Vertex Ordering Characterizations of Graphs of Bounded Asteroidal Number

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Abstract

Asteroidal Triple-free (AT-free) graphs have received considerable attention due to their inclusion of various important graphs families, such as interval and cocomparability graphs. The asteroidal number of a graph is the size of a largest subset of vertices such that the removal of the closed neighbourhood of any vertex in the set leaves the remaining vertices of the set in the same connected component. (AT-free graphs have asteroidal number at most 2.) In this paper, we characterize graphs of bounded asteroidal number by means of a vertex elimination ordering, thereby solving a long-standing open question in algorithmic graph theory. Similar characterizations are known for chordal, interval, and cocomparability graphs.

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1. Introduction

There are many ways to characterize various families of graphs including, intersection representations, forbidden subgraphs (induced or minors) and, the subject of this paper, vertex orderings. Chordal graphs have various different characterizations, including:

Forbidden induced subgraphs: No induced cycles of length four or more;

Intersection representation [17, 1]: Subtrees of a tree;

Vertex ordering [13, 16, 27]: for $G = (V, E)$, an ordering of V: $v_1, v_2, \ldots, v_i, \ldots, v_n$ is called a *Perfect Elimination Ordering* (**PEO**) if for every $i \in \{1, \ldots, n\}$, the vertex v_i is *simplicial* in $G_i = G[v_1, \ldots, v_i]$ (i.e., the neighbourhood of v_i in the subgraph of G induced by $\{v_1, \ldots, v_i\}$ is a clique).

As will be pointed out in Section 2.1 some subfamilies of chordal graphs, notably interval and unit interval graphs, also have a Vertex Ordering Characterization (VOC). More recently, various graph searches such as Generic Search, BFS, DFS and LBFS have been shown to have VOCs. The pattern of these characterizations led to the discovery of LDFS and the rediscovery of Maximal Neighbourhood Search (MNS); see Section 2.2 and [9].

In the early 1960s, Lekkerkerker and Boland [22] defined an *Asteroidal Triple (AT)* in a graph to be an independent triple of vertices such that between any two of them there is a path that avoids the neighbourhood of the third. See Figure 1 for various graphs containing asteroidal triples. A graph with no Asteroidal Triples is called *Asteroidal Triple-free (AT-free)*, and Lekkerkerker and Boland in [22] showed that a graph is an interval graph if and only if it is both chordal and AT-free. Starting in the mid 1990s, AT-free graphs have received considerable attention [10] and have been shown to exhibit various types of "linear structure".

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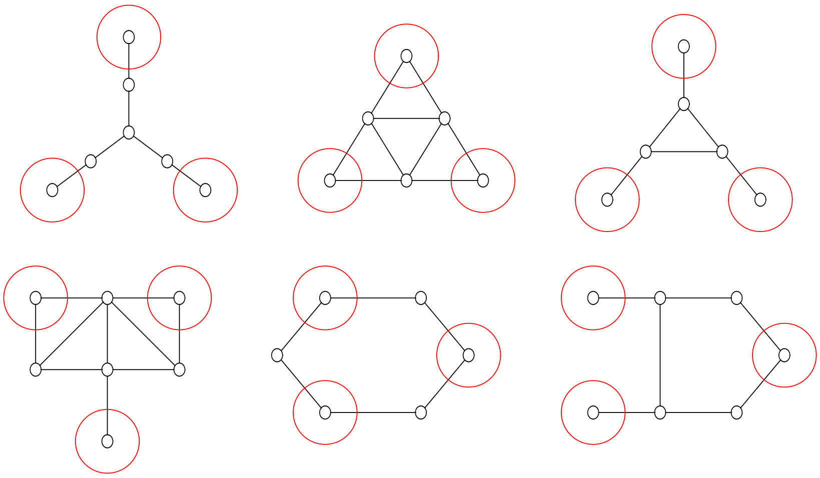


Figure 1: The AT vertices are circled.

In [19], the notion of AT-free graphs was generalized in the following way. Let G be a graph. A set $A \subseteq V(G)$ is *asteroidal*, if for all $a \in A$, the vertices of $A \setminus \{a\}$ belong to one component of $G - N[a]$, where $N[a]$ denotes the *closed neighbourhood* of a, namely $\{a\}$ together with its (open) neighbourhood $N(a)$. The *asteroidal number* of G, denoted by $a(G)$, is the size of a largest asteroidal set of G. Note that AT-free graphs are the ones that have asteroidal number at most two.

In this paper, we solve the long standing open problem of finding a VOC for AT-free graphs in particular, and graphs with bounded asteroidal number in general.

1.1. *Overview of the paper*

In Section 2 we give the required background on VOCs both for subfamilies of AT-free graphs and for graph searches; we also provide some examples of algorithmic results for these families resulting from graph searches. In Section 3 we present the VOC first for AT-free graphs, in particular, and then for graphs of bounded asteroidal number, in general. In both cases the proof is constructive. We then discuss the relationship of the new VOCs with existing VOCs, as well as whether the AT-free VOC can be attained via standard graph searches. The paper ends with a summary of our contributions and some open problems.

