Minimal Classes of Bipartite Graphs of Unbounded Clique-width

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Abstract

The celebrated result of Robertson and Seymour states that in the family of minor-closed graph classes the planar graphs constitute a unique minimal class of graphs of unbounded tree-width. When we study tree-width, the restriction to minor-closed graph classes is justified by the fact that the tree-width of a graph is never smaller than the tree-width of any of its minors. However, this is not the case with respect to clique-width, i.e. the clique-width of a graph can be (much) smaller than the clique-width of its minor. On the other hand, the clique-width of a graph is never smaller than the clique-width of any of its induced subgraphs. Therefore, when we study clique-width we may restrict ourselves to hereditary graph classes, i.e. those that are closed under taking induced subgraphs. The first two minimal hereditary classes of unbounded clique-width have been recently identified in [12]. In the present paper, we restrict our attention to bipartite graphs and identify two more minimal hereditary classes of graphs of unbounded clique-width.

Keywords: Clique-width; Rank-width; Hereditary class; Universal graph

1 Introduction

In this paper, we study clique-width. This is one of the representatives of the world of graph width parameters. This world is rich and includes both parameters studied in the literature for decades, such as path-width [18] or tree-width [19], and those that have been introduced recently, such as Boolean-width [5] or plane-width [10]. Graph width parameters find various applications in computer science and combinatorics (see e.g. [9]). In particular, many difficult algorithmic problems become tractable when restricted to graphs where one of these parameters is bounded by a constant.

The notion of clique-width belongs to the middle generation of graph width parameters. It generalizes the notion of tree-width in the sense that graphs of bounded tree-width have bounded clique-width, but not necessarily vice versa. The celebrated result of Robertson and Seymour states that in the family of minor-closed graph classes the planar graphs constitute a unique minimal class of graphs of unbounded tree-width [20]. This restriction to minor-closed graph classes is well justified when we study tree-width, because the tree-width of a graph G is never smaller than the tree-width of a minor of G. However, this is not the case with respect to clique-width, i.e. the clique-width of G can be (much) smaller than the clique-width of its minor. On the other hand, the clique-width of a graph is never smaller than the clique-width of any of its induced subgraphs. Therefore, when we study clique-width we may restrict ourselves to hereditary graph classes, i.e. those that are closed under taking induced subgraphs.

The first two minimal hereditary classes of unbounded clique-width have been identified recently in [12]. These are bipartite permutation graphs and unit interval graphs. More minimal classes can be obtained from these two by various graph operations that do not change the clique-width "too much", such as complementation, bipartite complementation, or local complementation. Moreover, with some care the first two minimal classes identified in [12] can also be related to each other by means of these operations. Taking into account these observations, in the present paper we restrict ourselves to hereditary classes of bipartite graphs.

The restriction to bipartite graphs can also be justified by the following arguments. It is known that the clique-width of graphs in a class X is bounded if and only if the rank-width is bounded [16]. Rank-width is a graph parameter defined on *bipartitions* of the input graph, and if the rank-width of a graph is high, then there is a bipartition of its vertex set such that the bipartite graph formed by the edges of the cut has a complex structure (measured by the rank of its adjacency matrix). Therefore, bipartite graphs are of fundamental importance in the study of both parameters, clique- and rank-width.

In the present paper, we identify two new minimal hereditary classes of bipartite graphs of unbounded clique-width. We call one of them *bichain graphs* and introduce it in Section 2, where we also report some preliminary results related to the topic of the paper. In Section 3, we prove that bichain graphs form a minimal hereditary class of graphs of unbounded clique-width. In Section 4, we prove a similar result for one more class of bipartite graphs.

2 Preliminaries

All graphs in this paper are simple, i.e. undirected, without loops and multiple edges. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of a graph G, respectively. Given a vertex $v \in V(G)$, we denote by $N_G(v)$ the *neighbourhood* of v, i.e. the set of vertices adjacent to v. The *degree* of v is the number of its neighbours.

Let G be a graph and $U \subseteq V(G)$ a subset of its vertices. Two vertices of U will be called U-*similar* if they have the same neighborhood outside of U. Clearly, U-similarity is an equivalence relation. The number of equivalence classes of U will be denoted $\mu_G(U)$. Also, by $G[U]$ we will denote the subgraph of G induced by U, i.e. the subgraph of G with vertex set U and two vertices being adjacent in $G[U]$ if and only if they are adjacent in G. We say that a graph H is an induced subgraph of G, or G contains H as an induced subgraph, if H is isomorphic to $G[U]$ for some $U \subseteq V(G)$. If no subset of $V(G)$ induces H, we say that G is H-free.

In a graph, a *clique* is a subset of pairwise adjacent vertices and an *independent set* is a subset of pairwise non-adjacent vertices. A graph G is a *split* graph if its vertices can be partitioned into an independent set and a clique, and G is a *bipartite* graph if its vertices can be partitioned into two independent sets (also called color classes or simply parts).

As usual, by P_n and C_n we denote a chordless path and a chordless cycle with n, respectively. To simplify the notion, we drop the subscript G from $N_G(v)$ and $\mu_G(U)$ if no confusion arises.

2.1 Clique-width

The notion of clique-width of a graph was introduced in [6] and is defined as the minimum number of labels needed to construct the graph by means of the four graph operations:

- creation of a new vertex v with label i (denoted $i(v)$),
- disjoint union of two labeled graphs G and H (denoted $G \oplus H$),
- connecting vertices with specified labels i and j (denoted $\eta_{i,j}$) and
- renaming label i to label j (denoted $\rho_{i \to j}$).

The clique-width of a graph G will be denoted $\mathrm{cwd}(G)$.

Every graph can be defined by an algebraic expression using the four operations above. This expression is called a k-expression if it uses k different labels. For instance, the cycle C_5 on vertices a, b, c, d, e (listed along the cycle) can be defined by the following 4-expression:

$$\eta_{4,1}(\eta_{4,3}(4(e) \oplus \rho_{4\to 3}(\rho_{3\to 2}(\eta_{4,3}(4(d) \oplus \eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))))).$$

Alternatively, any algebraic expression defining G can be represented as a rooted tree, whose leaves correspond to the operations of vertex creation, the internal nodes correspond to the $\oplus$-operations, and the root is associated with G. The operations η and ρ are assigned to the respective edges of the tree. Figure 1 shows the tree representing the above expression defining a C_5.

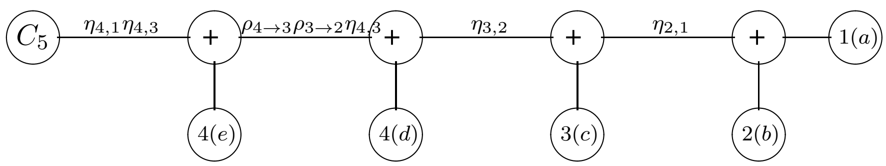

Figure 1: The tree representing the expression defining a C_5

In the course of our study we will need the following technical lemma proved in [12].

Lemma 1. [12] *Let $k \geq 2$ and ℓ be positive integers. Suppose that the vertex set of G can be partitioned into sets $V_1, V_2, \ldots$ where for each i,*

(1) $\mathrm{cwd}(G[V_i]) \leq k$,

(2) $\mu(V_i) \leq \ell$ *and* $\mu(V_1 \cup \cdots \cup V_i) \leq \ell$.

Then $\mathrm{cwd}(G) \leq k \cdot \ell$.

This lemma implies, in particular, that for any n, the clique-width of P_n is at most 4. This conclusion can be obtained by defining $V_i = \{v_i\}$, where v_i is the i-th vertex of the path P_n, in which case $k = 2$ and $\ell = 2$. This bound can be easily improved to 3 and we mention this example merely as an illustration of the lemma. In the proof or our main results, the reader will find more powerful applications of this lemma.

2.2 Bichain graphs

We will say that a set of vertices form a *chain* if their neighbourhoods form a chain with respect to the inclusion relationship, i.e. if they can be linearly ordered under this relation.

A bipartite graph will be called a k-*chain graph* if the vertices in each part of its bipartition can be split into at most k-chains.

Example.

1-chain graphs are known simply as *chain graphs*. A typical example of a chain graph is represented in Figure 2. The importance of this example is due to the fact that the represented graph contains all chain graphs with at most 5 vertices as induced subgraphs, i.e. it is 5-universal. Moreover, this example can be easily extended to a general construction of an n-universal chain graph. Such a graph has n vertices in each part of its bipartition and contains all chain graphs with at most n vertices as induced subgraphs, which can be easily proved by induction on n.

An important observation about chain graphs is that

(*) a bipartite graph is a chain graph if the vertices in *at least one* part of its bipartition form a chain. In other words, if the vertices in one part of a bipartite graph form a chain, then the vertices of the other part form a chain too.

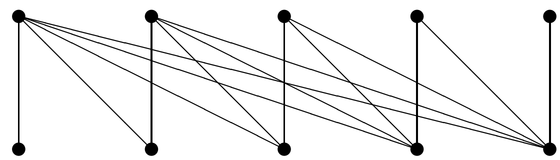

Figure 2: 5-universal chain graph

> The simple structure of chains graphs implies many nice properties. In particular, the clique-width of chain graphs is at most three [8] and they are well-quasi-ordered by the induced subgraph relation (i.e. the class of chain graphs does not contain infinite antichains with respect to this relation) [17].
>
> The boundedness of clique-width can also be derived from Lemma 1 by showing that the universal chain graph has bounded clique-width. This can be done by defining V_i to be the 2-element set containing the endpoints of the i-th vertical edge of the universal chain graph, in which case $k = 2$ and $\ell = 2$.

