A Decomposition Theorem for Chordal Graphs
and its Applications

Michel Habib\textsuperscript{1}  Juraj Stacho\textsuperscript{1,2}

\textit{LIAFA – CNRS and Université Paris Diderot – Paris VII}
\textit{Case 7014, 75205 Paris Cedex 13, France}

Abstract
We introduce a special decomposition, the so-called split-minors, of the reduced clique graphs of chordal graphs. Using this notion, we characterize asteroidal sets in chordal graphs and clique trees with minimum number of leaves.

Keywords: chordal graph, asteroidal set, split-minor, leafage, polynomial time

1 Introduction

In this paper, graph is always simple, undirected and loopless.

For a vertex $v$ of a graph $G$, we denote by $N(v)$ the neighbourhood of $v$ in $G$, i.e., the set of vertices $u$ such that $uv \in E(G)$; we denote by $N[v]$ the set $N(v) \cup \{v\}$. For a set $X \subseteq V(G)$, we denote by $G[X]$ the graph induced on $X$, i.e., graph $G[X] = (X, E \cap X \times X)$, and denote by $G - X$ the graph $G[V(G) \setminus X]$.

A complete subgraph or a clique of $G$ is a (not necessarily maximal) set of pairwise adjacent vertices of $G$. (For a complete terminology, see \cite{5}.)

A set $A$ of vertices of a graph $G$ is asteroidal, if for each vertex $v$ of $A$, the set $A \setminus \{v\}$ belongs to one connected component of $G - N[v]$.

\textsuperscript{1} Email: \{habib,jstacho\}@liafa.jussieu.fr
\textsuperscript{2} Author supported by the Foundation Sciences Mathématiques de Paris.
The asteroidal number $a(G)$ of $G$ is the size of a largest asteroidal set of $G$. Computing $a(G)$ is $NP$-hard already in planar graphs [3], but it is efficiently solvable in chordal graphs [3] (and other structured classes of graphs).

The leafage $l(G)$ of a connected chordal graph $G$ is the least number of leaves in a clique tree of $G$. If $G$ is disconnected, $l(G)$ is defined as the maximum leafage over all connected components of $G$. While testing $l(G) \leq k$ for $k \in \{2, 3\}$ is polynomial [4], the complexity of computing $l(G)$ is not known.

In this paper, we introduce a notion of a split-minor of the reduced clique graph of a chordal graph (Section 3). This novel tool allows us to obtain a total decomposition of the reduced clique graph well suited for algorithmic use. We apply this tool (Section 4) to characterize asteroidal sets of chordal graphs, and give a partial condition for the leafage.

2 The reduced clique graph

Let $G$ be a connected chordal graph. A clique tree of $G$ is any tree $T$ whose vertices are the maximal cliques of $G$ such that for every two maximal cliques $C, C'$, each clique on the path from $C$ to $C'$ in $T$ contains $C \cap C'$.

Two cliques $C, C'$ of $G$ form a separating pair, if every path from a vertex of $C \setminus C'$ to a vertex of $C' \setminus C$ contains a vertex of $C \cap C'$.

The reduced clique graph $\mathcal{C}_r(G)$ of $G$ is a graph whose vertices are the maximal cliques of $G$, and whose edges $CC'$ are between cliques $C, C'$ forming separating pairs. In addition, each edge $CC'$ of $\mathcal{C}_r(G)$ is labeled by $C \cap C'$.

The following is a fundamental result about reduced clique graphs.

**Theorem 2.1** [1] Every clique tree of $G$ is a maximum weight spanning tree of $\mathcal{C}_r(G)$ where the weight of each edge $CC'$ is defined as $|C \cap C'|$. Moreover, the reduced clique graph $\mathcal{C}_r(G)$ is precisely the union of all clique trees of $G$.

If $G$ is disconnected, the reduced clique graph $\mathcal{C}_r(G)$ of $G$ is defined as the disjoint union of the reduced clique graphs of its connected components.

3 Split-minors

Let $H$ be a graph, and let $e$ be an edge of $H$. By $H/_{e}$ we denote the graph obtained from $H$ by contracting the edge $e$ to a new vertex $v_e$ (i.e., removing $x$ and $y$, adding $v_e$, and connecting all neighbours of $x$ and neighbours of $y$ to $v_e$).

