline{Z}}$ by (ii). Further, $K^j_{\overline{Z}}K^j_Z$, $K^j_{\overline{Z}}L^j_Z$ and $K^j_{\overline{X}}L^j_X$ are not edges of G', since $K^j_{\overline{Z}} \big| K^j_Z$, $K^j_{\overline{Z}} \big| L^j_Z$, and $K^j_{\overline{X}} \big| L^j_X$ are in $\mathcal{Q}_I$. Thus, if L^j_X is adjacent to $K^j_{\overline{Z}}$, then by Lemma 16 applied to $\{K^j_X, L^j_X, K^j_{\overline{Z}}, K^j_{\overline{X}}, L^j_Z, K^j_Z\}$, we conclude that K^j_X is adjacent to $K^j_{\overline{X}}$, which is impossible since $K^j_{\overline{X}} \big| K^j_X$ is in $\mathcal{Q}_I$. Similarly, if K^j_X is adjacent to $K^j_{\overline{Z}}$, then by Lemma 15 applied to $\{K^j_X, K^j_{\overline{Z}}, K^j_{\overline{X}}, L^j_Z, K^j_Z\}$, we again conclude that K^j_X is adjacent to $K^j_{\overline{X}}$, a contradiction. So, we may assume that both K^j_X and L^j_X are not adjacent to $K^j_{\overline{Z}}$. Now, observe that L^j_Y is adjacent to $K^j_{\overline{Z}}$, K^j_Y, and the vertex K^j_X is adjacent to L^j_X, K^j_Y, since $\gamma^j_3 \in K^j_{\overline{Z}} \cap L^j_Y$, $\beta^j_{\overline{Y}} \in L^j_Y \cap K^j_Y$, $\beta^j_{\overline{X}} \in K^j_X \cap L^j_X$, and $\lambda^j \in K^j_Y \cap K^j_X$. Also, $K^j_{\overline{Y}}$ is adjacent to $K^j_{\overline{Z}}$ and L^j_X by (ii) and since $\gamma^j_2 \in K^j_{\overline{Y}} \cap L^j_X$. Further, $K^j_{\overline{Y}}K^j_Y$ and $K^j_{\overline{Y}}L^j_Y$ are not edges of G', since $K^j_{\overline{Y}} \big| K^j_Y$ and $K^j_{\overline{Y}} \big| L^j_Y$ are in $\mathcal{Q}_I$. Recall that K^j_X, L^j_X are not adjacent to $K^j_{\overline{Z}}$. This contradicts Lemma 16 when applied to $\{K^j_X, L^j_X, K^j_{\overline{Y}}, K^j_{\overline{Z}}, L^j_Y, K^j_Y\}$.

Thus, it is not the case that $X = Y = Z = 0$, and by the above also not $X = Y = 1$, nor $X = Z = 1$, nor $Y = Z = 1$. Therefore, either $X = 1$, $Y = Z = 0$, or $Y = 1$, $X = Z = 0$, or $Z = 1$, $X = Y = 0$.

This proves that σ is indeed a satisfying assignment for I, which concludes the proof. □

We are finally ready to prove Theorem 9.

PROOF OF THEOREM 9. Let G' be a minimal chordal sandwich of $(\mathrm{int}^*(\mathcal{Q}_I),\ \mathrm{forb}(\mathcal{Q}_I))$. By Lemma 21, there exists σ, a satisfying assignment for I, such that G_σ is a subgraph of G'. Thus, G' is also a chordal sandwich of $(G_\sigma, \mathrm{forb}(\mathcal{Q}_I))$, and hence, G^*_σ is a subgraph of G' by Lemma 19. But by Lemma 20, G^*_σ is chordal, and so G' is equal to G^*_σ by the minimality of G'. Conversely, if σ is a satisfying assignment for I, then the graph G^*_σ is chordal by Lemma 20. Moreover, $\mathrm{int}^*(\mathcal{Q}_I)$ is a subgraph of G^*_σ, by definition, and G^*_σ contains no edges of $\mathrm{forb}(\mathcal{Q}_I)$, also by definition. Thus, G^*_σ is a chordal sandwich of $(\mathrm{int}^*(\mathcal{Q}_I), \mathrm{forb}(\mathcal{Q}_I))$, and it is minimal by Lemma 19.