2. Background

If σ is an ordering of the vertices of graph G, we write $x <_\sigma y$ to indicate that x appears before y in the ordering σ. Throughout the paper we will assume our graphs are connected and that n and m respectively denote the number of vertices and edges in the graph. We now survey existing results for VOCs both for graph families in the AT-free graph hierarchy and for various graph searches.

2.1. *VOCs for graph families in the AT-free graph hierarchy*

First we define various AT-free subfamilies that will be discussed.

- G is an *interval graph* if it is the intersection graph of intervals of a line (equivalently subpaths of a path, thereby showing that interval graphs are chordal); namely, each vertex represents an interval and two vertices are adjacent if and only if their intervals intersect.

- G is a *unit interval graph* if it is an interval graph with an intersection representation by intervals of the same length (equivalent to *proper interval graphs* where no interval properly contains another [26]).

- G is a *cocomparability graph* if its complement $\overline{G}$ has a transitive orientation of its edges (i.e., if $x \to y$ and $y \to z$ then $x \to z$).

It is easy to see that unit interval graphs are strictly contained in interval graphs which are strictly contained in cocomparability graphs which are strictly contained in AT-free graphs; consider, respectively $K_{1,3}$, C_4, C_5. The VOCs of these three graph classes are as follows:

- Unit Interval VOC Theorem [23]: $G = (V, E)$ is a unit interval graph if and only if there is a vertex ordering (VO) σ of V such that for all $x <_\sigma y <_\sigma z$:

  if $xz \in E$, then $xy, yz \in E$. **(UI ORDER)**

- Interval VOC Theorem [25]: $G = (V, E)$ is an interval graph if and only if there is a VO σ of V such that for all $x <_\sigma y <_\sigma z$:

  if $xz \in E$, then $xy \in E$. **(I ORDER)**

- Cocomparability Graph VOC Theorem [21]: $G = (V, E)$ is a cocomparability graph if and only if there is a VO σ of V such that for all $x <_\sigma y <_\sigma z$:

  if $xz \in E$, then at least one of xy, yz is in E. **(COCOMP ORDER)**

Chang et al. in [2] generalized cocomparability graphs to *k-cocomparability graphs* by generalizing the COCOMP ORDER in the following way:

- k-cocomparability Graph VOC Theorem [2]: $G = (V, E)$ is a k-cocomparability graph if and only if there is a VO σ of V such that for all $x <_\sigma y <_\sigma z$:

  if $dist(x, z) \leq k$, then $dist(x, y) \leq k$ or $dist(y, z) \leq k$, **(k-COCOMP ORDER)**

  where $dist(\cdot, \cdot)$ denotes the distance in G (the length of a shortest path).

Note that cocomparability graphs are precisely the 1-cocomparability graphs. They also proved that AT-Free graphs are strictly contained in 2-cocomparability graphs; strict inclusion is illustrated by the 3-sun (the middle graph in the first row of Figure 1) where any ordering of the vertex set is a 2-COCOMP ORDER, since the distance between any two vertices is at most two.

The first attempt [8] at a VOC for AT-free graphs built off the COCOMP ORDER in the following way: consider a vertex ordering σ where for all $x <_\sigma y <_\sigma z$:

if $xz \notin E$, then every x-z path P satisfies $N[y] \cap P \neq \emptyset$. **(PO ORDER)**

The class of graphs (called *Path Orderable Graphs*) defined by this order strictly contains cocomparability graphs (consider C_5) and is contained in AT-free graphs. It was later shown that the latter inclusion is also strict (consider the graph in Figure 2, which is the smallest known AT-free graph that is not Path Orderable). To see this, suppose otherwise and let σ be a PO ORDER of the graph in Figure 2. Then in σ, note that 4 cannot be between 1 and 6 because of the path $1, 2, 6$ (similarly, 5 cannot be between 1 and 3). Further, the path $8, 7, 5, 6$ dictates that 3 is not between 7 and 6, or between 8 and 6, or between 7 and 5, which implies that 3 is not between 5 and 6 (similarly, 6 is not between 3 and 4). In a similar fashion, the path $3, 4, 5, 6$ dictates that 1 is not between 4 and 5. These five restrictions form a cycle showing that σ cannot exist. In [8] it was shown that recognizing Path Orderable graphs is NP-complete.

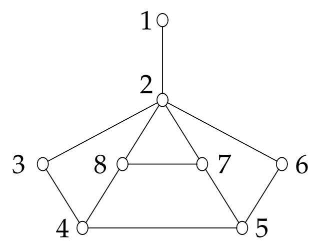


Figure 2: An AT-free graph that is not Path Orderable.

