Max Point-Tolerance Graphs

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Abstract

A graph G is a *max point-tolerance (MPT)* graph if each vertex v of G can be mapped to a *pointed-interval* (I_v, p_v) where I_v is an interval of $\mathbb{R}$ and $p_v \in I_v$ such that uv is an edge of G iff $I_u \cap I_v \supseteq \{p_u, p_v\}$. MPT graphs model relationships among DNA fragments in genome-wide association studies as well as basic transmission problems in telecommunications. We formally introduce this graph class, characterize it, study combinatorial optimization problems on it, and relate it to several well known graph classes. We characterize MPT graphs as a special case of several 2D geometric intersection graphs; namely, triangle, rectangle, L-shape, and line segment intersection graphs. We further characterize MPT as having certain linear orders on their vertex set. Our last characterization is that MPT graphs are precisely obtained by intersecting special pairs of interval graphs. We also show that, on MPT graphs, the maximum weight independent set problem can be solved in polynomial time, the coloring problem is NP-complete, and the clique cover problem has a 2-approximation. Finally, we demonstrate several connections to known graph classes; e.g., MPT graphs strictly contain interval graphs and outerplanar graphs, but are incomparable to permutation, chordal, and planar graphs.

Keywords: tolerance graphs, interval graphs, L-graphs, rectangle intersection graphs, outerplanar graphs, weighted independent set, coloring, clique cover.

1. Introduction

Interval graphs (namely, the intersection graphs of intervals on a line) are well-studied in computer science and discrete mathematics (see e.g.,[14, 17]). Many combinatorial problems which are NP-hard in general can be solved efficiently when restricted to interval graphs. For example, the maximum clique problem [17], the maximum weight independent set problem [16], and the coloring problem [21] can all be solved in linear time on interval graphs. The recognition problem is also solvable in linear time [2].

Due to their theoretical and practical significance many generalizations of interval graphs have been studied (see e.g.,[1, 8, 20, 22]). Particularly relevant to this work are *tolerance graphs*, first introduced in [22]. A graph is a *tolerance* graph (also known as a *min tolerance* graph) when every vertex v of G can be associated with an interval I_v (of the real number line: $\mathbb{R}$) and a tolerance value $t_v \in \mathbb{R}$ such that uv is an edge of G iff $|I_u \cap I_v| \geq \min\{t_u, t_v\}$. Similarly, a graph is a *max tolerance* graph when each vertex v of G can

be associated with an interval I_v and tolerance t_v such that $uv \in E(G)$ iff $|I_u \cap I_v| \geq \max\{t_u, t_v\}$. For a detailed study of tolerance graphs see [23].

In this paper we introduce the class of *max point-tolerance (MPT)* graphs. A graph G is an MPT graph when each vertex v of G can be represented by an interval I_v of $\mathbb{R}$ together with a point $p_v \in I_v$ such that two vertices u, v are adjacent iff both p_u and p_v belong to $I_u \cap I_v$; i.e., each pair of intervals can "tolerate" a non-empty intersection (without forming an edge) as long as at least one distinguished point is not contained in this intersection. We call such a collection $\{(I_v, p_v)\}_{v \in V(G)}$ of pointed intervals an MPT representation of G. Moreover, we also denote each (I_v, p_v) by a triplet (s_v, p_v, e_v) where s_v and e_v denote the start and end of I_v respectively.

MPT graphs have a number of practical applications. They can be used to detect loss of heterozygosity events in the human genome; see e.g., [24, 50]. In such applications an interval I represents the maximal boundary on a chromosome region from an individual that may carry a deletion and the point p represents a site in the considered region that shows evidence for a deletion. MPT graphs also model telecommunication networks; e.g., communication devices such that each device receives messages on a particular bandwidth (interval) and sends messages in a sub-band (point) of that bandwidth. In this situation the edges of the MPT graph correspond to devices with direct two-way communication.

Some classical optimization problems on MPT graphs correspond to practical problems. For example, when modeling genome-wide association studies, finding the chromosomal region showing the highest evidence for a massive loss of heterozygosity in a population of individuals involves solving the maximum clique problem and partitioning all evidence-of-deletion sites into the minimal number of deletions involves solving the minimum clique cover problem [3]. Similarly, in telecommunications, a minimum clique cover corresponds to partitioning the devices into a minimum collection of sets of fully-communicable devices and a maximum independent set is a largest set of non-communicable devices.

Interestingly, the maximum weight clique problem on a MPT graph was shown to be polynomially solvable due to the fact that an MPT graph can have at most $O(n^2)$ maximal cliques [3]. Additionally, the minimum weight clique cover problem was shown to be NP-complete for submodular cost functions [3, 10]. The complexity of the unweighted clique cover problem on MPT graphs remains unresolved.

Finally, closely related to MPT graphs is the class of *interval catch digraphs*. A digraph D is an *interval catch digraph* when each vertex v of D can be mapped to an interval I_v of $\mathbb{R}$ together with a point $p_v \in I_v$ such that uv is an arc of D iff $p_u \in I_v$. Notice that MPT graphs are precisely the underlying undirected graphs of the symmetric edges of interval catch digraphs. Interval catch digraphs have a vertex order characterization [36], an asteroidal-triple characterization [40], and a polynomial time recognition algorithm [41]. However, these results do not translate to MPT graphs.

Our Contributions: We provide characterizations of MPT graphs, utilize these characterizations for combinatorial optimization problems, and relate MPT graphs to well-known graph classes.

In section 2 we characterize MPT graphs as a special case of *L-graphs* (intersection graphs of L-shapes in the plane). This will imply that MPT is also a subclass of rectangle intersection graphs (also known as boxicity-2 graphs [43]) and of triangle intersection graphs. We also use this characterization to show that interval graphs and 2D ray graphs are strict subclasses of MPT graphs. We further characterize MPT graphs by certain linear vertex orders. In particular, we show that a graph $G = (V, E)$ is MPT iff the vertices of G can be linearly ordered by $<$ so that no quadruple $u, v, w, x \in V$ with $u < v < w < x$ has the edges uw and vx without the edge vw. Related to this ordering condition, we also describe MPT graphs as the intersection of two special interval graphs (see Theorem 5.5). Finally, MPT graphs are characterized as intersection graphs of certain line segments from cyclic line arrangements.

These characterizations are then used to study combinatorial optimization problems on MPT graphs.

Namely, we demonstrate that the *weighted independent set (WIS)* problem can be solved in polynomial time, the clique cover problem can be 2-approximated in polynomial time, and that the *coloring* problem is NP-complete but can be $\log(n)$-approximated in polynomial time. As part of the approximations, we show that the clique cover number $\gamma(G)$ is at most twice the independence number $\alpha(G)$ and that the chromatic number $\chi(G)$ is at most $O(\omega \log(\omega))$ where ω is the clique number[1].

Finally, we observe some structural results and compare MPT graphs to several well-known graph classes. For example, we observe that outerplanar graphs are proper subclass of MPT graphs and characterize them by a "contact" MPT representation. We additionally observe infinite families of forbidden induced subgraphs for MPT graphs which are constructed from non-interval and non-outerplanar graphs.

Related Work: While our results have been obtained independently, there are several places which overlap with some existing papers [35, 48, 49]. We will identify each of these as they are presented. Some of our results also appear in the Masters Thesis of our co-author Thomas Hixon [25].

Preliminaries: All graphs considered in this paper are simple, undirected, and loopless (unless otherwise stated). For a graph G with vertex set V and edge set E, we use the following notation. The symbols n and m denote $|V|$ and $|E|$ respectively. For a subset S of V, $G[S]$ denotes the subgraph of G induced by S and $G \setminus S$ denotes the subgraph of G induced by $V \setminus S$; i.e., $G \setminus S = G[V \setminus S]$. For a vertex $v \in V$, $N(v)$ denotes the neighborhood of v (i.e., the vertices in G which are adjacent to v).