This proves that by mapping each satisfying assigment σ to the graph G^*_σ, we obtain the required bijection. □

7. Perfect Phylogenies and Boolean Satisfiability

In this section, we prove Theorem 10. Let σ be a satisfying assignment for I; for each clause $\mathcal{C}_j = X \vee Y \vee Z$, either $X = 1$, $Y = Z = 0$, or $Y = 1$, $X = Z = 0$, or $Z = 1$, $X = Y = 0$. Consider $\mathcal{T}_\sigma = (T_I, \phi_\sigma)$ as defined in §4.

(We refer the reader to Fig. 5 and 6 for an illustration. We recommend the reader to observe this depiction when following the subsequent arguments.)

For each $i \in \{1 \dots n\}$, let $\mathcal{A}_i = \{a_i, a'_i, y'_i, z^{j_1}_i, \dots, z^{j_t}_i, c^{j_1}_i, \dots, c^{j_t}_i\}$ where $\Delta_i = \{j_1, \dots, j_t\}$, and for each $j \in \{1 \dots m\}$, let $\mathcal{B}_j = \{x^j_1, x^j_2, x^j_3, x^j_4, x^j_5, x^j_6, g^j_1, g^j_2, g^j_3, b^j_1, b^j_2, b^j_3, \ell^j\}$. (See Fig 6.)

It is not difficult to see that ϕ_σ defines a bijection between the elements of $\mathcal{X}_I$ and the leaves of T_I. For instance, for each $i \in \{1 \dots n\}$, we note that $\{\phi_\sigma(\alpha_{v_i}),\, \phi_\sigma(\alpha_{\overline{v_i}})\} = \{a_i,\, a'_i\}$, and for each $j \in \Delta_i$, either $\phi_\sigma(\beta^j_{v_i}) = c^j_i$ and $\phi_\sigma(\beta^j_{\overline{v_i}}) \in \{b^j_1, b^j_2, b^j_3\}$, or $\phi_\sigma(\beta^j_{\overline{v_i}}) = c^j_i$ and $\phi_\sigma(\beta^j_{v_i}) \in \{b^j_1, b^j_2, b^j_3\}$. Also, for each $j \in \{1 \dots m\}$, we have $\phi_\sigma(\lambda^j) = \ell^j$, and $\{\phi_\sigma(\gamma^j_1), \phi_\sigma(\gamma^j_2), \phi_\sigma(\gamma^j_3)\} = \{g^j_1, g^j_2, g^j_3\}$. Further, it can be readily verified that T_I is a ternary tree. Thus, $\mathcal{T}_\sigma = (T_I, \phi_\sigma)$ is indeed a ternary phylogenetic $\mathcal{X}_I$-tree. We show that it displays and is distinguished by $\mathcal{Q}_I$.

First, we show that $\mathcal{T}_\sigma$ displays $\mathcal{Q}_I$. We consider the quartet trees in $\mathcal{Q}_I$ one by one.

- consider $A_i \big| B$ for $i \in \{1 \dots n\}$. Recall that $A_i = \{\alpha_{v_i}, \alpha_{\overline{v_i}}\}$, $B = \{\delta, \mu\}$, and that $\{\phi_\sigma(\alpha_{v_i}), \phi_\sigma(\alpha_{\overline{v_i}})\} = \{a_i, a'_i\}$. Also, $\phi_\sigma(\delta) = y_0$ and $\phi_\sigma(\mu) = u_0$. Observe that $a_i, a'_i \in \mathcal{A}_i$. Hence, both a_i, a'_i are in one connected component of $T_I - y_i y'_i$ whereas y_0, u_0 are in another component. Thus, $\mathcal{T}_\sigma$ indeed displays $A_i \big| B$.

- consider $D^j_p \big| B$ for $j \in \{1 \dots m\}$ and $p \in \{1, 2, 3\}$. Recall that $D^j_p = \{\gamma^j_p, \lambda^j\}$, and $\phi_\sigma(\gamma^j_p) \in \mathcal{B}_j$, $\phi_\sigma(\lambda^j) \in \mathcal{B}_j$. Also, $B = \{\delta, \mu\}$ and $\phi_\sigma(\delta) = y_0$, $\phi_\sigma(\mu) = u_0$. Thus both $\phi_\sigma(\gamma^j_p)$, $\phi_\sigma(\lambda^j)$ are in one component of $T_I - u_j x^j_1$ whereas y_0, u_0 are in another component. This shows that $\mathcal{T}_\sigma$ displays $D^j_p \big| B$.