Let $X \cup Y$ be a partition of $V(H)$. We say that $X \cup Y$ is a split of $H$ if every vertex of $X$ with a neighbour in $Y$ has the same neighbourhood in $Y$.

We say that a graph $H'$ is a split-minor of $H$, if $H'$ can be obtained from $H$ by performing a sequence of the following three operations:
(S1) if $v$ is an isolated vertex, remove $v$.
(S2) if $e$ is an edge, contract $e$.
(S3) if $X \cup Y$ is a split, remove all edges between $X$ and $Y$.

Now, suppose that $H$ has labeled edges. We say that an edge $e = xy$ of $H$ is permissible, if for every triangle $x, y, z$, the edges $xz$ and $yz$ have the same label. If $e$ is a permissible edge of $H$, we denote by $H/e$ the graph with labeled edges obtained by contracting $e$ to a new vertex $v_e$ and assigning labels as follows: for every neighbour $z$ of $v_e$, the edge $v_ez$ is labeled using the label of $xz$ if $xz \in E(H)$, or using the label of $yz$ if $yz \in E(H)$, and all other edges are labeled using the same label they have in $H$. We say that a split $X \cup Y$ of $H$ is permissible, if all edges between $X$ and $Y$ have the same label.

We say that a graph $H'$ with labeled edges is a labeled split-minor of $H$ if $H'$ can be obtained from $H$ by a sequence of the following operations:

(L1) if $v$ is an isolated vertex, remove $v$.
(L2) if $e$ is a permissible edge, contract $e$.
(L3) if $X \cup Y$ is a permissible split, remove all edges between $X$ and $Y$.

If $e = CC'$ is an edge of $C_r(G)$, we denote by $G//e$ the graph obtained from $G$ by adding all possible edges between the vertices of $C$ and $C'$.

**Theorem 3.1** If $H'$ is a labeled split-minor of $H = C_r(G)$, then there exists a graph $G'$ such that $H' = C_r(G')$.

**Proof.** (Sketch) By induction. Let $H'$ be the graph obtained from $C_r(G)$ by one of the three operations. Suppose we apply (L1) to a vertex $C$ of $C_r(G)$. Then $C$ forms a connected component of $G$, and hence, $C_r(G - C) = H'$. Suppose we apply (L3) to a split $X \cup Y$. Then we let $V_X \subseteq V(G)$, respectively $V_Y \subseteq V(G)$, be the union of all maximal cliques that are elements of $X$, respectively $Y$. (Note that $V_X \cap V_Y \neq \emptyset$.) We let $G'$ be the disjoint union of $G[V_X]$ and $G[V_Y]$, and it follows that $C_r(G') = H'$. Finally, suppose we apply (L2) to an edge $e = CC'$. Then it is not difficult to show that $C_r(G//e) = H'$. □

An edge $e$ of $H$ is maximal, resp. minimal, if there is no edge $e'$ in $H$ whose label strictly contains $\subseteq$), resp. is strictly contained $\supset$ in, the label of $e$.

**Observation 3.2** Every maximal edge of $C_r(G)$ is permissible, and every minimal edge of $C_r(G)$ is an edge between sets $X, Y$ of a permissible split $X \cup Y$.

As a consequence of this observation we obtain the following theorem.

**Theorem 3.3** (Split-minor decomposition) Every reduced clique graph is totally decomposable with respect to (L1),(L3) and also with respect to (L2).
4 Applications

A vertex $v$ of $H$ is $S$-dominated, if $S$ is a subset of the label of every edge incident to $v$. An edge $e = xy$ with label $S$ is good, if $e$ is permissible in $H$, and no vertex of $H$ is $S$-dominated unless at least one of $x, y$ is $S$-dominated.

We say that $H'$ is a good split-minor of $H$, if $H'$ can be obtained from $H$ by a sequence of operations (L1),(L3), and the following:

(L2) if $e$ is a good edge, contract $e$.

**Theorem 4.1** If $H'$ is a good split-minor of $C_r(G)$, and $G'$ is the graph from Theorem 3.1 such that $H' = C_r(G')$, then $a(G') \leq a(G)$.