The class of 2-chain graphs, also called *bichain graphs*, is much richer and this is one of the two classes of our interest in this paper. The clique-width of bichain graphs is unbounded and this class is not well-quasi-ordered by induced subgraphs. Both conclusions follow readily from a relationship between bichain graphs and split permutation graphs. This relationship is described in Claim 1 below, where we also reveal a similar relationship between chain graphs and an important subclass of spit permutation graphs, known as *threshold graphs*.

Claim 1. *Let G be a split graph given together with a partition of its vertex set into a clique C and an independent set I, and let G^* be the bipartite graph obtained from G by deleting the edges of C. Then*

- *G is a threshold graph if and only if G^* is a chain graph,*
- *G is a split permutation graph if and only if G^* is a bichain graph.*

This claim follows from the definition of chain and bichain graphs and the fact that threshold graphs are graphs of Dilworth number 1, while split permutation graphs are split graphs of Dilworth number at most two [3].

In [11], it was shown that split permutation graphs have unbounded clique-width and they are not well-quasi-ordered by the induced subgraph relation. Together with Claim 1 this implies the same conclusions for bichain graphs. One of the main results of the present paper is that bichain (and therefore split permutation) graphs form a *minimal* hereditary class of unbounded clique-width. Our proof of minimality is based on a universal construction for bichain graphs. Before we describe this construction, we recall a few facts about bipartite permutation graphs.

2.3 Bipartite permutation graphs

A permutation graph G can be defined as the intersection graph of line segments whose endpoints are located on two parallel lines. Such a diagram is called an intersection model of G. For an illustration, see Figure 3 which represents a permutation graph (on the left) and its intersection model (on the right).

We emphasize that the graph represented in Figure 3 is not only a permutation graph, but also bipartite, and the vertices (line segments) in different parts of its bipartition are colored differently (black and white). This example is also typical in the sense that it suggests an idea of a universal construction for bipartite permutation graphs. Intuitively, this idea can be described in terms of the intersection model

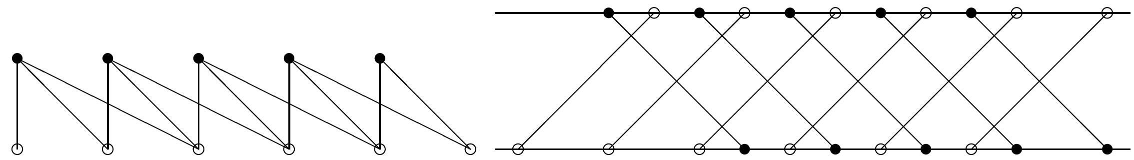

Figure 3: The graph on the left is the intersection graph of the diagram on the right

as follows: to be universal the intersection model must have “many” segments of both colors and each segment must intersect “many” segments of the opposite color.

The nature of permutation graphs suggests representing them as a “one-dimensional” structure, developing, for instance, from left to right as in Figure 3. Based on the above idea, which involves two parameters, we split the vertices of a universal bipartite permutation graph into layers and represent them as a two-dimensional structure. This representation allows us to speak about rows and columns of the universal graph, and we denote a universal bipartite permutation graph with n rows and n columns by $X_{n,n}$. An example of the graph $X_{n,n}$ for $n = 6$ (i.e. 6 columns and 6 rows) is shown on the left of Figure 4.

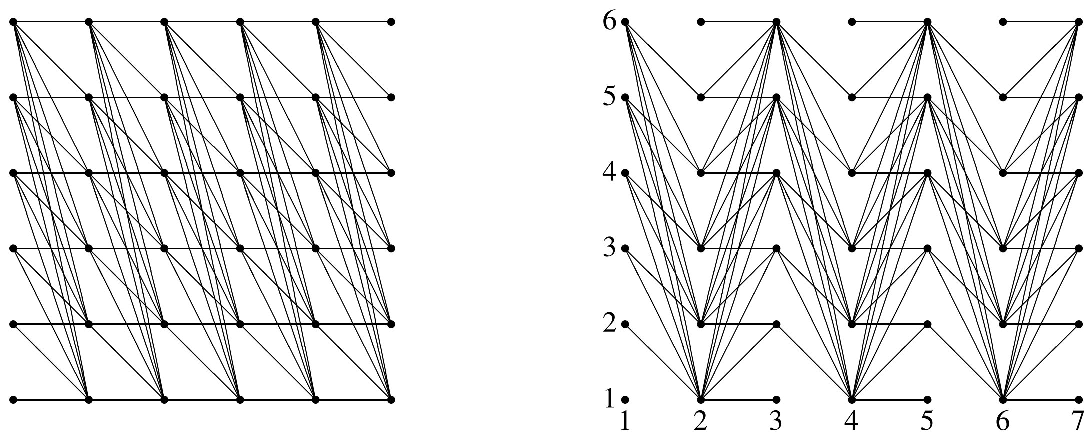

Figure 4: Graphs $X_{6,6}$ (left) and $Z_{7,6}$ (right). The graph $Z_{7,6}$ contains the edges shown in the picture and the “diagonal” edges connecting every even column i to every odd column $i' \geq i+3$ (these edges are not shown for clarity of the picture).

The following theorem was proved in [14].

Theorem 1. *The graph $X_{n,n}$ is an n-universal bipartite permutation graph, i.e. it contains every bipartite permutation graph with n vertices as an induced subgraph.*

The two-dimensional representation of the universal bipartite permutation graph also suggests an idea of why the clique-width is unbounded in this class. Observe that every row of $X_{n,n}$ is a chordless path, and hence, an algebraic expression defining the graph by means of bounded number of labels should develop “horizontally”. On the other hand, any two consecutive columns of $X_{n,n}$ induce an n-universal chain graph, and hence, suggested by Lemma 1, an algebraic expression defining the graph by means of bounded number of labels should develop “vertically”. Therefore, if the graph grows in both directions, the clique-width grows as well.

We call any graph of the form $X_{n,n}$ an X-grid. The proof of minimality of the class of bipartite permutation graphs with respect to clique-width is based on the universality of the X-grid and on the following theorem proved in [12].

Theorem 2. *For every n, the clique-width of $X_{n,n}$-free bipartite permutation graphs is bounded by a constant.*

We develop a similar approach for bichain graphs. To this end, in the next section we describe a universal construction for bichain graphs.

2.4 A universal bichain graph

Because of the relationship between bichain graphs and split permutation graphs described in Claim 1, constructing a universal bichain graph is equivalent to constructing a universal split permutation graph.

For split permutation graphs, one can employ a geometric approach as follows. Let G be a bipartite permutation graph and let A be one of its color classes. In the intersection model, the segments representing the vertices of A are mutually non-crossing. However, if we reverse the order of their bottom endpoints, then the segments become pairwise crossing and hence A becomes a clique. Therefore, this operation transforms G into a split permutation graph, say G'. One could expect that if G is a universal bipartite permutation graph, then G' is a universal split permutation graph. However, regardless of how complex the graph G is, the graph G' is necessarily a split graph of Dilworth number 1, i.e. a threshold graph. This is where the idea of layering a universal bipartite permutation graph comes to light. To keep the complex structure of G, one needs to do the transformation layer by layer.

A more straightforward approach to establishing a relationship between bipartite permutation graphs and split permutation graphs was proposed in [11]. In terms of bichain graphs, this approach can be described as follows.

Let $G = (A, B, E)$ be a bichain graph given together with a partition of its vertex set into two independent sets A and B. By definition, each of these sets can be further split into two chains, say $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. Since A_1 is a chain, the vertices of B in the subgraph of G obtained by the deletion of A_2 also form a chain according to Observation (*). In other words, the vertices of B can be linearly ordered with respect to A_1. Similarly, the vertices of B can be linearly ordered with respect to A_2. Two linear orders define a permutation on B, say π. Moreover, the permutation graph of π must be bipartite, since B can also be split into two chains (each defining an independent set in the permutation graph).

This relationship between bipartite permutation graphs on the one hand and bichain graphs (and split permutation graphs) on the other hand suggests the following idea for constructing a universal bichain (and hence split permutation) graph: build a bichain graph $G = (A, B, E)$ such that the permutation graph of π is a universal bipartite permutation graph. This idea was implemented in [2] and then improved in [4] as follows.

Denote by $Z_{n,k}$ the graph with the vertex set $\{z_{i,j} \ : \ 1 \le i \le n, 1 \le j \le k\}$ and with $z_{ij}z_{i'j'}$ being an edge if and only if

(1) i is odd and $i' = i + 1$ and $j > j'$,

(2) i is even and $i' = i + 1$ and $j \le j'$,

(3) i is even, i' is odd and $i' \ge i + 3$.

We call the edges of type 3 the *diagonal* edges. An example of the graph $Z_{n,k}$ with $n = 7$ and $k = 6$ is represented on the right of Figure 4, where for clarity of the picture we omitted the diagonal edges. Any graph of the form $Z_{n,k}$ will be called a Z-grid.

We call an induced subgraph G of $Z_{n,n}$ *row-sparse* if every row of $Z_{n,n}$ contains at most one vertex of G.

Theorem 3. [4] *The graph $Z_{n,n}$ is an n-universal bichain graph, i.e. it contains every bichain graph G on n vertices as an induced subgraph. Moreover, G is isomorphic to a row-sparse induced subgraph of $Z_{n,n}$.*

3 The class of bichain graphs is a minimal hereditary class of unbounded clique-width

In order to show that bichain graphs form a minimal hereditary class of unbounded clique-width, we need a number of preparatory results.