- consider $S^j_{v_i} \big| S^{j'}_{\overline{v_i}}$ where $i \in \{1 \dots n\}$ and $j, j' \in \Delta_i$. Recall that $S^j_{v_i} = \{\alpha_{v_i}, \beta^j_{v_i}\}$ and $S^{j'}_{\overline{v_i}} = \{\alpha_{\overline{v_i}}, \beta^{j'}_{\overline{v_i}}\}$. By symmetry, we may assume that $v_i = 1$. Then $\phi_\sigma(\alpha_{v_i}) = a_i$, $\phi_\sigma(\alpha_{\overline{v_i}}) = a'_i$, $\phi_\sigma(\beta^j_{v_i}) \in \mathcal{B}_j$, and $\phi_\sigma(\beta^{j'}_{\overline{v_i}}) = c^{j'}_i$. Let j_t denote the largest element in Δ_i. Then, both a'_i,$c^{j'}_i$ are in one component of $T_I - y'_i z^{j_t}_i$ whereas a_i and $\phi_\sigma(\beta^j_{v_i})$ are in a different component. This proves that $\mathcal{T}_\sigma$ displays $S^j_{v_i} \big| S^{j'}_{\overline{v_i}}$.

- consider $S^j_{v_i} \big| K^{j'}_{\overline{v_i}}$ and $S^j_{\overline{v_i}} \big| K^{j'}_{v_i}$ for $i \in \{1 \dots n\}$ and $j, j' \in \Delta_i$ where $j < j'$. Recall that $K^{j'}_{\overline{v_i}} \subseteq \{\beta^{j'}_{v_i}, \gamma^{j'}_1, \gamma^{j'}_2, \gamma^{j'}_3, \lambda^{j'}\}$, $K^{j'}_{v_i} \subseteq \{\beta^{j'}_{\overline{v_i}}, \gamma^{j'}_1, \gamma^{j'}_2, \gamma^{j'}_3, \lambda^{j'}\}$, $S^j_{v_i} = \{\alpha_{v_i}, \beta^j_{v_i}\}$ and $S^j_{\overline{v_i}} = \{\alpha_{\overline{v_i}}, \beta^j_{\overline{v_i}}\}$. Again, by symmetry, we may assume $v_i = 1$. So, $\phi_\sigma(\alpha_{v_i}) = a_i$, $\phi_\sigma(\alpha_{\overline{v_i}}) = a'_i$, $\phi_\sigma(\beta^j_{\overline{v_i}}) = c^j_i$, $\phi_\sigma(\beta^{j'}_{\overline{v_i}}) = c^{j'}_i$, $\phi_\sigma(\beta^j_{v_i}) \in \mathcal{B}_j$, and $\{\phi_\sigma(\beta^{j'}_{v_i}), \phi_\sigma(\gamma^{j'}_1), \phi_\sigma(\gamma^{j'}_2), \phi_\sigma(\gamma^{j'}_3), \phi_\sigma(\lambda^{j'})\} \subseteq \mathcal{B}_{j'}$. Let $j_1 < j_2 < \dots < j_t$ be the elements of Δ_i. Since $j \in \Delta_i$, let k be such that $j = j_k$. We conclude $k < t$, since $j < j'$ and $j' \in \Delta_i$. Thus, the elements of $\phi_\sigma(S^j_{\overline{v_i}})$ and $\phi_\sigma(K^{j'}_{v_i})$, respectively are in different components of $T_I - z^{j_k}_i z^{j_{k+1}}_i$. Further, observe that $\phi_\sigma(K^{j'}_{\overline{v_i}}) \subseteq \mathcal{B}_{j'}$, and since $j \neq j'$, the elements of $\phi_\sigma(S^j_{v_i})$ and $\phi_\sigma(K^{j'}_{\overline{v_i}})$ are in different components of $T_I - u_{j'} x^{j'}_1$. This proves that $\mathcal{T}_\sigma$ displays both $S^j_{v_i} \big| K^{j'}_{\overline{v_i}}$ and $S^j_{\overline{v_i}} \big| K^{j'}_{v_i}$.