2.2. VOCs for graph searches

We begin by introducing *Generic Search*, first described by Tarjan in the 1970s: start at an arbitrary vertex and at each subsequent stage of the search visit an unvisited vertex that is adjacent to some previously visited vertex. A VOC for a graph search X will be of the form:

"σ, an ordering of V, could be produced by an X search if and only if ...."

In characterizing various graph searches, we consider a vertex ordering σ produced by the search and for every $a <_\sigma b <_\sigma c$ such that $ac \in E$ but $ab \notin E$, we ask:

In the face of the edge ac and the nonedge ab, why is b before c? (see Figure 3.)

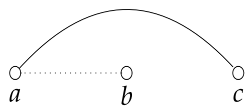


Figure 3: Ordering of vertices a, b, c.

As the following results from [9] show, Generic Search, BFS and DFS are characterized by the position of a vertex d which is a **neighbour** of b. Note that a is a **private neighbour** of c with respect to b (i.e., $ac \in E, ab \notin E$).

- Generic Search VOC Theorem:
  An ordering σ is a **Generic Search ORDER** if and only if for all $a <_\sigma b <_\sigma c$ where $ac \in E$, $ab \notin E$:
  there exists $d <_\sigma b$ such that $db \in E$.
- BFS VOC Theorem:
  An ordering σ is a **BFS ORDER** if and only if for all $a <_\sigma b <_\sigma c$ where $ac \in E$, $ab \notin E$:
  there exists $d <_\sigma a$ such that $db \in E$.
- DFS VOC Theorem:
  An ordering σ is a **DFS ORDER** if and only if for all $a <_\sigma b <_\sigma c$ where $ac \in E$, $ab \notin E$:
  there exists $a <_\sigma d <_\sigma b$ such that $db \in E$.

In the early 1970s, Rose, Tarjan and Lueker [28] introduced Lexicographic BFS (LBFS) and showed that any LBFS of a graph G is a PEO (perfect elimination ordering) of G if and only if G is chordal. This was not only the first linear time recognition algorithm of chordal graphs, but also the first time that a family of graphs was recognized by a specific graph search. LBFS initially assigns a null list to every vertex and then at every iteration chooses an unvisited vertex with the lexicographically largest list. The ith $(1 \leq i < n)$ chosen vertex appends the label $n - i + 1$ to the list of each of its unvisited neighbours.

The VOC characterization of LBFS (given below) is similar to the VOC of BFS (whence its name); the only difference between the BFS and LBFS VOCs is that the LBFS VOC requires the vertex d to be a **private neighbour** of b, with respect to c. In [9], both Generic Search and DFS are examined where d is similarly required to be a private neighbour of b, with respect to c. For Generic Search, this results in Maximal Neighbour Search (MNS) where vertex x chosen at a specific stage has the property that there is no unvisited vertex y where y's neighbourhood in the set of visited vertices strictly contains x's neighbourhood in the set of visited vertices. In the case of LDFS, the private neighbour condition applied to the DFS ordering was shown in [9] to result in a graph search which was named LDFS. It is only recently that applications of LDFS have been found as will be briefly discussed in Section 2.3.

The three searches are characterized as follows.

- MNS VOC Theorem:
  An ordering σ is an **MNS ORDER** if and only if for all $a <_\sigma b <_\sigma c$ where $ac \in E$, $ab \notin E$:
  there exists $d <_\sigma b$ such that $db \in E$ and $dc \notin E$.

- LBFS VOC Theorem [18, 15]:
  An ordering σ is an **LBFS ORDER** if and only if for all $a <_\sigma b <_\sigma c$ where $ac \in E$, $ab \notin E$:
  $$\text{there exists } d <_\sigma a \text{ such that } db \in E \text{ and } dc \notin E.$$
- LDFS VOC Theorem:
  An ordering σ is an **LDFS ORDER** if and only if for all $a <_\sigma b <_\sigma c$ where $ac \in E$, $ab \notin E$:
  $$\text{there exists } a <_\sigma d <_\sigma b \text{ such that } db \in E \text{ and } dc \notin E.$$

Thus any LBFS and any LDFS ORDER is also an MNS ORDER. In [28] it was shown that any MNS ORDER has the property that it is a PEO if and only if the graph is chordal (interestingly, it seems that the authors of [28] did not realize that the 4-vertex condition characterizes MNS). Thus, as corollaries, this holds not only for LBFS (as remarked earlier), but also for LDFS and MCS (Maximum Cardinality Search).