Lemma 2. *Let n, m be positive integers. Then $\mathrm{cwd}(Z_{n,m}) \leq 3n$.*

Proof. We decompose the Z-grid $Z_{n,m}$ into rows and then apply Lemma 1. For $j = 1, \ldots, m$, let U_j denote the j-th row of $Z_{n,m}$, i.e. $U_j = \{v_{ij} \mid i \in \{1, \ldots, n\}\}$. Since $|U_j| = n$, we have

- $\mu(U_j) \leq n$.

Also, it is not difficult to see that U_j induces a chain graph, and hence

- $\mathrm{cwd}(Z_{n,m}[U_j]) \leq 3$ for all $j \in \{1, \ldots, m\}$.

Finally, by direct inspection, the reader can easily check that

- $\mu(U_1 \cup U_2 \cup \cdots \cup U_j) \leq n$ for all $j \in \{1, \ldots, m\}$.

Now applying Lemma 1 with the partition $U_1, \ldots, U_m$ and using the above claims, we conclude that $\mathrm{cwd}(Z_{n,m}) \leq 3n$. □

Lemma 3. *Let n, m be positive integers and G be a row-sparse induced subgraph of $Z_{n,m}$. Denote by V_i the i-th column of $Z_{n,m}$, i.e. $V_i = \{v_{ij} \mid j \in \{1, \ldots, m\}\}$. If G contains no induced subgraph isomorphic to $Z_{n,n}$, then there exists a partition $V(G) = X \cup Y$ such that*

- *$|X \cap V_n| \leq n - 1$ and $|Y \cap V_1| \leq n - 1$,*
- *$\mu_{G[Y \cup V_i]}(X \cap V_i) \leq n$ and $\mu_{G[X \cup V_i]}(Y \cap V_i) \leq n$ for all $i \in \{1, \ldots, n\}$.*

Proof. Let G be a row-sparse induced subgraph of $Z_{n,m}$ that contains no induced $Z_{n,n}$. In order to find the desired partition of $V(G)$, we construct the following directed graph H:

(i) $V(H) = V(Z_{n,m}) \cup \{s, t\}$, where s, t are two new vertices.

(ii) $E(H)$ consists of

  - an arc from v_{ij} to $v_{i(j+1)}$ for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots m - 1\}$,
  - an arc from s to v_{1j} for each $j \in \{1, \ldots, m\}$ such that $v_{1j} \in V(G)$,
  - an arc from v_{nj} to t for each $j \in \{1, \ldots, m\}$ such that $v_{nj} \in V(G)$, and
  - two arcs between v_{ij} and $v_{(i-1)j}$, one in each direction, for each $i \in \{2, \ldots, n\}$ and $j \in \{1, \ldots m\}$ such that $v_{ij} \in V(G)$. For simplicity, in the proof we talk about *one* arc connecting v_{ij} and $v_{(i-1)j}$ in both directions.

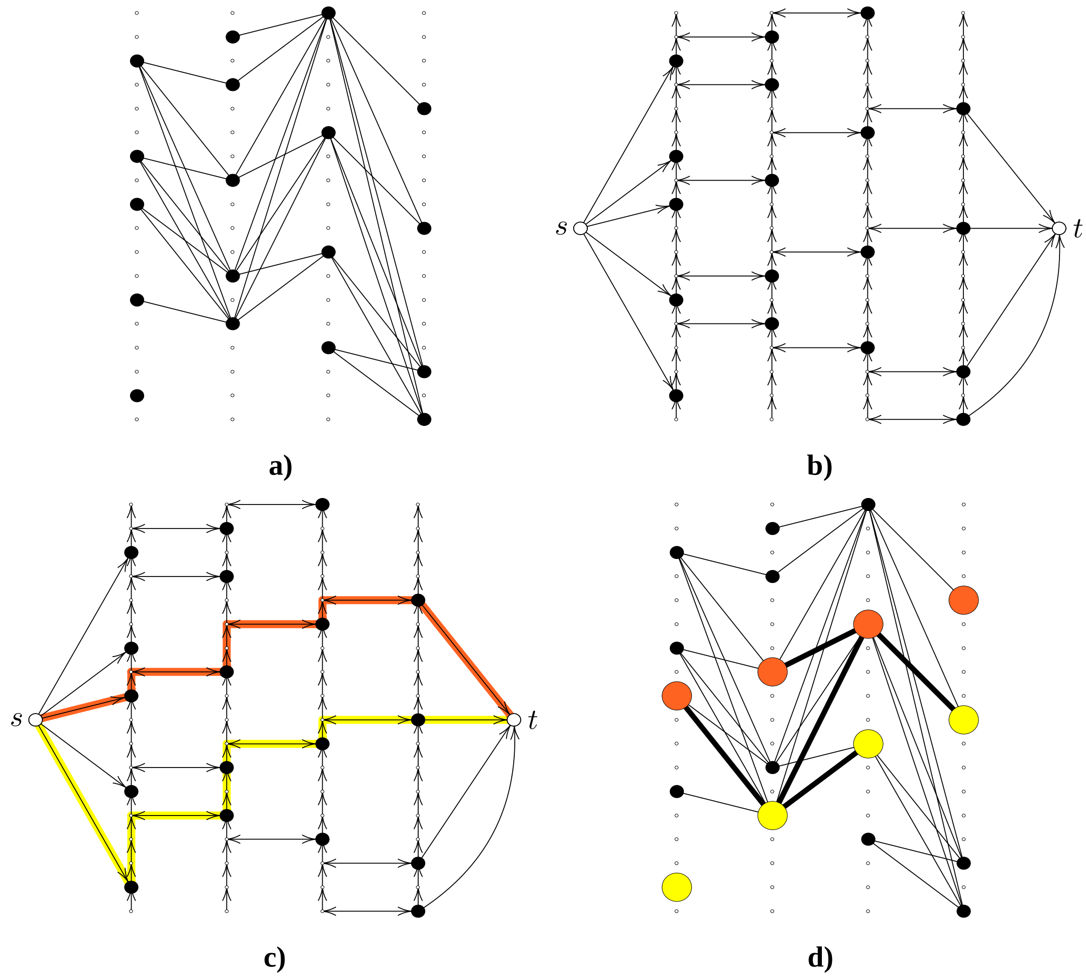

Figure 5: Example of the construction of the directed graph H: **a)** the subgraph G of $Z_{4,m}$; only edges of G shown, **b)** directed graph H, **c)** maximum st-flow; two disjoint paths, **d)** corresponding induced subgraph $Z_{4,2}$ in G.

An example of this construction is represented in Figure 5. We make an important observation that H is a directed *planar* graph. Indeed, note that H can be drawn in the plane by representing the vertices v_{ij} as points (i, j) in an $n \times m$ grid, and drawing s and t to the left and right of this grid, respectively. Then connecting adjacent vertices by straight lines clearly does not create crossings. (See Figure 5b for an example of this representation.)

The following claim about the graph H is very important for the proof of the lemma.

(3.1) *Every vertex v_{ij} in H with $i < n$ has degree at most 3*, since otherwise G is not a row-sparse subgraph of $Z_{n,m}$.

Now, let $P^1, P^2, \ldots, P^r$ be the largest collection of pairwise arc-disjoint directed st-paths in H. Using these paths, we will show how to obtain an induced subgraph of G isomorphic to $Z_{n,r}$, where r is the number of paths. Since G does not contain $Z_{n,n}$ as an induced subgraph, this will give us a bound on r and thus will provide us with an st-cut of H of capacity less than n. We then show that the cut induces the desired partition of G, which will prove the lemma.

Note that each path P^k goes from s to t and thus crosses each of the sets V_i. Thus we may define

$\phi(i,k)$ to be the index j such that v_{ij} is the first vertex on P^k that belongs to V_i (first when traversing P^k from s to t).

Since the paths $P^1,\dots,P^r$ are pairwise arc-disjoint, we have $\phi(1,k) \neq \phi(1,k')$ for distinct $k,k' \in \{1,\dots,r\}$. Thus without loss of generality, we may assume that the paths are ordered so that $\phi(1,1) < \phi(1,2) < \dots < \phi(1,r)$. We also assume that each path P^k crosses the last column V_n at exactly one vertex v_{nj} with $j = \phi(n,k)$, i.e. from this vertex the path moves directly to t. With these assumptions, we can conclude that

(3.2) *the paths $P^1, P^2, \dots, P^r$ are internally vertex-disjoint (i.e. s and t are the only common vertices of the paths)*, because if two paths have a common vertex v, then either v has degree 4 (contradicting Claim 3.1) or the paths have a common arc (contradicting the choice of the paths).

From this conclusion and the definition of H, taking also into account the planarity of H, the reader can easily derive the following claims about $\phi(i,k)$.

(3.3) $\phi(i,k) < \phi(i,k')$ *if and only if* $k < k'$.

(3.4) *Let* $j = \phi(i,k)$. *Then* $v_{ij} \in V(G)$.

(3.5) $\phi(i,k) < \phi(i',k)$ *if and only if* $i < i'$.

(3.6) *If* $i > 1$ *and* $k < k'$*, then* $\phi(i,k) < \phi(i-1,k')$.