2. Geometric representations of MPT graphs

In this section we relate MPT graphs to geometric intersection graphs. Specifically, we characterize MPT graphs as intersection graphs of axis-aligned L-shapes whose corner points form a line with negative slope (namely, *linear L-graphs* as defined below). Once we formalize this it will be easy to see that this implies that MPT graphs are a special subclass of boxicity-2 graphs and triangle intersection graphs. This equivalence is also observed in [48]. Later in this paper we use these characterizations to study combinatorial optimization problems on MPT graphs and to relate MPT graphs to classical graph classes.

An *L-shape* consists of a vertical line segment and a horizontal line segment with a *corner* that is the lowest point of the vertical segment and the left-most point of the horizontal segment. We define a *linear L-system* $\mathcal{L}$ to be a collection of L-shapes $\{L_1, \ldots, L_n\}$ in the plane such that the corner points of $L_1, \ldots, L_n$ are distinct and form a line with negative slope. We say that a graph G is a *linear L-graph* if G is the intersection graph of a linear L-system $\mathcal{L}$ and we refer to $\mathcal{L}$ as a linear L-system *of* G. We define *linear rectangle graphs* and *linear right-triangle graphs* similarly (i.e., with the lower-left corners of the shapes forming a line with negative slope; note: we always consider the lower-left corner of each triangle to be the right angle). In particular, it is easy to see that these three graph classes are the same; e.g., as in Figure 2.

Without loss of generality we assume that the corner points in all linear systems have the form $(c, -c)$ for some positive integer c. This allows us to specify each L-shape L in a linear L-system by (t_L, c_L, r_L) where: $-t_L$ is the y-coordinate of the top of L, $(c_L, -c_L)$ is the corner point of L, and r_L is the x-coordinate of the right-most point of L. Such an L-shape is given in Figure 1.

Theorem 2.1. *Max point-tolerance graphs are precisely linear L-graphs.*

Proof. Let $\{(s_1, p_1, e_1), \ldots, (s_n, p_n, e_n)\}$ be a MPT representation of a graph G. Consider the linear L-system $\mathcal{L} = \{L_1, \ldots, L_n\}$ where $t_{L_i} = -s_i$, $c_{L_i} = p_i$, and $r_{L_i} = e_i$. The theorem follows from the depiction of this construction given in Figure 3 □

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[1]The bound on $\chi(G)$ follows from [4] and one of our characterizations.
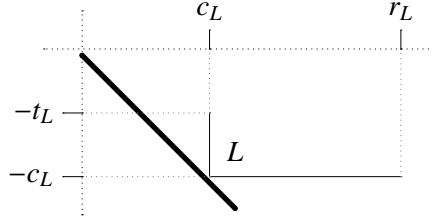
Figure 1: Anatomy of an L-shape in a linear L-system. Notice that we include a "platform" corresponding to the line $x + y = 0$ to emphasize the linearity of the system.
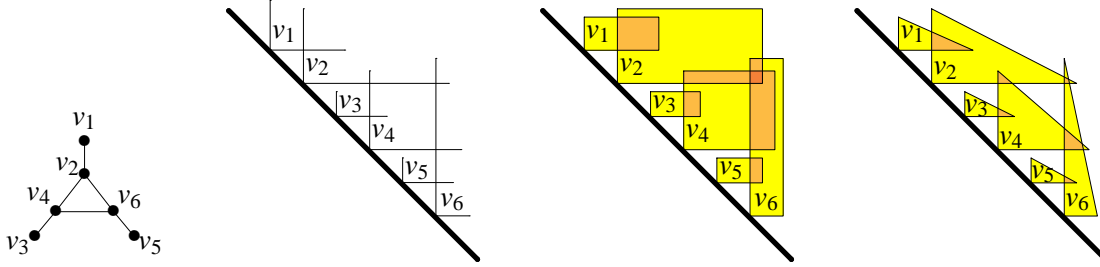

Figure 2: (from left-to-right) The net G, a linear L-system $\mathcal{L}$ of G, the linear rectangle-system corresponding to $\mathcal{L}$, and the linear right-triangle-system corresponding to $\mathcal{L}$.

3. L-systems of Interval Graphs

In this section we connect interval graphs with MPT graphs. We do this by demonstrating that every interval representation of a graph is equivalent to an *anchored* linear L-system (see Definition 3.1 and Lemma 3.2). Interval graphs have also been shown to be a subclass of MPT graphs in [48]. In fact, they show that rooted path graphs (a superclass of interval graphs) are indeed a subclass of MPT graphs, but they do not observe our characterization. We later use this characterization in our 2-approximation of clique cover and to identify an infinite family of non-MPT graphs.

Definition 3.1. A linear L-system $\mathcal{L}$ is *anchored* if there exists $A \in \mathbb{R}$ such that $t_L \leq A \leq c_L$ for every $L \in \mathcal{L}$. Note: we say that $\mathcal{L}$ is *anchored* at A and refer to A as the *anchor point* of $\mathcal{L}$.

Lemma 3.2. $G = (V, E)$ *is an interval graph iff G has an anchored linear L-system.*

PROOF. ($\Longrightarrow$) Let $\mathcal{I} = \{I_1, \ldots, I_n\}$ be an interval representation of G where s_{I_i} and e_{I_i} denote the starting and ending points of the interval I_i (respectively) for each $i \in \{1, \ldots, n\}$. Furthermore, (wlog) assume $s_{I_i} \geq 0$ and $s_{I_i} < s_{I_j}$ iff $i < j$. Consider the linear L-system $\mathcal{L} = \{L_1, \ldots, L_n\}$ such that $L_i = (0, s_{I_i}, e_{I_i})$; i.e., $\mathcal{L}$ is anchored at 0. Notice that, when two intervals I_i, I_j ($1 \leq i < j \leq n$) intersect, the corresponding L-shapes L_i, L_j will also intersect. Specifically, the horizontal segment of L_i will intersect the vertical segment of L_j (see Figure 4 (left)). Moreover, when two intervals are disjoint the corresponding L-shapes will be disjoint since their horizontal segments will not have any common x-coordinates (see Figure 4 (right)).

($\Longleftarrow$) Let $\mathcal{L} = \{L_1, \ldots, L_n\}$ be an anchored linear L-system of G. Consider the interval representation $\mathcal{I} = \{I_1, \ldots, I_n\}$ such that $I_i = (c_{L_i}, r_{L_i})$. The equivalence of $\mathcal{I}$ and $\mathcal{L}$ follows similarly to ($\Longrightarrow$). $\square$

Corollary 3.3. *Interval graphs are a strict subclass of MPT graphs.*

PROOF. This follows from Lemma 3.2 and the fact that a the graph in Figure 2 is an MPT graph but not an interval graph [34]. $\square$
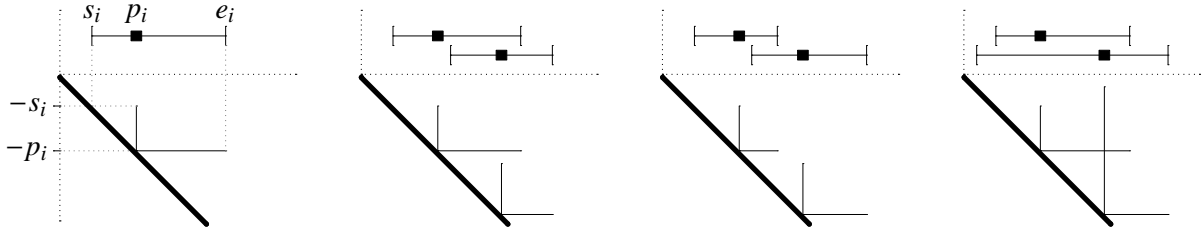
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Figure 3: Illustrating the equivalence between MPT representations and linear L-systems. From left-to-right: the L-shape corresponding to a pointed-interval, two examples of non-adjacent vertices as pointed-intervals and the corresponding linear Ls, and one example of adjacent vertices as pointed-intervals and the corresponding linear Ls.
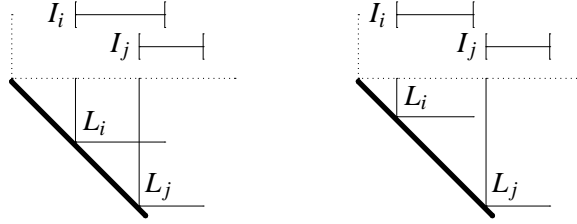

Figure 4: Illustrating the mapping between intervals and Ls for adjacent vertices (left) and non-adjacent vertices (right).