- consider $K^j_{\overline{v_i}} \big| F^{j'}$ and $K^j_{v_i} \big| F^{j'}$ for $i \in \{1 \dots n\}$ and $j < j'$ where $j \in \Delta_i$. Again, recall that $K^j_{\overline{v_i}} \subseteq \{\beta^j_{v_i}, \gamma^j_1, \gamma^j_2, \gamma^j_3, \lambda^j\}$, $K^j_{v_i} \subseteq \{\beta^j_{\overline{v_i}}, \gamma^j_1, \gamma^j_2, \gamma^j_3, \lambda^j\}$, and that $F^{j'} = \{\lambda^{j'}, \mu\}$. So, $\phi_\sigma(K^j_{\overline{v_i}}) \cup \phi_\sigma(K^j_{v_i}) \subseteq \mathcal{A}_i \cup \mathcal{B}_j$ whereas $\phi_\sigma(F^{j'}) \subseteq \mathcal{B}_{j'} \cup \{u_0\}$. Since $j < j' \leq m$, we conclude that $\phi_\sigma(K^j_{\overline{v_i}}) \cup \phi_\sigma(K^j_{v_i})$ and $\phi_\sigma(F^{j'})$ are in different components of $T_I - u_j u_{j+1}$. This proves that $\mathcal{T}_\sigma$ displays both $K^j_{\overline{v_i}} \big| F^{j'}$ and $K^j_{v_i} \big| F^{j'}$.

- consider $H_{v_{i'}} \big| S^j_{v_i}$, $H_{\overline{v_{i'}}} \big| S^j_{v_i}$, $H_{v_{i'}} \big| S^j_{\overline{v_i}}$, and $H_{\overline{v_{i'}}} \big| S^j_{\overline{v_i}}$ for $1 \leq i' < i \leq n$ and $j \in \Delta_i$. Recall that $H_{v_{i'}} = \{\alpha_{v_{i'}}, \delta\}$, $H_{\overline{v_{i'}}} = \{\alpha_{\overline{v_{i'}}}, \delta\}$, $S^j_{v_i} = \{\alpha_{v_i}, \beta^j_{v_i}\}$, and $S^j_{\overline{v_i}} = \{\alpha_{\overline{v_i}}, \beta^j_{\overline{v_i}}\}$. So, $\phi_\sigma(S^j_{v_i}) \cup \phi_\sigma(S^j_{\overline{v_i}}) \subseteq \mathcal{A}_i \cup \mathcal{B}_j$ whereas $\phi_\sigma(H_{v_{i'}}) \cup \phi_\sigma(H_{\overline{v_{i'}}}) \subseteq \mathcal{A}_{i'} \cup \{y_0\}$. Thus, since $i' < i \leq n$, we conclude that $\phi_\sigma(S^j_{v_i}) \cup \phi_\sigma(S^j_{\overline{v_i}})$ and $\phi_\sigma(H_{v_{i'}}) \cup \phi_\sigma(H_{\overline{v_{i'}}})$ are in different components of $T_I - y_{i'} y_{i'+1}$. This proves that $\mathcal{T}_\sigma$ displays all the four quartet trees $H_{v_{i'}} \big| S^j_{v_i}$, $H_{\overline{v_{i'}}} \big| S^j_{v_i}$, $H_{v_{i'}} \big| S^j_{\overline{v_i}}$ and $H_{\overline{v_{i'}}} \big| S^j_{\overline{v_i}}$.

- consider $H_{\overline{v_i}} \big| F^j$ and $H_{v_i} \big| F^j$ for $i \in \{1 \dots n\}$ and $j \in \{1 \dots m\}$. Recall that $H_{v_i} = \{\alpha_{v_i}, \delta\}$, $H_{\overline{v_i}} = \{\alpha_{\overline{v_i}}, \delta\}$, and $F^j = \{\lambda^j, \mu\}$. Hence, it follows that $\{\phi_\sigma(H_{\overline{v_i}}) \cup \phi_\sigma(H_{v_i})\} \subseteq \mathcal{A}_i \cup \{y_0\}$ and $\phi_\sigma(F^j) \subseteq \mathcal{B}_j \cup \{u_0\}$. Thus, we conclude that $\phi_\sigma(H_{\overline{v_i}}) \cup \phi_\sigma(H_{v_i})$ and $\phi_\sigma(F^j)$ are in different components of $T_I - y_n u_1$. This proves that $\mathcal{T}_\sigma$ displays both $H_{\overline{v_i}} \big| F^j$ and $H_{v_i} \big| F^j$.