With these technical claims in mind, we are finally ready to define the desired isomorphism showing that G contains an induced copy of $Z_{n,r}$. For each $k \in \{1,\dots,r\}$, define $W_k = \{v_{ij} \mid j = \phi(i,k),\ i \in \{1,\dots,n\}\}$. Define $W = W_1 \cup W_2 \cup \dots W_r$. Note that $W \subseteq V(G)$ by Claim 3.4.

(3.7) $G[W]$ *is isomorphic to* $Z_{n,r}$.

Proof. We define a mapping f between $V(Z_{n,r})$ and W as follows. For each $i \in \{1,\dots,n\}$ and $k \in \{1,\dots,r\}$, the mapping f maps $v_{ik} \in V(Z_{n,r})$ to the vertex $v_{ij} \in W_k$ where $j = \phi(i,k)$. Note that f is an injective mapping. Indeed, if $f(v_{ik}) = f(v_{i'k'})$, then $i = i'$ and $\phi(i,k) = \phi(i',k')$; thus $k = k'$ by Claim 3.3. Clearly, f is also surjective, by the definition of W. Now, recall that $W \subseteq V(G)$ and that G is an induced subgraph of $Z_{n,m}$. Thus, to prove that f is the desired isomorphism, it remains to show that $v_{ik}v_{i'k'} \in E(Z_{n,r})$ if and only if $v_{ij}v_{i'j'} \in E(Z_{n,m})$ for all $i,i' \in \{1,\dots,n\}$ and $k,k' \in \{1,\dots r\}$, where $j = \phi(i,k)$ and $j' = \phi(i',k')$. This is shown as follows.

Suppose first that $v_{ik}v_{i'k'} \in E(Z_{n,r})$. Then by the definition of $Z_{n,r}$, we may assume by symmetry that i is even, i' is odd, $i' \geq i-1$, and if $i' = i-1$, then $k < k'$, while if $i' = i+1$, then $k \leq k'$. Clearly, if $i' \geq i+3$, then $v_{ij}v_{i'j'} \in E(Z_{n,m})$ by the definition of $Z_{n,m}$. If $i' = i-1$, then we have $k < k'$ which yields $\phi(i,k) < \phi(i-1,k')$ by Claim 3.6. Thus $j < j'$ and we conclude that $v_{ij}v_{i'j'} \in E(Z_{n,m})$. Similarly, if $i' = i+1$, then $k \leq k'$ which yields $\phi(i,k) \leq \phi(i,k')$ by Claim 3.3. We also deduce $\phi(i,k') < \phi(i',k')$ by Claim 3.5, since $i < i'$. Put together, we have $j = \phi(i,k) \leq \phi(i,k') < \phi(i',k') = j'$. Thus $j < j'$ and we again conclude that $v_{ij}v_{i'j'} \in E(Z_{n,m})$.

Conversely, suppose that $v_{ij}v_{i'j'} \in E(Z_{n,m})$. This time, by symmetry, we shall assume that i is odd, i' is even, and either $i' \leq i-3$, or $i' = i+1$ and $j' < j$, or $i' = i-1$ and $j' \leq j$. Clearly, if $i' \leq i-3$, then $v_{ik}v_{i'k'} \in E(Z_{n,r})$. If $i' = i-1$ and $j' \leq j$, then we deduce $k' \leq k$ by Claim 3.6. Indeed, if $k < k'$, then Claim 3.6 yields $\phi(i,k) < \phi(i-1,k')$ which is $j < j'$, a contradiction. Therefore, since $k' \leq k$, it follows that $v_{ik}v_{i'k'} \in E(Z_{n,r})$, Similarly, if $i' = i+1$ and $j' < j$, then

$\phi(i,k) < \phi(i',k)$ by Claim 3.5. Thus if $k \le k'$, we have $\phi(i',k) \le \phi(i',k')$ by Claim 3.3 and so $j = \phi(i,k) < \phi(i',k) \le \phi(i',k') = j' < j$, a contradiction. We therefore conclude that $k' < k$ which again yields $v_{ik}v_{i'k'} \in E(Z_{n,r})$, as required.

From Claim 3.7, we deduce that $r < n$, since G does not contain an induced $Z_{n,n}$. By the Max-Flow-Min-Cut Theorem, this implies that in H there exists an st-cut (X^+, Y^+) with $s \in X^+$ and $t \in Y^+$ such that there are at most $n-1$ arcs in H going from X^+ to Y^+. We let $X = X^+ \cap V(G)$ and $Y = Y^+ \cap V(G)$.

We prove that $X \cup Y$ is the desired partition of $V(G)$. First, we observe that if $Y \cap V_1$ contains n vertices $y_1, \ldots, y_n$, then these are also vertices in G, since $Y \subseteq V(G)$. Thus H contains directed arcs from s to each of $y_1, \ldots, y_n$. Since $s \in X^+$ while $y_1, \ldots, y_n \in Y \subseteq Y^+$, these arcs constitute a set of n arcs going from X^+ to Y^+, contradicting our choice of the cut (X^+, Y^+). We conclude that

- $|Y \cap V_1| \le n-1$

as required. Similarly, if $X \cap V_n$ contains vertices $x_1, \ldots, x_n$, then H contains arcs from each of $x_1, \ldots, x_n$ to t. Since $t \in Y^+$, these n arcs go from X^+ to Y^+, again contradicting the choice of (X^+, Y^+). Thus

- $|X \cap V_n| \le n-1$.

The remaining two properties of the partition (X, Y) are proved below.

- $\mu_{G[Y \cup V_i]}(X \cap V_i) \le n$ for all $i \in \{1, \ldots, n\}$.

  For contradiction, let $X \cap V_i$ contains $n+1$ vertices $x_1, \ldots, x_{n+1}$ whose neighbourhoods in Y are all pairwise different. Without loss of generality, we may assume $N(x_1) \subseteq N(x_2) \subseteq \cdots \subseteq N(x_{n+1})$, since the vertices belong to the same column V_i. Thus we deduce $(N(x_1) \cap Y) \subset (N(x_2) \cap Y) \subset \cdots \subset (N(x_{n+1}) \cap Y)$, where all inclusions are proper, since the neighbourhoods of the vertices $x_1, \ldots, x_{n+1}$ in Y are all different.

  This implies that for all $j \in \{1, \ldots, n\}$, the set $N(x_{j+1}) \setminus N(x_j)$ contains a vertex of Y; let us denote it y_j. Note that $y_j \in V_{i-1} \cup V_{i+1}$, since by the definition of $Z_{n,m}$, the vertices x_j and x_{j+1} have same neighbourhoods in the columns $V_1, V_2, \ldots, V_{i-2}, V_{i+2}, V_{i+3}, \ldots, V_n$ and they are both non-adjacent to any other vertex in V_i.

  Let $\ell_1, \ldots, \ell_{n+1}$ be indices such that $x_1 = v_{i\ell_1}$, $x_2 = v_{i\ell_2}$, $\ldots$, $x_{n+1} = v_{i\ell_{n+1}}$. Similarly, let $k_1, \ldots, k_n$ be indices such that for all $j \in \{1, \ldots, n\}$, we have $y_j = v_{(i+1)k_j}$ if $y_j \in V_{i+1}$, and $y_j = v_{(i-1)k_j}$ if $y_j \in V_{i-1}$.

  Case 1. Suppose that i is odd. Since $N(x_1) \subseteq \cdots \subseteq N(x_{n+1})$, since i is odd, and since $y_j \in N(x_{j+1}) \setminus N(x_j)$ for all $j \in \{1, \ldots, n\}$, the definition of $Z_{n,m}$ yields $\ell_1 \le k_1 \le \ell_2 \le k_2 \le \cdots \le k_n \le \ell_{n+1}$. In fact, we can deduce $\ell_1 < k_1 < \ell_2 < k_2 < \cdots < k_n < \ell_{n+1}$, since $x_1, \ldots, x_{n+1}, y_1, \ldots, y_n$ are vertices in G, and we assume that G is a row-sparse induced subgraph of $Z_{n,m}$.

  Now, consider $j \in \{1, \ldots, n\}$. Recall that $\ell_j < k_j$. This implies that H contains a directed path Q_j from x_j to y_j; namely if $y_j \in V_{i+1}$, the path is $v_{i\ell_j}, v_{i(\ell_j+1)}, \ldots, v_{ik_j}, v_{(i+1)k_j}$, while if $y_j \in V_{i-1}$, the path is $v_{i\ell_j}, v_{(i-1)\ell_j}, v_{(i-1)(\ell_j+1)}, \ldots, v_{(i-1)k_j}$. For this, recall that $x, y \in V(G)$; so if $y_j \in V_{i+1}$, then H contains arcs in both directions between v_{ik_j} and $v_{(i+1)k_j}$, while if $y_j \in V_{i-1}$, then $i > 1$ and so H contains arcs between $v_{(i-1)\ell_j}$ and $v_{i\ell_j}$. Since $x_j \in X^+$ while $y_j \in Y^+$, the path Q_j contains an arc from X^+ to Y^+; let us denote this arc e_j.

  Notice that for all $j < j'$ the paths $Q_j, Q_{j'}$ are vertex-disjoint, since $\ell_j < k_j < \ell_{j'} < k_{j'}$. This implies that the arcs $e_1, \ldots, e_n$ are all distinct. But then the arcs $e_1, \ldots, e_n$ constitute a set of n

arcs from X^+ to Y^+, which contradicts our choice of the cut (X^+, Y^+). So Case 1 is impossible.