4. Combinatorial Optimization Problems

In this section we will discuss the weighted independent set (WIS) problem, clique cover (CC) problem, and the coloring problem on MPT graphs. In particular, we will show the WIS problem can be solved in $O(n^3)$ time, the CC problem can be 2-approximated in quadratic time, the coloring problem is NP-complete but can be $\log(n)$-approximated in linear time.

Throughout this section we consider an MPT graph $G = (V, E)$ together with a linear L-system $\mathcal{L} = \{L_1, \ldots, L_n\}$ of G where $i < j$ iff the corner point of L_i occurs to the left of L_j. Without loss of generality we shall assume that the corner point of L_i is $(i, -i)$ for each $i \in \{1, \ldots, n\}$; i.e., $p_i = i$ in the corresponding MPT representation and $L_i = (t_i, i, r_i)$.

4.1. Maximum Weight Independent Set

The IS problem, even for the unweighted case, is known to be NP-complete for: L-graphs, boxicity-2 graphs, and triangle intersection graphs since they contain the intersection graphs of vertical and horizontal line segments (also known as 2-DIR) and the problem is NP-complete on 2-DIR [32]. Prior to [32], the IS problem was known to be NP-complete on boxicity-2 graphs [15, 27]. However, for interval graphs, the WIS problem is known to be solvable in linear time from a superclass (e.g., *chordal* graphs [16]) of interval graphs. A graph is *chordal* when it has no induced k-cycle for all $k \geq 4$.

Notice that an independent set in an MPT graph corresponds to a collection of disjoint L-shapes in a linear L-system. We use this equivalence to solve the WIS problem on a vertex-weighted MPT graph in polynomial time algorithm via dynamic programming. Our approach is an independent rediscovery of a known algorithm to solve WIS on generalizations of interval graphs [35]. However we believe our presentation is much clearer for the context of MPT graphs. Also, there has been a recent $O(n^2)$ algorithm for this problem [49], but here we believe that the simplicity of our approach helps in understanding the structure of independent sets in MPT graphs and so we have included it.

We now discuss the key idea. Let $\mathcal{J}$ be a sub-collection of disjoint L-shapes of $\mathcal{L}$. We say that an L-shape L_i is *dominant* in $\mathcal{J}$ if it contains the right-most point among the L-shapes in $\mathcal{J}$; i.e., $L_i \in \mathcal{J}$ and

$r_i = \max_{L_j \in \mathcal{J}} r_j$. Consider a dominant L_i and some $L_j \in \mathcal{J}$ such that $j > i$. Notice that L_j cannot contain any points to the right of the line $x = r_j$ (since L_i is dominant). Moreover, L_j must occur strictly below the line $y = -i$ (since L_j's corner point is below L_i's corner point). Similarly, for $L_{j'} \in \mathcal{J}$ with $j' < i$, $L_{j'}$ again cannot contain any points to the right of the line $x = r_j$. Furthermore, $L_{j'}$ is contained strictly above the line $y = -i$. Thus, for an L-shape L_i, if L_i is dominant in a sub-collection $\mathcal{J}$ of disjoint L-shapes of $\mathcal{L}$, then the L-shapes which belong to $\mathcal{J}$ and precede L_i can be chosen independently of the L-shapes which belong to $\mathcal{J}$ and follow L_i.

The following notation is depicted in Figure 5. For $a, b \in \{1, \ldots, n\}$ and $a \le b$, let $\mathcal{L}_{0,n+1} = \mathcal{L}$ and $\mathcal{L}_{a,b} = \mathcal{L}_{0,b} \cap \mathcal{L}_{a,n+1}$ where:

- $\mathcal{L}_{0,b} = \{L_i : 1 \le i \le b - 1, r_i < r_b, \text{ and } L_i \cap L_b = \emptyset\}$; and
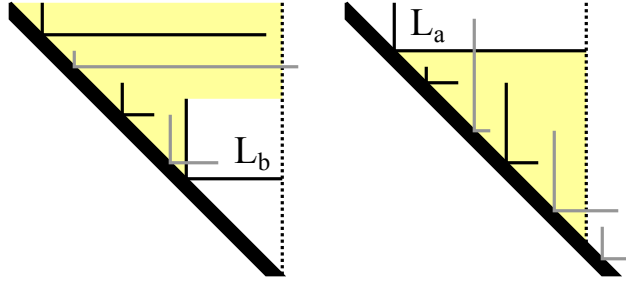- $\mathcal{L}_{a,n+1} = \{L_i : a + 1 \le i \le n, r_i < r_a, \text{ and } L_i \cap L_a = \emptyset\}$.

Figure 5: The L-shapes strictly contained in the shaded regions illustrate $\mathcal{L}_{0,b}$ (left) and $\mathcal{L}_{a,n+1}$ (right).

Let $opt[a, b]$ denote the maximum total weight of a collection of mutually disjoint L-shapes in $\mathcal{L}_{a,b}$. Notice that $opt[0, n + 1]$ is the maximum weight of an independent set in G. Furthermore, by the above discussion, we have the following recurrence for $opt[a, b]$:

$$opt[a, b] = \max_{L_i \in \mathcal{L}_{a,b}} (opt[a, i] + w(L_i) + opt[i, b])$$

It is easy to see that the collection of sets $\{\mathcal{L}_{a,b} : a, b \in \{0, \ldots, n + 1\}, a \le b\}$ can be computed in $O(n^3)$ time (since each of $\mathcal{L}_{a,b}$ can be computed in $O(n)$ time). Moreover, the size of the table opt is $O(n^2)$, and the time to compute each entry is $O(n)$. Thus, we have the following theorem.

Theorem 4.1. *For a vertex weighted MPT graph with a given linear L-system, a maximum weight independent set can be computed in $O(n^3)$ time.*

4.2. Clique Cover

The CC problem is known to be NP-complete on boxicity-2 graphs (from unit square intersection graphs [15]), and L-graphs (from circle graphs [29]). However it is solvable in polynomial time on interval graphs and outerplanar graphs.

In this subsection we describe a polynomial time 2-approximation algorithm for the CC problem on MPT graphs. Our approach uses ideas similar to the algorithm for hitting set in [7]. From our algorithm we will see that the clique cover number $\gamma(G)$ is at most twice the independence number $\alpha(G)$ for any MPT graph G. Recently it has been observed that a hitting set for a linear rectangle-system can be 2-approximated in polynomial time [49]. Such a hitting set also provides a corresponding clique cover of the same size and their proof implies the $2\alpha(G)$ bound. However, in this case our approach is faster and simpler.