- consider the clause $\mathcal{C}_j = X \vee Y \vee Z$ for $j \in \{1 \dots m\}$. Since σ is a satisfying assignment, and by the rotational symmetry between X, Y, and Z, we may assume that $X = 1$, $Y = 0$, and $Z = 0$. Let i_X be the index such that $X = v_{i_X}$ or $X = \overline{v_{i_X}}$, let i_Y be such that $Y = v_{i_Y}$ or $Y = \overline{v_{i_Y}}$, and let i_Z be such that $Z = v_{i_Z}$ or $Z = \overline{v_{i_Z}}$. Note that i_X, i_Y, i_Z are all distinct, since we assume that no variable appears more than once in the same clause. Thus we have that $\phi_\sigma(\beta^j_X) = b^j_1$, $\phi_\sigma(\beta^j_{\overline{Y}}) = b^j_2$, $\phi_\sigma(\beta^j_{\overline{Z}}) = b^j_3$, $\phi_\sigma(\gamma^j_1) = g^j_1$, $\phi_\sigma(\gamma^j_2) = g^j_2$, $\phi_\sigma(\gamma^j_3) = g^j_3$, and $\phi_\sigma(\lambda^j) = \ell^j$. (See Fig. 6c.) Also, $\{\phi_\sigma(\alpha_X), \phi_\sigma(\alpha_{\overline{X}}), \phi_\sigma(\beta^j_{\overline{X}})\} \subseteq \mathcal{A}_{i_X}$, $\{\phi_\sigma(\alpha_Y), \phi_\sigma(\alpha_{\overline{Y}}), \phi_\sigma(\beta^j_Y)\} \subseteq \mathcal{A}_{i_Y}$, and $\{\phi_\sigma(\alpha_Z), \phi_\sigma(\alpha_{\overline{Z}}), \phi_\sigma(\beta^j_Z)\} \subseteq \mathcal{A}_{i_Z}$.
  - consider $K^j_{\overline{X}} \big| K^j_X$ and $K^j_{\overline{X}} \big| L^j_X$. Recall that $K^j_{\overline{X}} = \{\beta^j_X, \gamma^j_1\}$, $K^j_X = \{\beta^j_{\overline{X}}, \lambda^j\}$, and $L^j_X = \{\beta^j_{\overline{X}}, \gamma^j_2\}$. Also, recall that $\phi_\sigma(\beta^j_{\overline{X}}) \in \mathcal{A}_{i_X}$. Thus it follows that $\phi_\sigma(K^j_X) \cup \phi_\sigma(L^j_X)$ and $\phi_\sigma(K^j_{\overline{X}})$ are in different components of $T_I - x^j_4 x^j_6$.
  - consider $K^j_{\overline{Y}} \big| K^j_Y$ and $K^j_{\overline{Y}} \big| L^j_Y$. Recall that $K^j_{\overline{Y}} = \{\beta^j_Y, \gamma^j_2\}$, $K^j_Y = \{\beta^j_{\overline{Y}}, \lambda^j\}$, and $L^j_Y = \{\beta^j_{\overline{Y}}, \gamma^j_3\}$ where $\phi_\sigma(\beta^j_Y) \in \mathcal{A}_{i_Y}$. Thus, $\phi_\sigma(K^j_Y) \cup \phi_\sigma(L^j_Y)$ and $\phi_\sigma(K^j_{\overline{Y}})$ are in different components of $T_I - x^j_1 x^j_2$.
  - consider $K^j_{\overline{Z}} \big| K^j_Z$ and $K^j_{\overline{Z}} \big| L^j_Z$. Recall that $K^j_{\overline{Z}} = \{\beta^j_Z, \gamma^j_3\}$, $K^j_Z = \{\beta^j_{\overline{Z}}, \lambda^j\}$, and $L^j_Z = \{\beta^j_{\overline{Z}}, \gamma^j_1\}$ where $\phi_\sigma(\beta^j_Z) \in \mathcal{A}_{i_Z}$. Thus, $\phi_\sigma(K^j_Z) \cup \phi_\sigma(L^j_Z)$ and $\phi_\sigma(K^j_{\overline{Z}})$ are in different components of $T_I - x^j_2 x^j_4$.
  - consider $S^j_Y \big| K^j_X$ and $S^j_Y \big| L^j_Z$. Recall that $S^j_Y = \{\alpha_Y, \beta^j_Y\}$, $K^j_X = \{\beta^j_{\overline{X}}, \lambda^j\}$ and $L^j_Z = \{\beta^j_{\overline{Z}}, \gamma^j_1\}$. Also, $\{\phi_\sigma(\alpha_Y), \phi_\sigma(\beta^j_Y)\} \subseteq \mathcal{A}_{i_Y}$ whereas $\phi_\sigma(\beta^j_{\overline{X}}) \in \mathcal{A}_{i_X}$. Thus, since $i_X \neq i_Y$, we conclude that $\phi_\sigma(S^j_Y)$ and $\phi_\sigma(K^j_X) \cup \phi_\sigma(L^j_Z)$ are in different components of $T_I - y_{i_Y} y'_{i_Y}$.
  - consider $S^j_Z \big| K^j_Y$ and $S^j_Z \big| L^j_X$. Recall that $S^j_Z = \{\alpha_Z, \beta^j_Z\}$, $K^j_Y = \{\beta^j_{\overline{Y}}, \lambda^j\}$, and $L^j_X = \{\beta^j_{\overline{X}}, \gamma^j_2\}$. Also, $\{\phi_\sigma(\alpha_Z), \phi_\sigma(\beta^j_Z)\} \subseteq \mathcal{A}_{i_Z}$, and $\phi_\sigma(\beta^j_{\overline{X}}) \in \mathcal{A}_{i_X}$. Thus, since $i_X \neq i_Z$, we conclude that $\phi_\sigma(S^j_Z)$ and $\phi_\sigma(K^j_Y) \cup \phi_\sigma(L^j_X)$ are in different components of $T_I - y_{i_Z} y'_{i_Z}$.
  - consider $S^j_X \big| K^j_Z$ and $S^j_X \big| L^j_Y$. Recall that $S^j_X = \{\alpha_X, \beta^j_X\}$, $K^j_Z = \{\beta^j_{\overline{Z}}, \lambda^j\}$ and $L^j_Y = \{\beta^j_{\overline{Y}}, \gamma^j_3\}$ where $\phi_\sigma(\alpha_X) \in \mathcal{A}_{i_X}$. Thus, $\phi_\sigma(S^j_X)$ and $\phi_\sigma(K^j_Z)$ are in different components of $T_I - x^j_4 x^j_5$, whereas $\phi_\sigma(S^j_X)$ and $\phi_\sigma(L^j_Y)$ are in different components of $T_I - x^j_2 x^j_3$.