Case 2. Suppose that i is even. We proceed similarly as in Case 1. Since i is even and G is row-sparse, we deduce $\ell_1 > k_1 > \ell_2 > k_2 > \cdots > k_n > \ell_{n+1}$. For $j \in \{1, \ldots, n\}$, we let Q_j be a directed path in H from x_{j+1} to y_j, namely if $y_j \in V_{i+1}$, then the path is $v_{i\ell_{j+1}}$, $v_{i(\ell_{j+1}+1)}$, $\ldots$, v_{ik_j}, $v_{(i+1)k_j}$, while if $y_j \in V_{i-1}$, the path is $v_{i\ell_{j+1}}$, $v_{(i-1)\ell_{j+1}}$, $v_{(i-1)(\ell_{j+1}+1)}$, $\ldots$, $v_{(i-1)k_j}$. The path Q_j contains an arc e_j from X^+ to Y^+, and the arcs $e_1, \ldots, e_n$ are all distinct. This again contradicts the choice of the cut (X^+, Y^+). So Case 2 is also impossible.

By analogy, we prove the remaining inequality:

- $\mu_{G[X \cup V_i]}(Y \cap V_i) \leq n$ for all $i \in \{1, \ldots, n\}$.

This completes the proof of Lemma 3. □

Lemma 4. *Let N, n be positive integers and G a row-sparse induced subgraph of $Z_{N,N}$. If G contains no induced subgraph isomorphic to $Z_{n,n}$, then* $\operatorname{cwd}(G) \leq 24n^3 - 12n^2$.

Proof. Let G be a row-sparse induced subgraph of $Z_{N,N}$ such that G does not contain $Z_{n,n}$ as an induced subgraph. We may assume $n \geq 2$ or else there is nothing to prove (G has no vertices). We may also assume that n divides N, since we can always enlarge N to achieve this. Let $V_1, \ldots, V_N$ denote the columns of $Z_{N,N}$, i.e. $V_i = \{v_{ij} \mid j \in \{1, \ldots, N\}\}$.

Let us denote $t = N/n$ and let us split $Z_{N,N}$ into t blocks, each containing n consecutive columns of the grid. Also, let R_i denote the set of vertices of G in the i-th block. More formally, for each $i \in \{1, \ldots, t\}$,

R_i is the set of vertices of G in columns $V_{(i-1)\cdot n+1}, V_{(i-1)\cdot n+2}, \ldots, V_{i\cdot n}$.

The application of Lemma 3 to $G[R_i]$ yields a partition $X_i \cup Y_i = R_i$, where

(i) $|X_i \cap V_{i\cdot n}| \leq n - 1$,

(ii) $|Y_i \cap V_{(i-1)\cdot n+1}| \leq n - 1$,

(iii) $\mu_{G[Y_i \cup V_j]}(X_i \cap V_j) \leq n$ and $\mu_{G[X_i \cup V_j]}(Y_i \cap V_j) \leq n$ for all $(i-1) \cdot n < j \leq i \cdot n$.

To prove the lemma, we use the sets X_i, Y_i ($1 \leq i \leq t$) to construct a partition $U_0, U_1, \ldots, U_t$ of $V(G)$ as follows: $U_0 = X_1$, $U_t = Y_t$, and for $i \in \{1, \ldots, t-1\}$, $U_i = Y_i \cup X_{i+1}$. Then the result follows by applying Lemma 1 to this partition. To complete the proof, we need to show that the partition possesses the desired properties.

Claim 2. $\mu_G(X_1) \leq n^2$ *and* $\mu_G(Y_t) \leq n^2$.

Proof. Assume $\mu_G(X_1) \geq n^2 + 1$. Then for some column V_j in the first block, the set $X_1 \cap V_j$ contains $n+1$ vertices $x_1, \ldots, x_{n+1}$ with pairwise different neighbourhoods in $V(G) \setminus X_1$. By (ii) we know that $j \leq n - 1$, i.e. V_j is not the last column of the first block. Hence the vertices $x_1, \ldots, x_{n+1}$ all have the same neighbourhood in $V(G) \setminus (X_1 \cup Y_1)$, by the definition of $Z_{N,N}$. Thus the vertices $x_1, \ldots, x_{n+1}$ have pairwise different neighbourhoods in Y_1. This contradicts (iii) and proves that $\mu_G(X_1) \leq n^2$. For the other inequality, the proof is analogous. □

Claim 3. *For all* $1 \leq i < t$*, we have* $\mu_G(X_{i+1} \cup Y_i) \leq 4n^2 - 2n$.

Proof. Suppose that $\mu_G(X_{i+1} \cup Y_i) \geq 4n^2 - 2n + 1$. Then by the Pigeonhole Principle, there are two possibilities.

Case 1. There exists a column V_j in the i-th block (i.e. with $(i-1)\cdot n < j \leq i\cdot n$) such that $V_j \cap Y_i$ contains at least $2n$ vertices $y_1, y_2, \ldots, y_{2n}$ with pairwise different neighbourhoods in $V(G) \setminus (X_{i+1} \cup Y_i)$.

Since the vertices $y_1, \ldots, y_{2n}$ all belong to the same column, we may assume, without loss of generality, that $N(y_1) \subseteq N(y_2) \subseteq \cdots \subseteq N(y_{2n})$. This implies that for each $k \in \{1, \ldots, 2n-1\}$, there exists $x_k \in N(y_{k+1}) \setminus N(y_k)$ such that $x_k \notin X_{i+1} \cup Y_i$, since the neighbourhoods of $y_1, \ldots, y_{2n}$ in $V(G) \setminus (X_{i+1} \cup Y_i)$ are pairwise different.

Note that the vertices $x_1, \ldots, x_{2n-1}$ are pairwise distinct. Further, note that all vertices in V_j have the same neighbours in all columns except for V_{j-1} and V_{j+1}. Thus it follows $x_1, \ldots, x_{2n-1}$ are vertices in $V_{j-1} \cup V_{j+1}$.

If V_j is the first column of the i-th block (i.e. if $j = (i-1)\cdot n + 1$), then we contradict (ii), since Y_i may contain at most $n-1$ vertices in V_j in this case. If V_j is neither the first nor the last column of i-th block (i.e. if $(i-1)\cdot n + 2 \leq j \leq i \cdot n - 1$), then vertices $x_1, \ldots, x_{2n-1}$ all belong to X_i, in which case we contradict (iii), because in this case the vertices of Y_i in the same column may have at most n pairwise different neighbourhoods in X_i. Assume now that V_j is the last column of the i-th block (i.e. $j = i \cdot n$). Then $x_1, \ldots, x_{2n-1}$ are vertices in $V_{i\cdot n-1} \cup V_{i\cdot n+1}$ and so they are vertices in $X_i \cup Y_{i+1}$. From (ii) we deduce that at most $n-1$ of them belongs to Y_{i+1}. Thus at least n among $x_1, \ldots, x_{2n-1}$ belongs to X_i. But then at least $n+1$ vertices among $y_1, \ldots, y_{2n}$ have pairwise different neighbourhoods in X_i, contradicting (iii).

Case 2. There exists j such that $V_j \cap X_{i+1}$ contains vertices $x_1, \ldots, x_{2n}$ with pairwise different neighbourhoods in $V(G) \setminus (X_{i+1} \cup Y_i)$. In this case, the proof coincides with that of Case 1, except that Xs and Ys switch roles, and we use (i) in place of (ii). ☐

Claim 4. *For all* $1 \leq i \leq t$*, we have* $\mu_G(R_1 \cup \cdots \cup R_{i-1} \cup X_i) \leq n^2 + n + 1$.

Proof. Let $S = R_1 \cup \cdots \cup R_{i-1} \cup X_i$. For contradiction, suppose that $\mu_G(S) \geq n^2 + n + 2$. Then S contains a collection of $n^2 + n + 2$ vertices with pairwise different neighbourhoods in $V(G) \setminus S$. We observe that all vertices in $V_1 \cup V_2 \cup \cdots \cup V_{(i-1)\cdot n-1}$ have the same neighbourhood in $V(G) \setminus S$. Thus the collection contains at most one vertex from the columns $V_1 \cup V_2 \cup \cdots \cup V_{(i-1)\cdot n-1}$. This implies, by the Pigeonhole principle, that there is a column V_j with $(i-1)\cdot n \leq j \leq i \cdot n$ such that $V_j \cap S$ contains at least $n+1$ vertices $x_1, \ldots, x_{n+1}$ with pairwise distinct neighbourhoods in $V(G) \setminus S$. Since the vertices $x_1, \ldots, x_{n+1}$ belong to the same column, we may assume, without loss of generality, that $N(x_1) \subseteq N(x_2) \subseteq \cdots \subseteq N(x_{n+1})$. From this we deduce that for each $k \in \{1, \ldots, n\}$, the set $(N(x_{k+1}) \setminus N(x_k)) \setminus S$ contains a vertex, denote it y_k. Observe that the vertices $y_1, \ldots, y_n$ are all distinct. Moreover, note that all vertices in V_j have the same neighbourhood in all columns except for V_{j-1} and V_{j+1}. Thus $y_1, \ldots, y_n$ are vertices in $V_{j-1} \cup V_{j+1}$.