Our algorithm begins with the linear L-system $\mathcal{L} = \{L_1, ..., L_n\}$. Recall that $\mathcal{L}$ is ordered according to the corner points of the L-shapes. From $\mathcal{L}$ we greedily select an independent set I. We then build a partial clique cover of G with one clique for each element of I. Finally, we consider the graph H which remains after removing these cliques and observe that it is an interval graph. Since H is an interval graph we can efficiently compute an optimal clique cover for it. This completes the overview of our algorithm. Notice that, since H will be an interval graph (i.e., a perfect graph), $\gamma(H) = \alpha(H)$. Thus, the size of the clique cover that we produce is $|I| + \alpha(H) \leq 2\alpha(G)$. We now describe our algorithm in detail.

First we construct the greedy independent set as follows. Let $I_1 = \{L_1\}$, and let $I_i = I_{i-1} \cup \{L_j\}$ such that L_j does not intersect any L-shape in I_{i-1} and j the smallest index satisfying this property. Let $I = \{L_{i_1}, ..., L_{i_k}\}$ be the maximal independent set constructed in this way such that $i_j < i_{j'}$ whenever $j < j'$. Since I is an independent set in G, we can see that k is at most the clique cover number of G. We will construct a partial clique cover using I and show that the remaining graph H will be an interval graph.

To this end, consider the following disjoint sets of vertices. For each $j \in \{1, \ldots, k-1\}$, let $C_j = \{v_\ell : i_j \leq \ell < i_{j+1}, \text{ and } r_\ell \geq i_{j+1}\}$. First we claim that each such C_j is a clique, and then we claim that removing all such C_js from G results in an interval graph H.

Claim 1: C_j is a clique.

Proof. Consider two vertices in C_j. Their corner points occur between the corners of L_{i_j} and $L_{i_{j+1}}$, their top points occur above the corner of L_{i_j} (otherwise one of them would be chosen into I instead of $L_{i_{j+1}}$), and their right points occur to the right of the corner of $L_{i_{j+1}}$. Thus, they must intersect; i.e., C_j is a clique. ☐

Claim 2: $H = G \setminus (\bigcup_{j=1}^{k} C_j)$ is an interval graph.

Proof. Consider v_p in H where $i_j \leq p < i_{j+1}$ and $1 \leq j < k$. First, due to our construction of I, either $v_p = v_{i_j}$ or v_p is a neighbor of some $v_{i_{j'}}$ where $i_{j'} \leq i_j$; i.e., the vertical segment of every such v_p intersects the line $y = i_j$. Second, we know that the right-most point of L_p is to the left of $L_{i_{j+1}}$ (since $v_p \notin C_j$). This implies that every neighbor v_q of v_p in H has $i_j \leq q < i_{j+1}$. Thus, H induced on its vertices between v_{i_j} and $v_{i_{j+1}}$ is an interval graph (since it has an anchored linear L-system anchored at i_j) and is a disjoint union of connected components of H.

The same argument applies to vertices v_p with $i_k \leq p$. This show that H is the disjoint union of interval graphs; i.e., H itself is an interval graph. ☐

Notice that the greedy independent set as well as the cliques C_j are easily generated in linear time. Moreover, the CC problem on interval graphs can be solved in linear time [26]. This leads to the main theorem of this subsection.

Theorem 4.2. *For an MPT graph G the clique cover number is at most twice the independence number. Also, when a linear L-system is given as input, the clique cover G can be 2-approximated in $O(n + m)$ time.*

4.3. Coloring

The coloring problem is known to be NP-complete on L-graphs (since circle graphs, also known as interval overlap graphs, are contained in L-graphs [1] and coloring circle graphs is NP-complete [19]), on boxicity-2 graphs [27], and on triangle intersection graphs (since they include planar graphs [9] and coloring is NP-complete on planar graphs [18]). On the other hand, the coloring problem can be solved in linear time on interval graphs [21] and outerplanar graphs [42].

In this section we will demonstrate that it is NP-complete to determine the chromatic number for MPT graphs, but it can be $\log(n)$-approximated in polynomial time. We will use $\chi(G)$ to denote the chromatic number of G.

Prior to proving the hardness result we observe that $\chi(G)$ can be $\log(n)$-approximated using known techniques. For any boxicity-2 graph G, the relationship between the $\chi(G)$ and $\omega(G)$ (the clique number) has been well-studied. The best results regarding this relationship are given in [4]. The relevant result for MPT graphs is as follows. For a boxicity-2 graph G with a rectangle system such that no rectangle contains another, $\chi(G)$ is $O(\omega(G)\log(\omega(G)))$ and this $\log(n)$-approximation of $\chi(G)$ can be computed in polynomial time. It is easy to see from our characterization of MPT graphs as linear boxicity-2 graphs, that this result applies directly to MPT graphs. Thus, the chromatic number of MPT graphs can be $\log(n)$-approximated in polynomial time.

We now turn to the hardness of coloring for MPT graphs. To do this we transform the hardness of coloring of circular-arc graphs to this class. *Circular-arc* graphs are the intersection graphs of arcs of a circle. Determining a minimum coloring of a circular-arc graph is known to be NP-hard [19]; i.e., it is NP-complete to determine whether a circular arc graph is k colorable when k is part of the input.

Theorem 4.3. *It is NP-complete to determine the chromatic number for MPT graphs.*

Proof. Consider a circular-arc graph $G = (V, E)$. We use n and m to denote $|V|$ and $|E|$ respectively. Now, for any $k > 2$, we will construct an MPT graph $G' = (V', E')$ such that: $|V'| = O(n)$, $|E'| = O(n^2)$, and $\chi(G) \le k$ iff $\chi(G') \le k$. Moreover, G' is easily constructed in $O(n^2)$ time. An example of this construction is depicted in Figure 6. The basic idea is that we "cut" the circular-arc representation at an arbitrary point p. This point corresponds to a clique and we split every vertex crossing this point into two vertices so that the result is an interval graph. This interval graph has an anchored linear L-system to which we add a clique consisting of k vertices. This clique will ensure that in any coloring of this constructed graph, the two copies of every split vertex have the same color. We now present the formal proof.

Consider an arbitrary circular-arc representation $\mathcal{A}$ of G (such a representation can be constructed in $O(n + m)$ time [37]). Let p be a fixed point on the circle of $\mathcal{A}$ and let $\mathcal{A}_p = \{A_1, \ldots, A_\ell\}$ be the arcs of $\mathcal{A}$ that include p. The vertices $\{v_1, \ldots, v_\ell\}$ corresponding to $\mathcal{A}_p$ form a clique in G (since the arcs all share the point p). Hence, if no arcs pass through the point p, then G is an interval graph; i.e., G is an MPT graph and so we can let $G' = G$ and we are done. Similarly, if $\ell > k$, then $\chi(G) > k$ and we are done; i.e., we simply let G' be a clique on ℓ vertices. Thus we may assume $1 \le \ell \le k$.

We now form an interval graph H from G by "cutting" the circular-arc representation $\mathcal{A}$ at the point p. Formally, for some small enough $\epsilon > 0$ and each $i \in \{1, \ldots, \ell\}$, we replace the arc $A_i = (s_i, e_i)$ with two arcs $A_i^1 = (s_i, p - \epsilon)$, and $A_i^2 = (p + \epsilon, e_i)$ and consider H as the resulting intersection graph. In particular, each vertex v_i is replaced by two vertices v_i^1 and v_i^2 corresponding to the arcs A_i^1 and A_i^2 respectively. Notice that $|V(H)| = n + \ell$ and $|E(H)| = m + \binom{\ell}{2}$. Since there are no arcs passing through the point p in this circular-arc representation of H, the graph H is an interval graph. Thus, by Lemma 3.2, H has an anchored linear L-system.