**Proof.** (Sketch) By induction. The case of (L1) and (L3) is easy to handle. Suppose that $C_r(G')$ is obtained by contracting a good edge $e = CC'$ using (L2). Let $S = C \cap C'$. Then $G' = G/e$ is obtained from $G$ by making $C \cup C'$ a clique. Let $A$ be a largest asteroidal set of $G'$. Suppose that $A$ is not asteroidal in $G$. We may assume $|A| \geq 3$. Hence, there exist $a_1, a_2, a_3 \in A$ such that $a_1$ and $a_3$ are in different connected components of $G - N_G[a_2]$. Since $A$ is asteroidal in $G'$, we deduce $S \subseteq N_G[a_2]$, and there is no other vertex $a \in A$ with $S \subseteq N_G[a]$. Hence, we let $C^*$ be a maximal clique that contains $\{a_2\} \cup S$. It now follows that both $CC^*$ and $C'^*C^*$ are edges of $C_r(G)$ with label $S$.

Next, we let $A' = A \setminus \{a_2\}$ and observe that $A'$ is asteroidal in both $G'$ and $G$. After that, we deduce that both $C$ and $C'$ are not $S$-dominated, and hence, since $e$ is a good edge, $C^*$ is also not $S$-dominated. Therefore, there exists a maximal clique $C^{**}$ such that $C'^*C^{**}$ is an edge of $C_r(G)$ with label $S^{**} \not\supseteq S$. We let $a''_2$ be any vertex of $C^{**} \setminus C^*$, and let $A'' = A' \cup \{a''_2\}$.

Finally, we show that $A''$ is an asteroidal set of both $G'$ and $G$. \hfill $\square$

A $k$-star is a graph obtained by taking a vertex with $k$ neighbours forming an independent set. A labeled $k$-star is a $k$-star whose edges are labeled.

**Theorem 4.2** $a(G) < k$ iff no labeled $k$-star is a good split-minor of $C_r(G)$.

**Proof.** (Sketch) If a labeled $k$-star $C_r(G')$ is a good split-minor of $C_r(G)$, it is not difficult to show that the label of no edge of $C_r(G')$ is contained in another edge. Hence, $a(G') = k$, and by Theorem 4.1, we deduce $a(G) \geq k$.

Now, let $A = \{a_1, \ldots, a_k\}$ be an asteroidal set of $G$. Let $C_1, \ldots, C_k$ be maximal cliques of $G$ containing $a_1, \ldots, a_k$, respectively. Let $T$ be a clique tree of $G$, and $T'$ be the subtree of $T$ formed by taking all paths between $C_1, \ldots, C_k$. Since $A$ is asteroidal, $C_1, \ldots, C_k$ are leaves of $T'$. Let $T$ be chosen so that $T'$ is smallest possible. Let $C_iC'_i$ be the (unique) edge incident to $C_i$ in $T'$.
We observe that removal of any minimal edge of $T$ yields two trees whose vertices form a permissible split of $C_r(G)$. Hence, by minimality of $T'$, we can remove or contract all edges whose label does not appear on $T'$. Finally, we contract all edges of $T'$ other than $C_iC'_i$, and obtain a $k$-star.

**Corollary 4.3** If $C_r(G) \cong C_r(G')$ as labeled graphs, then $a(G') = a(G)$.

For the leafage of chordal graphs, we have the following similar statement (proof omitted). We say that a vertex $v$ of $H$ is $S$-bounded, if $v$ is incident to an edge labeled with $S$, and the label of every other edge incident to $v$ is a subset ($\subseteq$) of $S$. An edge $e = xy$ with label $S$ is nice, if $e$ is maximal in $H$ and no vertex of $H$ is $S$-bounded unless at least one of $x, y$ is $S$-bounded.

We say that $H'$ is a nice split-minor of $H$, if $H'$ can be obtained from $H$ by a sequence of operations (L1),(L3), and the following:

(L2$''$) if $e$ is a nice edge, contract $e$.

**Theorem 4.4** If $C_r(G')$ is a nice split-minor of $C_r(G)$, then $l(G') \leq l(G)$.

Unfortunately, we do not have a characterization of $l(G)$ similar to Theorem 4.2, since the minimal forbidden split-minors for the leafage are not easy to describe. However, we can describe other conditions that allow computing the leafage of $G$ from its reduced clique graph. (More in [2].)

We close by noting that the restrictions introduced in the operations (L2$'$) and (L2$''$) still allow for a total decomposition of reduced clique graphs.

**Theorem 4.5** Every reduced clique graph is totally decomposable with respect to (L2$'$) and also with respect to (L2$''$).

**References**