This implies that if $i > 1$ and $j = (i-1)\cdot n$, then the vertices $y_1, \ldots, y_n$ belong to V_{j+1}, since $V_{j-1} \cap V(G) \subseteq S$. But this contradicts (ii). Similarly, if $j = i \cdot n$, then the vertices $x_1, \ldots, x_{n+1}$ belong to $X_i \cap V_{i\cdot n}$, which is impossible by (i) since the set $X_i \cap V_{i\cdot n}$ can contain at most $n-1$ vertices. Finally, assume that $(i-1)\cdot n < j < i \cdot n$. Then the vertices $x_1, \ldots, x_{n+1}$ belong to X_i and the vertices $y_1, \ldots, y_n$ to Y_i. This contradicts (iii), because in this case a column of X_i can have at most $n-1$ vertices with pairwise different neighbourhood in Y_i. ☐

As the last bit of the proof of Lemma 4, we observe that any set in the partition $U_0, U_1, \ldots, U_t$ of $V(G)$ occupies at most $2n$ consecutive columns of the grid and hence $\mathrm{cwd}(G[U_i]) \leq 6n$ by Lemma 2. Combining this inequality with Claims 2, 3, 4, we conclude by Lemma 1 that $\mathrm{cwd}(G) \leq 24n^3 - 12n^2$. ☐

Now we are ready to prove the main result of the section.

Theorem 4. *The class of bichain graphs is a minimal hereditary class of unbounded clique-width.*

Proof. Consider a bichain graph H. We prove that every bichain graph that does not contain an induced copy of H has bounded clique-width, more specifically has clique-width bounded by a function of $|V(H)|$.

Consider a bichain graph G with no induced H. By Theorem 3, the graph H is an induced subgraph of $Z_{n,n}$ where $n = |V(H)|$. Since H is an induced subgraph of $Z_{n,n}$ and G does not contain an induced H, it also does not contain an induced $Z_{n,n}$. By Theorem 3, the graph G is a row-sparse induced subgraph of $Z_{N,N}$ for some N. The two facts together allow us to conclude by Lemma 3 that $\mathrm{cwd}(G) \leq 24n^3 - 12n^2$.

This proves that the clique-width of G is bounded by a function depending only on H, as required. Therefore, the class of bichain graphs is indeed a minimal hereditary class of unbounded clique-width. □

3.1 An alternative proof

We observe that the proof given above is conceptually similar to that given in [12] for bipartite permutation graphs. In this section, we develop an entirely different approach to proving Theorem 4, which *reduces* the problem to bipartite permutation graphs. This approach is of independent interest, because it provides a new tool for proving results in this area. The proof given above allows us to reduces the description of the alternative approach to a sketch.

The alternative proof is done by means of so-called pivoting operation (to be defined later), which does not change rank-width, and hence, does not change clique-width "too much". We will show that a Z-grid can be pivoted into an X-grid. To this end, we need an intermediate construction called Y-grid.

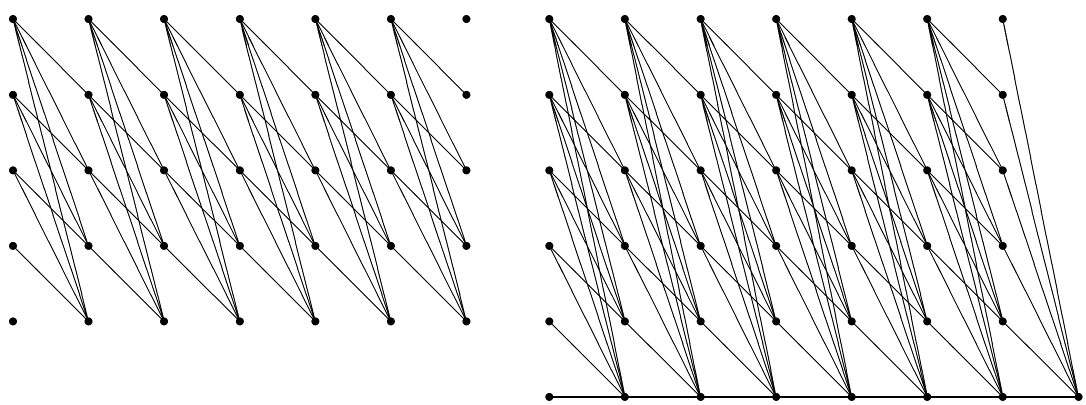

Figure 6: Graphs $Y_{7,5}$ (left) and $Y^+_{7,5}$ (right)

A Y-grid is one more grid-like graph. A graph of this form with n columns and k rows is denoted $Y_{n,k}$ and an example of this graph with $n = 7$ and $k = 5$ is shown in Figure 6 (left). By adding to $Y_{n,k}$ an extra line at the bottom as shown in Figure 6 (right) we obtain a grid which we denote $Y^+_{n,k}$. The example shown in Figure 6 (right) represents the graph $Y^+_{n,k}$ with $n = 7$ and $k = 5$ (note that $Y^+_{n,k}$ contains $k + 1$ rows). We will refer to graphs of the form $Y_{n,k}$ and $Y^+_{n,k}$ as Y-grids and Y^+-grids, respectively.

It is not difficult to see that a Y-grid is simply a modification of an X-grid obtained by shifting the vertices within each column so that every horizontal line of the X-grid turns 45° clockwise. Therefore, any Y-grid is a bipartite permutation graph and hence can be embedded into an X-grid. With a bit of care, one can verify that a Y^+-grid is embeddable into an X-grid as well. Figure 7 (left) illustrates how $Y^+_{3,3}$ can be embedded into $X_{6,6}$. On the other hand, an X-grid can be embedded into a Y-grid and hence into a Y^+-grid, as exemplified in Figure 7(right).

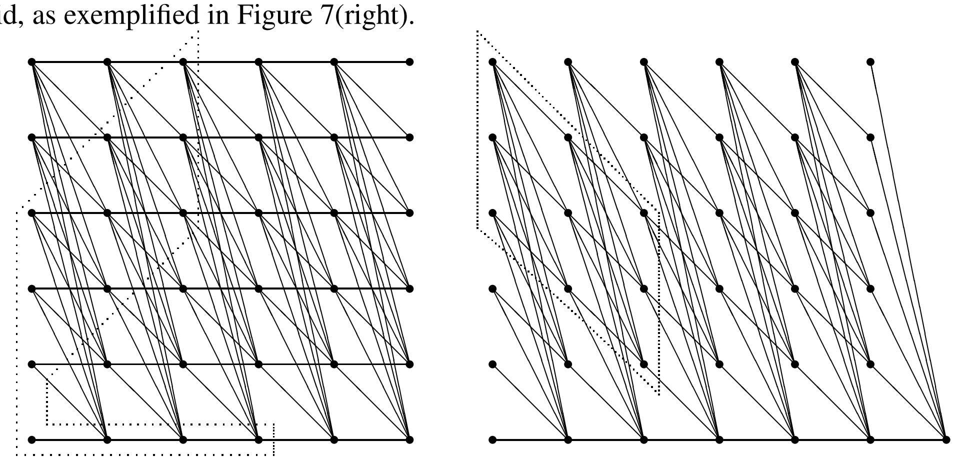

Figure 7: On the left, the graph $X_{6,6}$ contains the graph $Y^+_{3,3}$ surrounded by a dotted boundary. On the right, the graph $Y^+_{6,5}$ contains the graph $X_{3,3}$ surrounded by a dotted boundary.

Both examples can be easily generalized to the following statement.

Lemma 5. *The graph $X_{2n,2n}$ contains $Y_{n,n}$ and $Y^+_{n,n}$ as induced subgraphs, and the graphs $Y_{2n,2n}$ and $Y^+_{2n,2n}$ contain $X_{n,n}$ as an induced subgraph.*

Let ab be an edge in a bipartite graph, $A = N(a) - \{b\}$ and $B = N(b) - \{a\}$. The operation of pivoting consists in complementing the edges between A and B. i.e. replacing every edge xy ($x \in A$ and $y \in B$) with a non-edge and vice versa.

We want to show that a Z-grid can be transformed by a sequence of pivoting operations into a bipartite permutation graph. Since the pivoting operation is very sensitive, we apply it to a particular form of the Z-grid. First, we restrict ourselves to graphs of the form $Z_{n,k}$ only with odd values of n. Second, we extend $Z_{n,k}$ in a specific way by adding to it an extra bottom line and denote the resulting graph by $Z^+_{n,k}$. Formally speaking, the graph $Z^+_{n,k}$ can be obtained from the graph $Z_{n+2,k+1}$ by deleting the bottom-left vertex and all vertices of the last two columns except the bottom vertices. An example of the graph $Z^+_{n,k}$ with $n = 7$ and $k = 5$ is shown on the left of Figure 8. The bottom line is labelled by 0. We will call the bottom line of the graph $Z^+_{n,k}$ the pivoting line, which is explained by the following lemma.