Finally, we add a clique of size k to H so that the result is an MPT graph G' and in any k-coloring of G', the vertices v_i^1 and v_i^2 must be assigned the same color. To this end, we define $G' = (V', E')$ as follows:

$V' = V(H) \cup \{u_1, \ldots, u_k\}$,

$E' = E(H) \cup \{u_t v_i^j : j \in \{1, 2\}, i \in \{1, \ldots, \ell\}, t \in \{i + 1, \ldots, k\}\} \cup \{u_i u_j : i, j \in \{1, \ldots, k\}, i \ne j\}$.

We show that G' has a k-coloring iff $\chi(G) \le k$.

$\implies$ Notice that the vertices v_1^1 and v_1^2 are adjacent to the same clique of size $k - 1$ in G'. Thus, in any k-coloring of G', v_1^1 and v_1^2 must be assigned the same color. Inductively, it is easy to see that v_i^1 and v_i^2 must also receive the same color in any k-coloring of G'. Specifically, u_i, v_i^1, and v_i^2 will receive the same color for every $i \in \{1, \ldots, \ell\}$. Thus, any k-coloring of G' provides a k-coloring of G.

$\Longleftarrow$ We can extend any k-coloring $f : V(G) \to \{1, \ldots, k\}$ of G to a k-coloring $f' : V(G') \to \{1, \ldots, k\}$ of G' as follows. For every $v \in V(G) \setminus \{v_1, \ldots, v_\ell\}$, set $f'(v) = f(v)$. For each $i \in \{1, \ldots, \ell\}$, set $f'(u_i) = f'(v_i^1) = f'(v_i^2) = f(v_i)$, and then choose $f'(u_{\ell+1}), \ldots, f'(u_k)$ so that $\{f'(u_{\ell+1}), \ldots, f'(u_k)\} = \{1, \ldots, k\} \setminus \{f(v_1), \ldots, f(v_\ell)\}$. It is easy to see that f' is a k-coloring of G'. This completes the proof of the claim.

All that remains is to show that G' has the appropriate size and that it is an MPT graph. Notice that $|V(G')| = n + \ell + k \leq 3n$ and $|E(G')| = m + \binom{\ell}{2} + \binom{k}{2} + (k - \ell) * 2\ell + \sum_{t=1}^{\ell-1} 2t \leq 2n^2$. Thus, G' has the appropriate size. Furthermore, we can construct an MPT representation of G' by starting from an anchored linear L-system of H and adding L-shapes for the new clique "above" this anchored linear L-system (see Figure 6). Thus, G' is an MPT graph.

From the above construction we can see that determining the chromatic number for MPT graphs is NP-hard, since it is NP-hard to determine the chromatic number for circular-arc graphs. $\qquad\square$
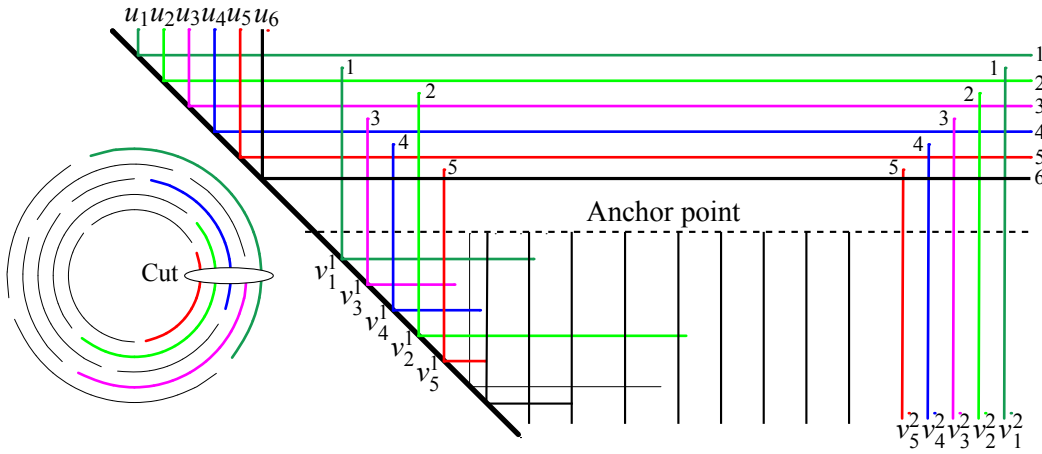

Figure 6: Sample construction from the proof of Theorem 4.3 where the "cut" contains 5 vertices and $k = 6$.

This leaves open the k-coloring problem for fixed $k \geq 3$. In particular, note that in the above construction it was necessary that the number of colors k was part of the input, since for fixed k, the k-coloring problem is solvable in polynomial time on circular-arc graphs [19].

5. Other Characterizations

In this section we characterize MPT graphs by linear vertex orders, the intersection of interval graphs, and as a restricted class of segment graphs.

5.1. Vertex Ordering

Several well known graph classes have been characterized by special linear orders on their vertices; e.g., interval graphs (see Definition 5.1 and Theorem 5.2), unit interval graphs [46], chordal graphs [11], and co-comparability graphs [33]. In this section we characterize MPT graphs as graphs with MPT-orders (see Definition 5.3 and Theorem 5.4). This ordering characterization has been independently observed [48]. We then use this ordering to show that a graph is an MPT graph iff it is the intersection of two "special" interval graphs (see Theorem 5.5).

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Definition 5.1. An *I-order* of a graph G with vertices $v_1, \ldots, v_n$ is an ordering $v_1 < v_2 < \cdots < v_n$ such that: for every $u < v < w$, if $uw \in E(G)$, then $uv \in E(G)$.

Theorem 5.2. *[39, 44, 45] G is an interval graph iff G has an I-order. Moreover for any interval representation $\mathcal{I}$ of a graph G, ordering the vertices of G by the left end-points of their intervals results in an I-order of G.*

Definition 5.3. An *MPT-order* of a graph G with vertices $v_1, \ldots, v_n$ is an ordering $v_1 < v_2 < \cdots < v_n$ such that: for every $u < v < w < x$, if $uw, vx \in E(G)$, then $vw \in E(G)$.

Notice that MPT-order is a generalization of I-order. In particular, let σ be an I-order of a graph G. Now suppose we have $u, v, w, x \in V(G)$ such that $u <_\sigma v <_\sigma w <_\sigma x$ and $uw, vx \in E(G)$. Since σ is an I-order with $v <_\sigma w <_\sigma x$ and $vx \in E(G)$, the edge vw is forced. Thus, σ is an MPT-order; i.e., every I-order is also an MPT-order. We now prove that MPT graphs are characterized as the graphs with MPT-orders.

Theorem 5.4. *$G = (V, E)$ is an MPT graph iff G has an MPT-order (i.e., the vertices of G can be ordered by $<$ so that for every $u, v, w, x \in V$, if $u < v < w < x$ and $uw, vx \in E$, then $vw \in E$).*

PROOF. ($\Longrightarrow$) Let $\{(s_v, p_v, e_v) : v \in V\}$ be an MPT representation of G. Order the vertices of G such that vertex v comes before vertex u if $p_v \leq p_u$. Now, consider any four distinct vertices u, v, w, x where $u < v < w < x$ and $uw, vx \in E$. Then, it is easy to realize that, due to the considered ordering, it holds that $s_w \leq p_u \leq p_v$ and $e_v \geq p_x \geq p_w$, which implies $vw \in E$.