This proves that $\mathcal{T}_\sigma$ displays $\mathcal{Q}_I$. It remains to prove that $\mathcal{T}_\sigma$ is distinguished by $\mathcal{Q}_I$. We analyze the edges of T_I.

- consider the edge $y_i y'_i$ for $i \in \{1 \dots n\}$. Recall that $A_i = \{\alpha_{v_i}, \alpha_{\overline{v_i}}\}$ and $B = \{\delta, \mu\}$. By definition, we have $\phi_\sigma(A_i) = \{a_i, a'_i\}$ and $\phi_\sigma(B) = \{y_0, u_0\}$. Note that every connected subgraph of T_I that contains both y_0 and u_0 must also contain y_i, since it lies on the path between u_0 and y_0 in T_I. Likewise, every connected subgraph of T_I that contains a_i, a'_i also contains y'_i. This shows that the edge $y_i y'_i$ is distinguished by $A_i \big| B$ which is in $\mathcal{Q}_I$.
- consider the edge $u_j x^j_1$ for $j \in \{1 \dots m\}$. By the definition of ϕ_σ, we observe that there exists $p \in \{1, 2, 3\}$ such that $\phi_\sigma(\gamma^j_p) = g^j_2$. We recall that $B = \{\delta, \mu\}$ and $D^j_p = \{\gamma^j_p, \lambda^j\}$. Thus, $\phi_\sigma(B) = \{y_0, u_0\}$ and $\phi_\sigma(D^j_p) = \{g^j_2, \ell^j\}$. Since g^2_j is adjacent to x^j_1, and u_j lies on the path between y_0 and u_0, it follows that the edge $u_j x^j_1$ is distinguished by $D^j_p \big| B$ which is in $\mathcal{Q}_I$.
- consider $i \in \{1 \dots n\}$ and let $j_1 < j_2 < \dots < j_t$ be the elements of Δ_i. Let $W \in \{v_i, \overline{v_i}\}$ be such that $W = 1$. Then we have $\phi_\sigma(\alpha_W) = a_i$, $\phi_\sigma(\alpha_{\overline{W}}) = a'_i$, and $\phi_\sigma(\beta^j_{\overline{W}}) = c^j_i$ for all $j \in \Delta_i$. Recall that $S^j_{\overline{W}} = \{\alpha_{\overline{W}}, \beta^j_{\overline{W}}\}$