Lemma 6. *The sequence of pivoting operations applied to the bottom edges of the graph $Z^+_{n,k}$ in the order from left to right starting from the second edge transforms $Z^+_{n,k}$ into the graph $Y^+_{n,k}$.*

Proof. The result follows by direct inspection. The only peculiarity of this transformation is that after the pivoting, every vertex of the bottom line in an odd column i moves to column $i - 2$, as illustrated in Figure 8. $\square$

3.1.1 On more proof of Theorem 4

Let $\mathcal{A}$ be a proper hereditary subclass of bichain graphs, i.e. a subclass obtained by forbidding at least one bichain graph. According to Theorem 3, every graph $G \in \mathcal{A}$ with n vertices can be embedded into a

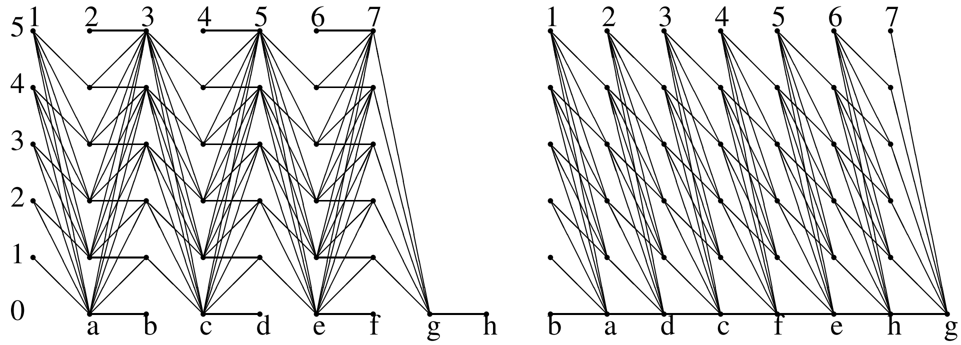

Figure 8: The graph $Z^+_{7,5}$ (on the left, with diagonal edges omitted) transforms into the graph $Y^+_{7,5}$ (on the right) by pivoting on the edges cd, ef, gh.

Z-grid $Z_{n,n}$. We extend $Z_{n,n}$ to $Z^+_{n,n}$ by adding to it a pivoting line. We also add the pivoting line to G and denote the resulting graph by G_+.

By pivoting on the edges of the pivoting line, we transform G_+ into a graph which we denote by G^*_+, and by deleting the pivoting line from G^*_+ we obtain a graph denoted by G^*. According to Lemma 5 and Theorem 1, G^*_+ (and hence G^*) is a bipartite permutation graph.

For an arbitrary bipartite permutation graph H, let $x(H)$ be the maximum n (the x-number) such that H contains $X_{n,n}$ as an induced subgraph, and for an arbitrary bichain graph H we denote by $z(H)$ the maximum n (the z-number) such that H contains $Z_{n,n}$ as an induced subgraph. We also denote

$$\mathcal{A}_+ = \{G_+ \ : \ G \in \mathcal{A}\}$$

$$\mathcal{A}^*_+ = \{G^*_+ \ : \ G_+ \in \mathcal{A}_+\}$$

$$\mathcal{A}^* = \{G^* \ : \ G^*_+ \in \mathcal{A}^*_+\}$$

Assume the x-number is unbounded for graphs in $\mathcal{A}^*$, i.e. assume that graphs in $\mathcal{A}^*$ contain arbitrarily large induced copies of $X_{n,n}$. Then, by Lemma 5, they also contain arbitrarily large induced copies of $Y_{n,n,}$. With the help of Lemma 6 it is not difficult to see that if a set of vertices induces in a graph $G^* \in \mathcal{A}^*$ a large copy of $Y_{n,n,}$, then the same set of vertices induces in $G \in \mathcal{A}$ a graph containing a large copy of $Z_{n,n}$ (we talk about embeddings of G and G^* into a Z-grid and Y-grid, respectively). Therefore, if the x-number is unbounded for graphs in $\mathcal{A}^*$, then the z-number is unbounded for graphs in $\mathcal{A}$. But then, by Theorem 3, $\mathcal{A}$ must contain all bichain graphs, contradicting our assumption. This contradiction shows that the x-number is bounded by a constant, say k, for graphs in $\mathcal{A}^*$, i.e. these graphs are $X_{k,k}$-free. Therefore, graphs in $\mathcal{A}^*_+$ are $H_{k+1,k+1}$-free, since by adding one line of the grid we can increase the x-number by at most one. By Theorem 2 this implies that graphs in $\mathcal{A}^*_+$ have bounded clique-width. Therefore, they also have bounded rank-width. Since pivoting does not change rank-width [15], graphs in $\mathcal{A}_+$ also have bounded rank, and hence, bounded clique-width. As a result, graphs in $\mathcal{A}$ have bounded clique-width.

4 One more minimal class of bipartite graphs of unbounded clique-width

Let us denote by F_n the bipartite complement of a 1-regular graph with $2n$ vertices. In other words, F_n is a bipartite graph in which every vertex has exactly one non-neighbour in the opposite part. Further, let $F_{k,n}$ be the graph with $k \times n$ vertices arranged in k rows, each of length n, in which every two consecutive rows induce an F_n. Throughout the section we denote by $v_{i,j}$ the vertex of $F_{n,n}$ in row i and column j.

Let $\mathcal{F}$ be the hereditary closure of the set $\{F_{n,n} : n \geq 1\}$, i.e the set of graphs containing all graphs of the form $F_{n,n}$ and all their induced subgraphs. By definition, $\mathcal{F}$ is a hereditary class. In this section, we prove that $\mathcal{F}$ is a minimal hereditary class of unbounded clique-width. We start by showing that clique-width is unbounded in $\mathcal{F}$.

Theorem 5. *The clique-width of the graph $F_{n,n}$ is at least $\lfloor n/2 \rfloor$.*

Proof. Let $\mathrm{cwd}(F_{n,n}) = t$. Denote by τ a t-expression defining $F_{n,n}$ and by $\mathit{tree}(\tau)$ the rooted tree representing τ. The subtree of $\mathit{tree}(\tau)$ rooted at a node x will be denoted $\mathit{tree}(x, \tau)$. This subtree corresponds to a subgraph of $F_{n,n}$, which will be denoted $F(x)$. The label of a vertex v of the graph $F_{n,n}$ at the node x is defined as the label that v has immediately prior to applying the operation x.

Let a be a lowest $\oplus$-node in $\mathit{tree}(\tau)$ such that $F(a)$ contains a full row of V. Denote the children of a in $\mathit{tree}(\tau)$ by b and c. Let us color all vertices in $F(b)$ blue and all vertices in $F(c)$ red, and the remaining vertices of $F_{n,n}$ yellow. Note that by the choice of a the graph $F_{n,n}$ contains a non-yellow row (i.e. a row each vertex of which is non-yellow), but none of its rows is entirely red or blue. We denote a non-yellow row of $F_{n,n}$ by r. Without loss of generality we assume that $r \leq \lceil n/2 \rceil$ and that the row r contains at least $n/2$ red vertices, since otherwise we could consider the rows in reverse order and swap colors red and blue.

Observe that edges of $F_{n,n}$ between different colored vertices are not present in $F(a)$. Therefore, if a non-red vertex distinguishes two red vertices u and v, then u and v must have different labels at the node a. We will use this fact to show that $F(a)$ contains a set U of at least $\lfloor n/2 \rfloor$ vertices with pairwise different labels at the node a. Such a set can be constructed by the following procedure.

1. Set $i = r,\ U = \emptyset$ and $J = \{j : v_{r,j} \text{ is red}\}$.
2. Set $K = \{j \in J : v_{i+1,j} \text{ is non-red}\}$.
3. If $K \neq \emptyset$, add the vertices $\{v_{i,k} : k \in K\}$ to U. Remove members of K from J.
4. If $J = \emptyset$, terminate the procedure.
5. Increase i by 1. If $i = n$, choose an arbitrary $j \in J$, put $U = \{v_{m,j} : r \leq m \leq n-1\}$ and terminate the procedure.
6. Go back to Step 2.

It is not difficult to see that this procedure must terminate. To complete the proof, it suffices to show that whenever the procedure terminates, the size of U is at least $\lfloor n/2 \rfloor$ and the vertices in U have pairwise different labels at the node a

First, suppose that the procedure terminates in Step 5. Then U is a subset of red vertices from at least $\lfloor n/2 \rfloor$ consecutive rows of column j. Consider two vertices $v_{l,j}$, $v_{m,j} \in U$ with $l < m$. According to the above procedure, $v_{m+1,\,j}$ is red. Since $F_{n,n}$ does not contain an entirely red row, there must exist a non-red vertex w in row $m+1$. According to the structure of $F_{n,n}$, vertex w is adjacent to $v_{m,j}$ and non-adjacent to $v_{l,j}$. We conclude that $v_{l,j}$ and $v_{m,j}$ have different labels. Since $v_{l,j}$ and $v_{m,j}$ have been chosen arbitrarily, the vertices of U have pairwise different labels.

Now suppose that the procedure terminates in Step 4. By analyzing Steps 2 and 3, it is easy to deduce that U is a subset of red vertices of size at least $\lfloor n/2 \rfloor$. Suppose that $v_{l,j}$ and $v_{m,k}$ are two vertices in U with $l \leq m$. The procedure certainly guarantees that $j \neq k$ and that both $v_{l+1,j}$ and $v_{m+1,k}$ are non-red. If $m \in \{l, l+2\}$, then it is clear that $v_{l+1,j}$ distinguishes vertices $v_{l,j}$ and $v_{m,k}$, and therefore these vertices have different labels. If $m \notin \{l, l+2\}$, we may consider vertex $v_{m-1,k}$ which must be red. Since $F_{n,n}$ does not contain an entirely red row, the vertex $v_{m,k}$ must have a non-red neighbor w in row

$m-1$. But w is not a neighbor of $v_{l,j}$, trivially. We conclude that $v_{l,j}$ and $v_{m,k}$ have different labels, and therefore, the vertices of U have pairwise different labels. The proof is complete. $\square$

By Theorem 5, the clique-width of graphs in $\mathcal{F}$ is unbounded. Now we turn to proving that $\mathcal{F}$ is a minimal hereditary class of unbounded clique-width. First, with the help of Lemma 1 we derive the following conclusion.