($\Longleftarrow$) Let $G = (V, E)$ be a graph with ordered vertex set $V = \{v_1, \ldots, v_n\}$ such that for any $i, j, k, \ell \in \{1, \ldots, n\}$, if $i < j < k < \ell$ and $v_i v_k, v_j v_\ell \in E$ then $v_k v_j \in E$ (i.e., $v_1 < \cdots < v_n$ is an MPT-order). We now construct an MPT representation of G based on this ordering. For each $i \in \{1, \ldots, n\}$, let:

- $s_i = \min\{i, j\}$ where j is the **smallest** index such that $v_j v_i$ is an edge in G.
- $p_i = i$
- $e_i = \max\{i, j\}$ where j is the **largest** index such that $v_i v_j$ is an edge in G.

Clearly $\mathcal{I} = \{(s_i, p_i, e_i) : i \in \{1, \ldots, n\}\}$ is an MPT representation in which every edge of G is captured. Now we need to demonstrate that this representation does not include any edges which are not edges of G. Suppose that for some $j, k \in \{1, \ldots, n\}$, $j < k$, $v_j v_k \notin E$ but $s_k \leq p_j$ and $e_j \geq p_k$. Since $s_k \leq p_j$ there must be v_i with $i < j$ such that $v_i v_k \in E$. Similarly, there must be v_ℓ with $\ell > k$ such that $v_j v_\ell \in E$. However, we now have $i < j < k < \ell$ with $v_i v_k, v_j v_\ell \in E$ but $v_j v_k \notin E$; i.e., a contradiction to the vertex order. Thus $\mathcal{I}$ is an MPT representation of G. $\qquad \square$

Notice that, since every I-order is an MPT-order and every graph with an MPT-order is an MPT graph, we have an alternate proof of Corollary 3.3; i.e., that every interval graph is an MPT graph. Also, since the order of vertices in an MPT-order corresponds to the order of the points in an MPT representation, they also correspond to the order of the corner points in a linear L-system of an MPT graph.

We conclude this section by further characterizing MPT graphs as the intersection of two related interval graphs.

Theorem 5.5. *$G = (V, E)$ is an MPT graph with MPT-order $\sigma = (v_1 < \ldots < v_n)$ iff there are interval graphs $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ such that $E = E_1 \cap E_2$, σ is an I-order of H_1, and the reverse of σ (i.e., $v_n < \cdots < v_1$) is an I-order of H_2.*

PROOF. ($\Longrightarrow$) Let $\sigma = v_1 < \cdots < v_n$ be an MPT-order of G, and let $\mathcal{L}$ be the linear L-system of G constructed in the proof of Theorem 5.4 using this order; i.e., $c_{v_i} < c_{v_j}$ iff $i < j$. Construct an anchored linear L-system $\mathcal{L}_1$ by extending the horizontal segment of every L-shape in $\mathcal{L}$ to the right beyond the corner of the right-most L-shape in $\mathcal{L}$. Similarly, construct an anchored linear L-system $\mathcal{L}_2$ by extending the vertical segment of every L in $\mathcal{L}$ so that it reaches above the corner of the left-most L-shape in $\mathcal{L}$. By Lemma 3.2, each of $\mathcal{L}_1$ and $\mathcal{L}_2$ corresponds to an interval representation. Let $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ be the interval graphs specified by $\mathcal{L}_1$ and $\mathcal{L}_2$. Notice that, by Theorem 5.2, σ is an I-order of H_1 and the reverse of σ is an I-order of H_2. Thus, we just need to ensure that $E_1 \cap E_2 = E$. Clearly $E \subseteq E_1 \cap E_2$ from our construction of $\mathcal{L}_1$ and $\mathcal{L}_2$. Moreover there can be no edge in both E_1 and E_2 which is not in E simply due to how these "extra" edges come into existence (see Figure 7).
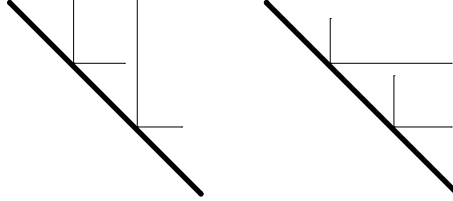

Figure 7: (left) L_u, L_v such that $uv \in E_1 \setminus E$. (right) L_u, L_v such that $uv \in E_2 \setminus E$.

($\Longleftarrow$) Let $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ such that $V = \{v_1, \ldots v_n\}$, $\sigma = (v_1 < \cdots < v_n)$ is an I-order of H_1, and the reverse of σ is an I-order of H_2. We now claim that σ is an MPT-order of $G = (V, E_1 \cap E_2)$. Consider $1 \leq i < j < k < \ell \leq n$ where $v_i v_k, v_j v_\ell \in E_1 \cap E_2$. Notice that $v_j v_k \in E_1$ since σ is an I-order of H_1 and $v_j v_\ell \in E_1$. Similarly, $v_j v_k \in E_2$ since the reverse of σ is an I-order of H_2 and $v_i v_\ell \in E_2$. Thus, $v_j v_k \in E_1 \cap E_2$ as needed. $\qquad\square$

5.2. Cyclic Segment Graphs

In this section we characterize MPT graphs as intersection graphs of line segments from a cyclic line arrangement. A *line arrangement* is simply a collection of lines in the plane (see [13] for more on line arrangements). A line arrangement $\mathcal{A}$ is *cyclic* when there is a convex function f (e.g., a parabola) such that every line in $\mathcal{A}$ is tangent to f. We define *cyclic segment graphs* as the intersection graphs of line segments where the underlying line arrangement is cyclic with respect to some function f and each segment contains a point on f. In the following theorem we prove that cyclic segment graphs are precisely MPT graphs. This follows easily from our characterization of MPT graphs via MPT-orders (see Theorem 5.4).

Theorem 5.6. *MPT graphs are precisely cyclic segment graphs.*

PROOF. Let $\sigma = (v_1 < \ldots < v_n)$ be an MPT-order of an MPT graph G. We will construct a cyclic segment representation of G by mapping each vertex to a segment of a line tangential to the parabola $y = x^2$. First, we assign each v_i the tangent line ℓ_i of the parabola for the point (i, i^2). Now, to choose the segment of ℓ_i for the vertex v_i we consider the left-most and right-most vertices, say $v_{i_{\min}}$ and $v_{i_{\max}}$, from $N(v_i) \cup \{v_i\}$. In particular, we let the segment S_i for v_i be defined as the segment of ℓ_i starting from $\ell_{i_{\min}}$ and ending on $\ell_{i_{\max}}$. Note, if $i = i_{\min}$ ($i = i_{\max}$) then we simply use the point (i, i^2) as the starting (ending) point of the segment ℓ_i. Clearly each S_i passes through the point (i, i^2). Thus, we have constructed a valid cyclic segment representation. Consider an edge $v_i v_j$ of G with $i < j$. From our construction, S_i passes through the line ℓ_j in order to reach $S_{i_{\max}}$. Similarly, S_j passes through the line ℓ_i in order to reach $S_{j_{\min}}$. Thus, S_i and S_j intersect. Now, suppose that S_i and S_j intersect ($i < j$), but $v_i v_j$ is not an edge of G. In order for these segments to intersect, each must need to "reach over" the other. In particular, this means that there

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is v_p and v_q such that $p < i < j < q$, and $v_i v_q$ and $v_j v_p$ are edges in G; i.e., this violates the MPT-order. Therefore, every MPT graph has a cyclic segment representation.

To construct an MPT-order from a cyclic segment representation one simply uses the order of the tangent points and the proof follows similarly to the above. □

6. Non-MPT graphs and More Subclasses of MPT graphs

In this section we observe two additional strict subclasses of MPT graphs (namely, outerplanar graphs and 2D ray graphs). We further observe infinite families of graphs which are not MPT graphs.

6.1. Outerplanar graphs

In this section we consider outerplanar graphs as a restricted form of MPT graphs. In particular, we consider linear L-contact-systems and demonstrate that the graphs of these contact systems are precisely outerplanar graphs. It has been independently observed that outerplanar graphs are a subclass of MPT graphs [48]. Their proof is completely different from ours and does not provide the characterization we have observed.