and $K^j_W \subseteq \{\beta^j_{\overline{W}}, \gamma^j_1, \gamma^j_2, \gamma^j_3, \lambda^j\}$ where $\{\phi_\sigma(\gamma^j_1), \phi_\sigma(\gamma^j_2), \phi_\sigma(\gamma^j_3), \phi_\sigma(\lambda^j)\} \subseteq \mathcal{B}_j$ for all $j \in \Delta_i$. Thus, for each $k \in \{1 \ldots t-1\}$, it follows that $\phi_\sigma(\beta^{j_k}_{\overline{W}})$ is adjacent to $z^{j_k}_i$ whereas $\phi_\sigma(\beta^{j_{k+1}}_{\overline{W}})$ is adjacent to $z^{j_{k+1}}_i$. This proves that the edge $z^{j_k}_i z^{j_{k+1}}_i$ is distinguished by $S^{j_k}_{\overline{W}} \,\big|\, K^{j_{k+1}}_W$. Similarly, recall that $S^j_W = \{\alpha_W, \beta^j_W\}$ where $\phi_\sigma(\beta^j_W) \in \mathcal{B}_j$ and $\phi_\sigma(\alpha_W)$ is adjacent to y'_i. Thus, the edge $z^{j_t}_i y'_i$ is distinguished by $S^{j_t}_W \,\big|\, S^{j_t}_{\overline{W}}$. Further, if $i \geq 2$, then we recall that $H_{v_{i-1}} = \{\alpha_{v_{i-1}}, \delta\}$ where $\phi_\sigma(\alpha_{v_{i-1}}) \in \mathcal{A}_{i-1}$ and $\phi_\sigma(\delta) = y_0$. Thus $y_{i-1}y_i$ is distinguished by $H_{v_{i-1}} \,\big|\, S^{j_t}_W$.

- consider $j \in \{1, \ldots m\}$ where $\mathcal{C}_j = X \vee Y \vee Z$. By the rotational symmetry, we may assume that $X = 1$ and $Y = Z = 0$. Thus $\phi_\sigma(\beta^j_X) = b^j_1$, $\phi_\sigma(\beta^j_{\overline{Y}}) = b^j_2$, $\phi_\sigma(\beta^j_{\overline{Z}}) = b^j_3$, $\phi_\sigma(\gamma^j_1) = g^j_1$, $\phi_\sigma(\gamma^j_2) = g^j_2$, $\phi_\sigma(\gamma^j_3) = g^j_3$, and $\phi_\sigma(\lambda^j) = \ell^j$. (Again see Fig. 6c.) Recall that $K^j_Y = \{\beta^j_{\overline{Y}}, \lambda^j\}$ and $K^j_{\overline{Y}} = \{\beta^j_Y, \gamma^j_2\}$ where $\phi_\sigma(\beta^j_Y) \notin \mathcal{B}_j$. This shows that the edge $x^j_1 x^j_2$ is distinguished by $K^j_{\overline{Y}} \,\big|\, K^j_Y$. Recall that $S^j_X = \{\alpha_X, \beta^j_X\}$, $L^j_Y = \{\beta^j_{\overline{Y}}, \gamma^j_3\}$, and $K^j_Z = \{\beta^j_{\overline{Z}}, \lambda^j\}$ where $\phi_\sigma(\alpha_X) \notin \mathcal{B}_j$. Thus, the edge $x^j_2 x^j_3$ is distiguished by $S^j_X \,\big|\, L^j_Y$ whereas the edge $x^j_4 x^j_5$ is distinguished by $S^j_X \,\big|\, K^j_Z$. Recall that $K^j_{\overline{Z}} = \{\beta^j_Z, \gamma^j_3\}$ and $L^j_Z = \{\beta^j_{\overline{Z}}, \gamma^j_1\}$ where $\phi_\sigma(\beta^j_Z) \notin \mathcal{B}_j$. Thus, the edge $x^j_2 x^j_4$ is distinguished by $K^j_{\overline{Z}} \,\big|\, L^j_Z$. Recall that $K^j_X = \{\beta^j_{\overline{X}}, \lambda^j\}$ and $K^j_{\overline{X}} = \{\beta^j_X, \gamma^j_1\}$ where $\phi_\sigma(\beta^j_{\overline{X}}) \notin \mathcal{B}_j$. Thus, the edge $x^j_4 x^j_6$ is distinguished by $K^j_{\overline{X}} \,\big|\, K^j_X$. Further, if $j < m$, recall that $F^{j+1} = \{\lambda^{j+1}, \mu\}$ where $\phi_\sigma(\lambda^{j+1}) \in \mathcal{B}_{j+1}$ and $\phi_\sigma(\mu) = u_0$. Thus $u_j u_{j+1}$ is distinguished by $K^j_X \,\big|\, F^{j+1}$.
- consider the edge $y_n u_1$ and recall that $H_{v_n} = \{\alpha_{v_n}, \delta\}$ and $F^1 = \{\lambda^1, \mu\}$. So, $\phi_\sigma(H_{v_n}) \subseteq \mathcal{A}_n \cup \{y_0\}$ and $\phi_\sigma(F^1) \subseteq \mathcal{B}_1 \cup \{u_0\}$. Thus, the edge $y_n u_1$ is distinguished by $H_{v_n} \,\big|\, F^1$.