Proposition 1. *The clique-width of $F_{k,n}$ is at most $2k$.*

Proof. Denote by V_i the i-th column of $F_{k,n}$. Since each column induces an independent set, it is clear that $\mathrm{cwd}(G[V_i]) \leq 2$ for every i. Trivially, $\mu(V_i) \leq k$, since $|V_i| = k$. Also, denoting $W_i := V_1 \cup ... \cup V_i$, it is not difficult to see that $\mu(W_i) \leq k$ for every i, since the vertices of the same row in W_i are W_i-similar. Now the conclusion follows from Lemma 1. $\square$

Now we use Lemma 1 and Proposition 1 to prove the following result.

Lemma 7. *For any fixed $k \geq 1$, the clique-width of $F_{k,k}$-free graphs in the class $\mathcal{F}$ is bounded by a function of k.*

Proof. Let k be a fixed number and G be a $F_{k,k}$-free graph in $\mathcal{F}$. By definition of $\mathcal{F}$, the graph G is an induced subgraph of $F_{n,n}$ for some n. For convenience, assume that n is a multiple of k, say $n = tk$. The vertices of $F_{n,n}$ that induce G will be called *black* and the remaining vertices of $F_{n,n}$ will be called *white*. Also, we will refer to the set of vertices of G in the same row of $F_{n,n}$ as a layer of G.

For $1 \leq i \leq t$, let us denote by W_i the subgraph of $F_{n,n}$ induced by the k consecutive rows $(i-1)k+1, (i-1)k+2, \ldots, ik$. For simplicity, we will use the term 'row r of W_i' when referring to the row $(i-1)k+r$ of $F_{n,n}$. We partition the vertices of G into subsets $V_1, V_2, ..., V_t$ according to the following procedure:

1. Set $V_j = \emptyset$ for $1 \leq j \leq t$. Add every black vertex of W_1 to V_1 . Set $i = 2$.
2. For $j = 1, \ldots, n$,
   - if column j of W_i is entirely black, then add the first vertex of this column to V_{i-1} and the remaining vertices of the column to V_i.
   - otherwise, add the (black) vertices of column j preceding the first white vertex to V_{i-1} and add the remaining black vertices of the column to V_i.
3. Increase i by 1. If $i = t+1$, terminate the procedure.
4. Go back to Step 2.

Let us show that the partition $V_1, V_2, ..., V_t$ given by the procedure satisfies the assumptions of Lemma 1 with l and m depending only on k.

The procedure clearly assures that each $G[V_i]$ is an induced subgraph of $W_i \cup W_{i+1}$. By Proposition 1, we have $\mathrm{cwd}(W_i \cup W_{i+1}) = \mathrm{cwd}(F_{2k,n}) \leq 4k$. Since the clique-width of an induced subgraph cannot exceed the clique-width of the parent graph, we conclude that $\mathrm{cwd}(G[V_j]) \leq 4k$, which shows condition (1) of Lemma 1.

To show condition (2) of Lemma 1, let us call a vertex $v_{m,j}$ of V_i boundary if either $v_{m-1,j}$ belongs to V_{i-1} or $v_{m+1,j}$ belongs to V_{i+1} (or both). It is not difficult to see that a vertex of V_i is boundary if it belongs either to the second row of an entirely black column of W_i or to the first row of an entirely black column of W_{i+1}. Since the graph G is $F_{k,k}$-free, the number of columns of W_i which are entirely black is at most $k-1$. Therefore, the boundary vertices of V_i introduce at most $2(k-1)$ equivalence classes in V_i.

Now consider two non-boundary vertices $v_{m,j}$ and $v_{m,p}$ in V_i from the same row. It is not difficult to see that $v_{m,j}$ and $v_{m,p}$ have the same neighborhood outside V_i. Therefore, the non-boundary vertices of the same row of V_i are V_i-similar, and hence the non-boundary vertices give rise to at most $2k$ equivalence classes in V_i. Thus, $\mu(V_i) \leq 4k-2$ for all i.

An identical argument shows that $\mu(V_1 \cup ... \cup V_i) \leq 3k-1 \leq 4k-2$ for all i. Therefore, by Lemma 1, we conclude that $\mathrm{cwd}(G) \leq c(k) := 16k^2 - 8k$, which completes the proof. □

Theorem 6. *$\mathcal{F}$ is a minimal hereditary class of graphs of unbounded clique-width.*

Proof. Let X be a proper hereditary subclass of $\mathcal{F}$ and $H \in \mathcal{F} - X$. Since H is an induced subgraph of $F_{k,k}$ for some k, each graph in X is $F_{k,k}$-free. Therefore, by Lemma 7, the clique-width of graphs in X is bounded by a constant. □

5 Conclusion

In the present paper, we identified two new minimal hereditary classes of graphs of unbounded clique-width. We believe that there are many more such classes and that the problem of identifying all of them can be done through the notion of well-quasi-orderability. In [7], it was conjectured that the clique-width is bounded in every hereditary class of graphs which is well-quasi-ordered by the induced subgraph relation. In other words, every class of graphs of unbounded clique-width contains an infinite antichain with respect to this relation. We further conjecture that every *minimal* class of unbounded clique-width contains a *canonical* antichain (i.e. an antichain which, in a sense, is unique). For the first two minimal classes (bipartite permutation and unit interval graphs) this was verified in [13]. In a separate publication, we show this for the class of bichain graphs.

In the problem of identifying infinite antichains with respect to the induced subgraph relation, the main tool is the notion of so-called *factor graphs*: these are graphs that are given together with a linear order of its vertices and any embedding of one graph of this type into another graph must follow the order. A trivial example of factor graphs are chordless paths, and the canonical antichains of both bipartite permutation and unit interval graphs are based on these graphs. In the case of bichain graphs (that do bot contain large chordless paths) the factor graphs are induced by the diagonal vertices of the Z-grid.

The problem of identifying infinite antichains with respect to the induced subgraph relation can be solve through a language-theoretic approach, because the induced subgraph relation on factor graphs is equivalent to the factor containment relation on words. For languages, this problem was recently solved in [1]. Once an infinite antichain with respect to the induced subgraph relation is identified, a minimal class of unbounded clique-width can be created by means of simple building blocks (that include chain graphs and other minimal classes in the factorial range of growth of the family of hereditary classes of graphs) stringed along this antichain.

References

[1] A. Atminas, V.V. Lozin, and M. Moshkov, Deciding WQO for factorial languages, *Lecture Notes in Computer Science*, 7810 (2013) 68–79.

[2] A. Atminas, S. Kitaev, V. V. Lozin, A. Valyuzhenich, Universal graphs and universal permutations, *Discrete Mathematics, Algorithms and Applications*, 5 (2013), no. 4, 1350038, 15 pp.

[3] C. Benzaken, P.L. Hammer, D. de Werra, Split graphs of Dilworth number 2, *Discrete Math*. 55 (1985) 123–127.

[4] R, Brignall, V. Lozin, J. Stacho, Bichain graphs: geometric model and universal graphs, *submitted*.

[5] B.-M. Bui-Xuan, J. A. Telle, and M. Vatshelle, Boolean-width of graphs, *Theoretical Computer Science*, 412(39) (2011) 5187–5204.

[6] B. Courcelle, J. Engelfriet, G. Rozenberg, Handle-rewriting hypergraph grammars, *Journal of Computer and System Sciences* 46 (1993) 218–270.

[7] J. Daligault, M. Rao and S. Thomassé, Well-Quasi-Order of Relabel Functions, *Order* 27 (2010) 301–315.

[8] M.C. Golumbic, U. Rotics, On the clique-width of some perfect graph classes, *International Journal of Foundations of Computer Science*, 11 (2000) 423–443.

[9] G. Gottlob, P. Hlinený, S.-I. Oum, D. Seese, Width Parameters Beyond Tree-width and Their Applications, *The Computer Journal*, 51 (2008) (3) 326–362.

[10] M. Kaminski, P. Medvedev, M. Milanič, The plane-width of graphs, *Journal of Graph Theory* 68 (2011) 229–245.

[11] N. Korpelainen, V.V. Lozin, and C. Mayhill, Split permutation graphs, *Graphs and Combinatorics*, accepted.

[12] V.V. Lozin, Minimal classes of graphs of unbounded clique-width, *Annals of Combinatorics*, 15 (2011) 707–722.

[13] V.V. Lozin, C. Mayhill, Canonical antichains of unit interval and bipartite permutation graphs, *Order*, 28 (2011) 513–522.

[14] V.V. Lozin, G. Rudolf, Minimal universal bipartite graphs *Ars Combinatoria* 84 (2007) 345–356.

[15] S.-i. Oum, Rank-width and vertex-minors. *J. Combin. Theory Ser. B* 95 (2005), no. 1, 79-100.

[16] S.-i. Oum, P. Seymour, Approximating clique-width and branch-width. *J. Combin. Theory Ser. B* 96 (2006) 514-528.

[17] M. Petkovšek, Letter graphs and well-quasi-order by induced subgraphs. *Discrete Math*. 244 (2002), no. 1-3, 375-388.

[18] N. Robertson and P.D. Seymour, Graph minors. I. Excluding a forest, Journal of Combinatorial Theory, Series B 35 (1) (1983) 39-61.

[19] N. Robertson and P.D. Seymour, Graph minors III: Planar tree-width, Journal of Combinatorial Theory, Series B, 36 (1) (1984) 49-64.

[20] N. ROBERTSON and P.D. SEYMOUR, Graph minors. V. Excluding a planar graph, *J. Combinatorial Theory Ser. B* 41 (1986) 92–114.