A graph is *outerplanar* if it has a crossing-free embedding in the plane such that all vertices are on the same face. Moreover, an outerplanar graph is said to be maximal when it is not a proper subgraph of any outerplanar graph with the same number of vertices. We will demonstrate that outerplanar graphs are precisely the *linear contact L-graphs* (see Definition 6.1 and Theorem 6.3). For more information on contact L-graphs see [6, 5, 30].

Definition 6.1. A graph G is a *linear L contact graph* when it has a linear L-system $\mathcal{L}$ such that no two L-shapes "cross-over" each other; i.e., for L-shapes $L_u = (t_u, c_u, r_u), L_v = (t_v, c_v, r_v)$, if $L_u \cap L_v \neq \emptyset$ and $c_u < c_v$, then either $c_u = t_v$ or $r_u = c_v$. Moreover, we say such an L contact system is *equilateral* when, for each L-shape, the vertical and horizontal segments have the same length.

We will use the following characterization of maximal outerplanar graphs related to 2-trees (which follows easily from [38]). A *2-tree* is a graph that can be constructed by starting from an edge and iteratively adding vertices with exactly two adjacent neighbors. Semi-squares will also play a role throughout this section. A *semi-square* is a right-triangle whose vertical and horizontal sides are the same length (i.e., the lower-left "half" of a square). Moreover, it is known that max tolerance graphs are precisely semi-square intersection graphs [28].

Theorem 6.2. *Let G be a maximal outerplanar graph. For any edge $v_1 v_2$ of the outerface of G, the vertices of G can be ordered $v_1, \ldots, v_n$ such that v_i ($2 < i \leq n$) has exactly two neighbors, u and v, in $G_{i-1} = G[\{v_1, \ldots, v_{i-1}\}]$ and uv is and edge of G. We refer to such an order as an* outerplanar-order.

Theorem 6.3. *Every maximal outerplanar graph G is a linear equilateral-L contact graph.*

PROOF. Consider an outerplanar order $v_1, v_2, \ldots, v_n$ of G. We iteratively build the linear equilateral-L contact system as follows. Let $L_{v_1} = (-1, 0, 1)$ and $L_{v_2} = (0, 1, 2)$ be the L-shapes for v_1 and v_2 respectively. Clearly L_{v_1} and L_{v_2} contact each other at the point $(1, 0)$, both terminate at this point, are equilateral, and their corner points lie on the line $y = -x$. Moreover, the semi-square defined by the points $(0, 0), (1, 0), (1, -1)$ is: empty, its diagonal is a segment of the line $y = -x$, and the point $(1, 0)$ is the point of contact between L_{v_1} and L_{v_2}. Now assume that we have a linear equilateral-L contact system $\mathcal{L}_{i-1}$ for $v_1, \ldots, v_{i-1}$ such that every

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edge uv on the outerface of G_{i-1} corresponds to an empty semi-square as in the base case. We extend this representation to a representation of G_i as follows. From the outerplanar ordering, the vertex v_i is adjacent in G_{i-1} to precisely some pair u, v such that uv is an edge on the outerface of G_{i-1}. Thus, we have an empty semi-square $(x, -x), (x+d, -x), (x+d, -x-d)$ where the point $(x+d, -x)$ corresponds is the contact point of L_u and L_v. Consider $\mathcal{L}_i = \mathcal{L}_{i-1} \cup \{L_{v_i}\}$ where $L_{v_i} = (x, x+d/2, x+d)$. Without loss of generality L_{v_i} contacts L_u at the point $(x+d/2, -x)$ and L_v at the point $(x+d, -x-d/2)$. Moreover, these new contact points form the appropriate empty and disjoint semi-squares as needed. Finally, since the semi-square corresponding to uv was empty before inserting L_{v_i}, the L-shape L_{v_i} does not intersect any other L-shapes. Thus, $\mathcal{L}_i$ is a linear equilateral L contact system as needed. □

Similarly to how linear L-graphs are equivalent to linear boxicity-2 graphs and linear right-triangle graphs, we have the following corollary regarding linear contact graphs.

Corollary 6.4. *The following graph classes are equivalent: outerplanar, linear L contact, linear equilateral-L contact, linear semi-square contact, linear square contact.*

PROOF. Since maximal outerplanar graphs are linear equilateral L contact graphs (by Theorem 6.3), all outerplanar graphs are linear equilateral L contact graphs. In particular, one may simply adjust an equilateral L a by small amount to remove any individual contact with another L such that no other contact is altered.

Moreover, given any of the contact representations listed, one can easily construct an outerplanar drawing of the graph. In particular, each vertex v is located at its corresponding corner point on the line $y = -x$ and the edges uv are drawn by tracing L_u and L_v to the corresponding contact point. Clearly all vertices lie on the outside of such a drawing and this can be done so that no edges intersect. □

6.2. 2D Ray Graphs

A graph is a *2D ray* graph when it is an intersection graph of rays in the plane where the rays have at most two directions and parallel rays do not intersect (i.e., this is a bipartite graph class). For more information on this graph class see [31, 47]. We observe that 2D ray graphs form a strict subclass of bipartite MPT graphs and that they play an interesting role in the structure of neighborhoods of vertices of MPT graphs.

Proposition 6.5. *2D ray graphs are a strict subclass of bipartite MPT graphs.*

PROOF. Notice that, without loss of generality, we may assume that any 2D ray representation of a graph G only uses $\downarrow$ and $\leftarrow$ as the two directions its rays follow. With this in mind it is easy to see that this representation is in fact a linear L-system. In particular, we can imagine a line with negative slope that intersects all the rays and occurs "below" and to the "left" of any point of intersection between two rays. Thus, by stopping all rays on this line we have a linear L-system of G. Additionally, this inclusion is strict since a 6-cycle is not a 2D ray graph [31], but it is easily constructed as a linear L-graph. □

Recall that MPT graphs have been shown to have $O(n^2)$ maximal cliques [3]. Moreover, every complete bipartite graph is a 2D ray graph. Thus, $O(n^2)$ is a tight bound (up to a multiplicative constant) on the number of maximal cliques in MPT graphs.

We now consider the neighborhood of a single vertex and observe the following connection to 2D ray graphs and interval graphs.

Proposition 6.6. *If G is an MPT graph and v is a vertex of G, then the neighborhood of v can be partitioned into V_L and V_R such that:*

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- *G[V_L] and G[V_R] are interval graphs; and*
- *the bipartite graph induced by the edges connecting vertices from V_L to V_R is a 2D ray graph.*

Proof. Let $\mathcal{L}$ be a linear L-system of G. We set V_L as the neighbors of v whose corner points occur prior to v's corner point and define V_R to be the remaining neighbors of v. Notice that the corner points of vertices in V_R will occur after v in $\mathcal{L}$. The L-shapes of V_L clearly form an anchored linear L-system and as such correspond to an interval representation. Thus, by Lemma 3.2, $G[V_L]$ and (similarly) $G[V_R]$ are interval graphs. Moreover, by considering $\mathcal{L}$, one can easily see that the bipartite graph induced by the edges connecting vertices from V_L to V_R is a 2D ray graph. In particular, the horizontal segments of V_L correspond to ← rays and the vertical segments of V_R correspond to ↓ rays. □

6.3. Non-MPT graphs

In this section we observe some structural properties of MPT graphs that allow us to identify infinite families of non-MPT graphs. These non-MPT graphs will allow us to compare MPT graphs to planar and permutation graphs.