2.3. Interplay between these graph searches and graph families

Applications of graph searches to the various families previously mentioned include recognition, diameter estimation and efficient solutions of problems that are NP-complete in general. The first, and most celebrated, such recognition algorithm is the previously mentioned chordal graph recognition algorithm from [28]. More recently it has been shown that unit interval graphs [3] and interval graphs [12] can be recognized by multi-sweep LBFS algorithms using respectively 3 and 6 sweeps. In such multi-sweep algorithms it is common for a sweep to be applied in a "+" fashion to a previously obtained vertex ordering σ. Under this paradigm when the search has tied vertices and is required to make a choice, it chooses the rightmost tied vertex in σ. It has also been shown that the last vertex of an LBFS of graphs in these various families has eccentricity that is within a small constant of the graph's diameter [5, 14, 15].

In the study of connected AT-free graphs, one of the strongest witnesses of "linear structure" is the existence of a *dominating pair of vertices* [10], namely a pair of vertices $\{x, y\}$ such that every x-y path dominates the graph in the sense that for any such path P, every vertex of the graph is either on P or has a neighbour on P. Vertices 1 and 4 form a dominating pair for the graph in Figure 2. In [11], the authors showed that the following very simple LBFS algorithm produces such a dominating pair:

1. Let σ be an arbitrary LBFS, and let x be its last vertex.
2. Let τ be an arbitrary LBFS that starts at x, and let y be its last vertex.
3. Return $\{x, y\}$.

Such an application of LBFS outside the chordal graph family was quite unexpected and produced various LBFS properties of AT-free graphs. In particular, vertices y, z are called *unrelated with respect to vertex x* if there exist an x-y path P that is outside z's neighbourhood and an x-z path Q that is outside y's neighbourhood. For the graph in Figure 2, the vertices 1 and 4 are unrelated with respect to the vertex 6 because of the paths $6, 2, 1$ and $6, 5, 4$. Vertex x is called *admissible* if no pair of vertices is unrelated with respect to x.

Admissible elimination: For $G = (V, E)$, an ordering of V: $v_1, v_2, \ldots, v_i, \ldots, v_n$ is called an *Admissible Elimination Ordering* **(AEO)** if for every $i \in \{1, \ldots, n\}$, the vertex v_i is admissible in $G_i = G[v_1, \ldots, v_i]$.

Similar to the theorem of Rose, Tarjan, and Lueker [28] that **any** LBFS of a chordal graph is a PEO, it was shown in [11] that any LBFS of an AT-free graph is an AEO. Unfortunately, the converse is not true, as shown by the graph in Figure 4 (with AT $\{2, 3, 4\}$), where the LBFS 1 2 3 4 5 6 7 is an AEO. However, it was shown in [7] that a graph G is AT-free if and only if **every** LBFS of G is an AEO.

As mentioned in the previous subsection, LDFS was discovered by applying the private neighbour requirement to the vertex d in the DFS VOC, and only recently have applications of LDFS been discovered. In particular, a cocomparability graph has a COCOMP order that is also an LDFS and on such an order, some very simple greedy algorithms that work for interval graphs, also work for cocomparability graphs. This happens for the minimum path cover problem [4], the longest path problem [24] and the maximum independent set and minimum clique cover problems [6]. Furthermore, by doing a direct translation of the minimum path cover algorithm to the associated poset, one has a simple certifying algorithm for the bump number problem of posets [4].

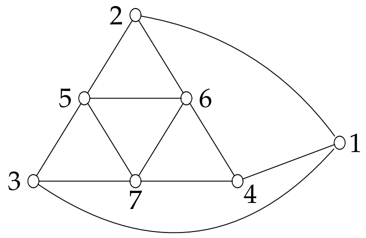


Figure 4: A graph with an AT $\{2,3,4\}$ that has an LBFS that is an AEO.

3. Results

We now proceed to the main results of the paper. Since the result for AT-free graphs is easier to state and prove (and will be of greater interest to some readers), we will first concentrate on AT-free graphs and then generalize the result to graphs of bounded asteroidal number. The structure of this section is as follows.

In the next two subsections, we introduce some useful tools and notation, and then separately discuss a VOC for AT-free graphs and graphs of bounded asteroidal number.

3.1. Lexicographic ordering

To construct the orderings we shall select vertices based on special labels that we assign to the vertices. We compare these labels using the lexicographic (dictionary) ordering. To make our arguments clearer (and more precise), we briefly discuss some useful properties of lexicographic orderings.

We use the lexicographic ordering on sequences of integers defined as follows.

Definition 1. Let $a_1, \ldots, a_s$ and $b_1, \ldots, b_t$ be two sequences of integers. Let i be the largest index such that $i \leq \min\{s,t\}$ and $a_1 = b_1$, $a_2 = b_2$, $\ldots$, $a_i = b_i$; if $a_1 \neq b_1$, then define i as 0.

We write $(a_1, \ldots, a_s) \prec (b_1, \ldots, b_t)$ and say that the sequence $a_1, \ldots, a_s$ is *lexicographically smaller* than $b_1, \ldots, b_t$ if $i < t$ and either $i = s$ or $a_{i+1} < b_{i+1}$.