This concludes the proof of Theorem 10. □

Finally, we have all pieces to prove Theorem 1.

8. Proof of Theorem 1

The problem is clearly in CoNP as it can be defined by the formula "$\mathcal{T}$ displays $\mathcal{Q}$, and for every X-tree $\mathcal{T}'$, if $\mathcal{T}'$ displays $\mathcal{Q}$, then $\mathcal{T}'$ is isomorphic to $\mathcal{T}$". For this, note that isomorphism of labelled trees admits a polynomial-time algorithm [2], and checking if a given X-tree displays a given quartet tree $\{a, b\} \,\big|\, \{c, d\}$ can be done easily (by testing if the path between the leaves labelled a and b is disjoint from the path between the leaves labelled c and d).

To prove CoNP-hardness, consider an instance I of ONE-IN-THREE-3SAT and a satisfying assignment σ for I. We construct the collection $\mathcal{Q}_I$ of quartet trees, as well as the ternary phylogenetic tree $\mathcal{T}_\sigma$ as described in §4. Clearly, constructing $\mathcal{Q}_I$ and $\mathcal{T}_\sigma$ takes polynomial time. By combining Theorem 8 with Theorems 9 and 10, we obtain that σ is the unique satisfying assignment of I if and only if $\mathcal{T}_\sigma$ is the only phylogenetic tree that displays $\mathcal{Q}_I$. Since, by Theorem 2, it is CoNP-hard to determine if an instance of ONE-IN-THREE-3SAT has a unique satisfying assignment, it is therefore CoNP-hard to decide, for a given phylogenetic tree $\mathcal{T}$ and a collection of quartet trees $\mathcal{Q}$, whether or not $\mathcal{Q}$ defines $\mathcal{T}$. That concludes the proof of Theorem 1. □

9. Concluding remarks

In this paper, we have shown that determining whether a given phylogenetic tree represents the unique evolution of a given collection of species is a CoNP-complete problem.

In addition, we proved that the unique minimal chordal sandwich problem is CoNP-complete. This is interesting from the perspective of applications that deal with incomplete data, where sandwich problems [17] allow one to approximate or complete the dataset, assuming *a priori* that it should posses specific properties (like being from a specific structured family of graphs). Deciding uniqueness in this context serves as a test of quality of the sandwich, namely it allows one to see whether there are alternative explanations of the dataset or not. Here, we provide complexity for the case of having a unique minimal sandwich that is a chordal graph. Following this direction, it would be interesting to consider the complexity of uniqueness of other sandwich problems, especially those with interesting applications. For instance, for interval sandwich (DNA physical mapping) or cograph sandwich (genome comparison) problems. Note that the decision problem for the former is NP-complete [18] while it is polynomial for the latter [6, 17].

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