Proposition 6.7. *If G is an MPT graph with non-adjacent vertices u and v, then $G[N(u) \cap N(v)]$ is an interval graph.*

Proof. Consider the relative position of u and v in the linear L-system of G. Without loss of generality they must occur as in Figure 8. In each possibility the corner point of any common neighbour of u and v occurs in the shaded region; i.e., in every linear L-system of G, the L-shapes corresponding to $N(u) \cap N(v)$ form an anchored linear L-system (anchored to v's L-shape). Therefore, by Lemma 3.2, $G[N(u) \cap N(v)]$ is an interval graph.
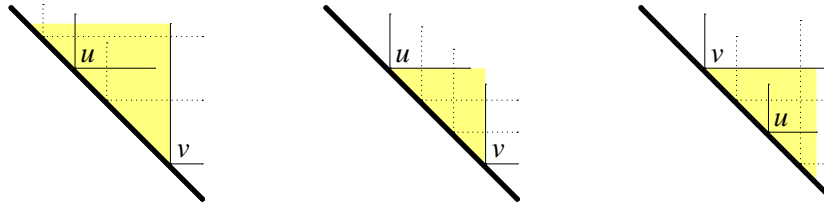

Figure 8: The three ways to represent two non-adjacent vertices u, v in a linear L-model. Notice that any common neighbor of u and v must have its corner point in the shaded region.

□

Notice that Proposition 6.7 is tight. In particular, if one adds a independent set I to an interval graph G such that every element of I is adjacent to every vertex in G, then the resulting graph G' is an MPT graph. Specifically, by Lemma 3.2, G has an anchored linear L-system $\mathcal{L}$. We form a linear L-system for G' as follows. Starting from $\mathcal{L}$, one simply adds a set of $|I|$ horizontal segments such that the first one occurs "just below" the anchor point of $\mathcal{L}$, and each subsequent segment occurs "just below" the previous segment. Each such segment will intersect every L-shape of in $\mathcal{L}$ (since they are anchored) and they are disjoint from each other. Thus, this is a linear L-system of G'; i.e., G' is an MPT graph. This leads to the following observation regarding minimal forbidden induced subgraphs for MPT graphs. The set of minimal forbidden induced subgraphs of interval graphs is known [34] and is infinite.

Observation 6.8. *If H is a minimal forbidden induced subgraph for interval graphs and G is obtained from H by adding two non-adjacent universal vertices x and y to H; i.e., $V(G) = V(H) \cup \{x, y\}$ and $E(G) = E(H) \cup \{zu : z \in \{x, y\}$ and $u \in V(H)\}$, then either G, G \ \{x\}, or H is a minimal forbidden induced subgraph of MPT graphs.*

By Proposition 6.7 we see that $K_{2,2,2}$, the graph formed by taking a 4-cycle together with two non-adjacent vertices adjacent to each vertex of the cycle, is not an MPT graph. However, it is easy to see that this graph is a permutation graph as well as a planar graph (see Figure 9). Moreover, non planar graphs (e.g., the 5-clique) and non permutation graphs (e.g., the graph in Figure 2) are both MPT graphs. Thus we have the following observation.

Figure 9: A planar drawing of the non-MPT graph $K_{2,2,2}$ together with a permutation representation of it.

Observation 6.9. *The MPT graph class is incomparable with both planar graphs and permutation graphs.*

The minimal forbidden induced subgraphs of MPT graphs include many more graphs than those built from the graphs non-interval graphs. For example, we will show that the full subdivision of any non-outerplanar graph is also not an MPT graph. The *full subdivision G* of a graph *H* is the graph obtained from *H* by subdividing every edge of *G*. It is known that any string representation $\mathcal{S}$ of the full-subdivision *H* of a planar graph *G* is combinatorially equivalent to some planar embedding of *G* [12]. In particular, in $\mathcal{S}$, each edge *e* of *G* corresponds to a string S_e which connects exactly the two strings corresponding to vertices incident with *e* and S_e does not intersect any other strings. From this it is easy to see that the full-subdivision of any non-planar graph is not a string graph [12].

A graph *G* is an *outer-string* graph when it has a string representation such that, for a fixed circle *C*, every string is contained within *C* and exactly one endpoint of each string belongs to *C*. It is easy to see that outer-string graphs are a superclass of MPT graphs; i.e., we can easily replicate a linear L-system with an outer-string representation. Moreover, in Lemma 6.10, we observe that the full-subdivision of a non-outerplanar graph is not an outer-string graph and, consequently, not an MPT graph.

Lemma 6.10. *If H is a non-outerplanar graph and G is the graph obtained from H by subdividing every edge of H, then G is not an outer-string graph (i.e., G is not an MPT graph).*

Proof. Since *G* is not outerplanar, any string representation $\mathcal{S}$ of *H* necessarily contains a string S_v such that S_v is contained in a region enclosed by the strings of a set *X* of non-neighbors of *v*. In particular, it is not possible to draw a circle onto $\mathcal{S}$ so that both S_v and every string of the vertices in *X* satisfy the outer-string property. □

Even with the set of forbidden graphs we have observed, there are yet many more which are not captured (e.g., the complement of a 7-cycle). Thus, a complete description of the minimal forbidden induced subgraphs for MPT graphs remains an open problem.

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7. Concluding Remarks

In this paper we have introduced max point-tolerance graphs. We have characterized this class and demonstrated inclusions with respect to well-known graph classes. Our results are summarized in Theorems 7.2 and 7.3 below. We also solved the WIS problem in polynomial time, 2-approximated the clique cover problem in polynomial time, showed the NP-completeness of the coloring problem, and $\log(n)$-approximated the coloring problem in polynomial time.

Interesting open problems remain for this graph class. Perhaps the most interesting is that of recognition. Our characterizations of this graph class provide a variety of ways to approach this problem. Several combinatorial optimization problems remain open for this graph class. Two particularly interesting ones are: k-coloring (for fixed k) and unweighted clique cover. Additionally, one may be interested to further study the relationships with existing graph classes.

Another direction of research would be to study *min* point-tolerance graphs. In particular, just as there are min tolerance graphs and max tolerance graphs one can consider *min point-tolerance (mPT)* graphs.

Definition 7.1. A graph $G = (V, E)$ is a *min point-tolerance (mPT)* graph if each vertex v of G can be mapped to a *pointed-interval* (I_v, p_v) where I_v is an interval of $\mathbb{R}$ and $p_v \in I_v$ such that uv is an edge of G iff either $p_u \in I_v$ or $p_v \in I_u$.

It should be noted that there already exists a directed graph class utilizing this definition, namely, the interval catch digraphs [40, 41] mentioned in the introduction. The min point-tolerance graphs are precisely the undirected graphs underlying interval catch digraphs. In contrast, max point-tolerance graphs are precisely the undirected graphs underlying the bi-directed edges of interval catch digraphs.

Theorem 7.2. *The max point-tolerance graph class strictly includes interval graphs, outerplanar graphs, and 2D ray graphs.*

Theorem 7.3. *For a graph $G = (V, E)$, the following are equivalent:*

- *G is a max point-tolerance graph.*
- *G is a linear L-graph (equivalently, a linear rectangle-graph or a linear right-triangle-graph).*
- *The vertices of G can be ordered by $<$ so that for every $u, v, w, x \in V(G)$, if $u < v < w < x$ and $uw, vx \in E(G)$, then vw is an edge of G.*
- *There are two interval graphs $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ such that $E = E_1 \cap E_2$ and the vertices of G can be ordered by $<$ so that for every $u < v < w$ if $uw \in E_1$ then $uv \in E_1$ and if $uw \in E_2$ then $wv \in E_2$.*
- *G is a cyclic segment graph.*
- *There is an interval catch digraph $D = (V, A)$ such that the bi-directed arcs of D are precisely the edges of G.*

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