We write $(a_1, \ldots, a_s) \preceq (b_1, \ldots, b_t)$ if either $(a_1, \ldots, a_s) \prec (b_1, \ldots, b_t)$ or the two sequences are identical.

For instance $(1,3,1) \prec (1,3,1,2) \prec (1,3,2) \prec (2,1) \prec (3)$, where (3) is *lexicographically largest* among these sequences. Note that $\prec$ is a total order. We shall need the following straightforward observations.

Lemma 1. *Let $a_1, \ldots, a_s$ and $b_1, \ldots, b_t$ be two sequences of integers. If there exists $j \leq \min\{s,t\}$ such that $(a_1, \ldots, a_j) \prec (b_1, \ldots, b_j)$, then $(a_1, \ldots, a_s) \prec (b_1, \ldots, b_t)$.*

PROOF. Consider the largest $i \leq j$ such that $a_1 = b_1$, $\ldots$, $a_i = b_i$. Since $(a_1, \ldots, a_j) \prec (b_1, \ldots, b_j)$, we have $i < j$ and $a_{i+1} < b_{i+1}$ by definition. So, since $j \leq \min\{s,t\}$, we deduce $i < t$ and $i \neq s$. Thus $(a_1, \ldots, a_s) \prec (b_1, \ldots, b_t)$ by definition. □

Lemma 2. *Suppose that $b_1 \geq b_2 \geq \ldots \geq b_t$ and let $f_1, \ldots, f_s$ be distinct integers from $\{1, \ldots, t\}$. Then $(b_{f_1}, b_{f_2}, \ldots, b_{f_s}) \preceq (b_1, \ldots, b_t)$.*

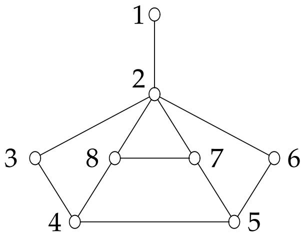
For example consider $b_1 = 5 \geq b_2 = 5 \geq b_3 = 3 \geq b_4 = 2 \geq b_5 = 2 \geq b_6 = 1$. With $f_1 = 1$, $f_2 = 3$, $f_3 = 4$, $f_4 = 2$, $f_5 = 6$, we see that $(b_{f_1}, \cdots, b_{f_s})$ is $(5,3,2,5,1)$ and also that $(5,3,2,5,1) \prec (5,5,3,2,2,1)$.

PROOF. Since $f_1, \ldots, f_s$ are distinct, the mapping $f : i \mapsto f_i$ is injective. Thus $s \leq t$.

Let $i \leq s$ be the largest index such that $b_{f_1} = b_1$, $b_{f_2} = b_2$, $\ldots$, $b_{f_i} = b_i$; if $b_{f_1} \neq b_1$, then let $i = 0$. If $i = s < t$, then $(b_{f_1}, \ldots, b_{f_s}) \prec (b_1, \ldots, b_t)$ by definition. If $i = s = t$, then the two sequences are actually identical, since f is injective. Thus $(b_{f_1}, \ldots, b_{f_s}) \preceq (b_1, \ldots, b_t)$ when $i = s$. So we may assume $i < s$.

Let $A = \{j \mid b_j > b_{i+1}\}$. We now show that $A \subseteq \{f_j \mid 1 \leq j \leq i\}$. Since $b_1 \geq b_2 \geq \ldots \geq b_t$, we have $A \subseteq \{1, \ldots, i\}$. Moreover, since $b_1 = b_{f_1}$, $b_2 = b_{f_2}$, $\ldots$, $b_i = b_{f_i}$, we conclude that $\{f_j \mid j \in A\} \subseteq A$. In other words, $f(A) \subseteq A$ and since f is injective, we find that $f(A) = A$. Thus $A = \{f_j \mid j \in A\} \subseteq \{f_j \mid 1 \leq j \leq i\}$.

From this, we now deduce that $b_{f_j} \leq b_{i+1}$ for all $j > i$. Since $b_{f_{i+1}} \neq b_{i+1}$, we conclude $b_{f_{i+1}} < b_{i+1}$ which yields $(b_{f_1}, b_{f_2}, \ldots, b_{f_s}) \preceq (b_1, \ldots, b_t)$, as required. □



$comp(1) = (6)$
$comp(2) = (2)$
$comp(3) = (4, 1)$
$comp(4) = (4)$
$comp(5) = (4)$
$comp(6) = (4, 1)$
$comp(7) = (2, 1, 1)$
$comp(8) = (2, 1, 1)$

Figure 5: The graph from Figure 2 and its $comp$ labels.

3.2. Components of the non-neighbourhood

In order to deal with asteroidal sets, we need a way to handle the structure of the non-neighbourhoods of vertices of G. For this purpose, we assign to every vertex v of G a label $comp(v)$ defined as follows.

Definition 2. Let v be a vertex of G, and let $C_1, C_2, \ldots, C_t$ be the connected components of $G - N[v]$ where $|C_1| \geq |C_2| \geq \ldots \geq |C_t|$. Then define

$$comp(v) = (|C_1|, |C_2|, \ldots, |C_t|).$$

The $comp$ labels for the graph from Figure 2 are presented as Figure 5. Note that the $comp$ labels are sequences of integers. Thus we can compare them lexicographically using the relations $\prec$ and $\preceq$ as defined earlier. In particular, we can define the following ordering of the vertices of G.

Definition 3. An ordering $v_1, v_2, \cdots, v_n$ of the vertices of a graph G is called a **LEXCOMP ORDER** if $comp(v_i) \preceq comp(v_{i+1})$ for all $1 \leq i < n$.

As an example, for the graph in Figure 5, the LEXCOMP ORDERs can be represented as 2 [7 8] [4 5] [3 6] 1, where the square brackets indicate that choice could be made. A straightforward BFS algorithm computes a LEXCOMP ORDER of a given graph in time $O(n \times m)$; recall that our graphs are assumed to be connected.

As will be seen in the next two subsections, LEXCOMP ORDERs play a critical role in our vertex order characterizations of AT-free graphs in particular, and graphs of bounded asteroidal number in general. The main useful feature of the $comp$ labels is presented in the following lemma.

Lemma 3. *Let v be a vertex of G and let $C_1, C_2, \ldots, C_t$ be the connected components of $G - N[v]$ where $|C_1| \geq |C_2| \geq \ldots \geq |C_t|$. Then for every $i \in \{1, \ldots, t\}$ and every $x \in C_i$ such that $comp(x) \preceq comp(v)$, each of $C_1, \ldots, C_{i-1}$ is also a connected component of $G - N[x]$.*

PROOF. Suppose that the claim fails for some $i \in \{1, \ldots, t\}$ and $x \in C_i$ such that $comp(x) \preceq comp(v)$. In other words, there exists $j < i$ such that the set C_j is not a connected component of $G - N[x]$. Choose j to be smallest with this property.

Since $x \in C_i$ and $j < i$, the vertex x is neither in C_j nor has a neighbour in C_j. (Note that $C_1, \ldots, C_t$ are distinct connected components.) So C_j induces a connected subgraph in $G - N[x]$. But C_j itself is not a connected component of $G - N[x]$. Therefore, there exists a connected component D of $G - N[x]$ that properly contains C_j. Thus $|C_j| < |D|$ implying $(|C_1|, \ldots, |C_{j-1}|, |C_j|) \prec (|C_1|, \ldots, |C_{j-1}|, |D|)$. From this, by Lemma 1, we obtain $comp(v) = (|C_1|, \ldots, |C_t|) \prec (|C_1|, \ldots, |C_{j-1}|, |D|)$.

Moreover, the minimality of j implies that $C_1, \ldots, C_{j-1}$ are connected components of $G - N[x]$. Thus D is not one of $C_1, \ldots, C_{j-1}$, since it contains C_j but none of $C_1, \ldots, C_{j-1}$ could since $C_1, \ldots, C_j$ are distinct connected components of $G - N[v]$. In other words, $C_1, \ldots, C_{j-1}, D$ are distinct connected components of $G - N[x]$. So Lemma 2 and the definition of $comp(x)$ produce $(|C_1|, \ldots, |C_{j-1}|, |D|) \preceq comp(x)$.

Putting the two together gives us $comp(v) \prec comp(x)$, a contradiction. $\square$

3.3. AT-free Graphs

We can now proceed to the main theorems. Recall that non-adjacent vertices x, y are *unrelated* in a graph G with respect to a vertex z if there exists an x-z path in $G - N[y]$ and an y-z path in $G - N[x]$.

We write $\mathcal{I}(x,y)$ to denote the set of all z such that x and y are unrelated in G with respect to z. Note that $\mathcal{I}(x,y) = \mathcal{I}(y,x)$.

For the graph in Figure 2; the non-empty $\mathcal{I}$-sets for this graph are: $\mathcal{I}(1,4) = \mathcal{I}(4,1) = \{6,7\}$; $\mathcal{I}(1,5) = \mathcal{I}(5,1) = \{3,8\}$; $\mathcal{I}(3,5) = \mathcal{I}(5,3) = \{8\}$; $\mathcal{I}(3,6) = \mathcal{I}(6,3) = \{7,8\}$; $\mathcal{I}(4,6) = \mathcal{I}(6,4) = \{7\}$.

In a related fashion, we write $\mathcal{J}(z)$ to denote the set of all unordered pairs $\{x,y\}$ such that $z \in \mathcal{I}(x,y)$. (In other words, $\mathcal{J}(z)$ contains the pairs of vertices that are unrelated with respect to z.)

For the graph in Figure 2, the non-empty $\mathcal{J}$-sets are: $\mathcal{J}(3) = \{\{1,5\}\}$; $\mathcal{J}(6) = \{\{1,4\}\}$; $\mathcal{J}(7) = \{\{1,4\},\{3,6\},\{4,6\}\}$; $\mathcal{J}(8) = \{\{1,5\},\{3,5\},\{3,6\}\}$. Observe that if A is an asteroidal set of size greater than two, then for all $z \in A$, every pair of vertices from $A \setminus \{z\}$ is in $\mathcal{J}(z)$.

Using this notation, we define an ordering of G as follows.

Definition 4. An ordering σ of the vertices of G is an **AT-free ORDER** if every vertex z and every pair $\{x,y\} \in \mathcal{J}(z)$ is such that $z <_\sigma x$ or $z <_\sigma y$.

Before proving the VOC theorem for AT-free graphs it is worth noting the difference between an AT-free ORDER and an AEO (Admissible Elimination Ordering – see Section 2.3). An AEO forbids a vertex z having an unrelated pair $\{x,y\}$ where all vertices of the z-x and z-y paths are before z in the ordering. On the other hand, an AT-free ORDER forbids a vertex z having an unrelated pair $\{x,y\}$ where x and y are before z in the ordering; note, there is no restriction on the location of the interior vertices of the z-x and z-y paths.

What now follows is the VOC theorem for AT-free graphs.

Theorem 4. *G is AT-free if and only if G admits an AT-free ORDER.*

Proof. For the backward direction, suppose that σ is an AT-free ORDER of G, but G contains an asteroidal triple $\{x,y,z\}$. By symmetry, we may assume $x <_\sigma y <_\sigma z$. Since $\{x,y,z\}$ is an asteroidal triple in G, there exists an x-z path in $G - N[y]$ and also a y-z path in $G - N[x]$. From this, we deduce that $\{x,y\} \in \mathcal{J}(z)$. However $x <_\sigma z$ and $y <_\sigma z$ contradicting our assumption that σ is an AT-free ORDER.

For the converse, we define the *unrelated pairs graph* H as follows:

(i) $V(H) = V(G)$,

(ii) $E(H) = \left\{(x,y) \,\middle|\, \mathcal{I}(x,y) \neq \emptyset\right\}$,

(iii) Each edge $e = (x,y) \in E(H)$ is assigned the label $label(e) = \mathcal{I}(x,y)$.

Note that the label on an edge (x,y) of H contains all vertices z where z is unrelated to $\{x,y\}$. The unrelated pairs graph H for the graph in Figure 2 appears as Figure 6.

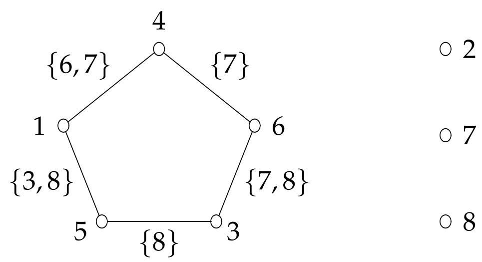


Figure 6: The unrelated pairs graph for the graph in Figure 2.

The converse is a consequence of the following claim.

Claim 1. If G is AT-free, then in every induced subgraph H' of H, there exists a vertex $z \in V(H')$ such that $z \notin label(e)$ for all $e \in E(H')$. In particular, a vertex in $V(H')$ of largest comp value has this property.

In other words, we claim that H' contains a vertex z that does not appear in the label of any edge of H'. To prove this, we choose z to be a vertex of H' with lexicographically largest value of $comp(z)$ among the vertices of H'. We show that no edge of H' has z in its label which will prove the claim.

Suppose otherwise, and let $e = (x, y)$ be an edge of H' with $z \in label(e)$. Since H' is an induced subgraph of H, we deduce $e \in E(H)$. Thus $\{x, y\} \in \mathcal{J}(z)$. This means that there exists an x-z path in $G - N[y]$ and also a y-z path in $G - N[x]$. In particular, both x and y are vertices in $G - N[z]$.

Let $C_1, \ldots, C_t$ be the connected components of $G - N[z]$ where $|C_1| \geq |C_2| \geq \ldots \geq |C_t|$. Recall that $comp(z) = (|C_1|, \ldots, |C_t|)$. Since x, y are vertices in $G - N[z]$, there exist indices i, j such that $x \in C_i$ and $y \in C_j$.

Suppose first that $i = j$. Then we claim that $\{x, y, z\}$ is an AT in G. Indeed, recall that there exists a y-z path in $G - N[x]$, and also an x-z path in $G - N[y]$. The remaining x-y path in $G - N[z]$ is provided by walking from x to y in $C_i = C_j$. Since G is assumed to be AT-free, this is impossible, and we conclude $i \neq j$.

By symmetry, we may assume that $j < i$. Recall that z was chosen to have lexicographically largest $comp(z)$ among the vertices of H'. Note that $x \in V(H')$, since $e = (x, y)$ is an edge of H'. Thus $comp(x) \preceq comp(z)$ and by Lemma 3, each of $C_1, \ldots, C_{i-1}$ is a connected component in $G - N[x]$. In particular, C_j is a connected component of $G - N[x]$, since $j < i$. However, this is impossible. Recall that $y \in C_j$ and that there exists a y-z path P in $G - N[x]$. Thus $z \in C_j$ which contradicts the fact that C_j is a connected component of $G - N[z]$. This proves Claim 1.

We construct the required ordering σ of $V(G)$ using the following algorithm.

Input: a graph $G = (V, E)$
Output: ordering $\sigma = \sigma(1), \sigma(2), \ldots, \sigma(n)$ of the vertices of G

construct the unrelated pairs graph H
initialize $H' \leftarrow H$
for $i = |V|$ *downto* 1 **do**
  pick a vertex v of H' that has largest comp value among all vertices of H' $(\star)$
  $\sigma(i) \leftarrow v$
  remove v from H'
end for

Clearly, the ordering σ is well-defined. We now verify that σ is an AT-free ORDER.

Let $z \in V(G)$, let $\{x, y\} \in \mathcal{J}(z)$, and consider the edge $e = (x, y) \in H$. The definition of H implies that $z \in label(e)$. Consider the graph H' in the algorithm when z is chosen by rule $(\star)$. By Claim 1, this choice means that z is not in the label of any edge of H'. Therefore, e is not an edge of H'. Since H' is an induced subgraph of H, this implies that at least one of x, y is not a vertex of H', namely, x or y was removed earlier in the algorithm. In other words, $z <_\sigma x$ or $z <_\sigma y$ as required.

This proves that σ is an AT-free ORDER of G which concludes the proof. □

Corollary 5. *A graph G is AT-free if and only if an arbitrary LEXCOMP ORDER of G is an AT-free ORDER.*

PROOF. This follows immediately from Claim 1. □

For the graph in Figure 5, the LEXCOMP ORDERs are 2 [7 8] [4 5] [3 6] 1, where the square brackets indicate that choice could be made. These, however, are not the only AT-free ORDERs of this graph; consider for instance the order 1 3 2 8 5 7 6 4.

3.4. Graphs with Bounded Asteroidal Number

We now generalize the AT-free ORDER and the VOC proof to graphs of bounded asteroidal number $k \geq 2$. We define a k-asteroidal ORDER and prove that it characterizes graphs of asteroidal number at most k.

The proof of this will follow essentially the same steps as the VOC proof for the AT-free ORDER (from the previous section), with only a few minor differences. Namely, in place of the unrelated pairs graph, we will now consider sets of unrelated pairs and use the corresponding hypergraph in the elimination process. The process will again be guided by the same *comp* vectors which will again be chosen in non-increasing lexicographic order. In the remainder, we shall assume that $k \geq 2$ is a fixed integer.

As before, for nonadjacent vertices x, y of G, we let $\mathcal{I}(x, y)$ denote the set all z such that x and y are unrelated with respect to z.

For a vertex z, we let $\mathcal{J}_k(z)$ denote the collection of all independent sets $S \subseteq V(G)$ with $|S| = k$ such that $z \in I(x, y)$ for all $x, y \in S$. (That is, $\mathcal{J}_k(z)$ contains all sets of cardinality k in which every pair of vertices is unrelated with respect to z.)

Observe that if A is an asteroidal set of size $k+1$, then $A \setminus \{z\} \in \mathcal{J}_k(z)$ for all $z \in A$.

Definition 5. An ordering σ of the vertices of G is a k**-asteroidal ORDER** if for every vertex z and set $S \in \mathcal{J}_k(z)$ there exists $x \in S$ with $z <_\sigma x$.

Recall that $a(G)$ denotes the size of a largest asteroidal set in G.

Theorem 6. *$a(G) \leq k$ if and only if G admits a k-asteroidal ORDER.*

PROOF. For the backward direction, suppose that σ is a k-asteroidal ORDER of G, but G contains an asteroidal set A of size $k+1$. (Note that if G contains a larger asteroidal set, then any $k+1$ subset of it is also asteroidal.) Let z be the last vertex of A in σ, that is, z satisfies $x <_\sigma z$ for all $x \in A \setminus \{z\}$. Clearly, $A \setminus \{z\} \in \mathcal{J}_k(z)$. But $x <_\sigma z$ for all $x \in A \setminus \{z\}$ contradicting our assumption that σ is a k-asteroidal ORDER.

For the converse, we define the *unrelated pairs hypergraph* $\mathcal{H}$ as follows:

(i) $V(\mathcal{H}) = V(G)$,
(ii) $E(\mathcal{H}) = \left\{ S \in V(G)^k \;\middle|\; \exists z \in V(G) \text{ such that } S \in \mathcal{J}_k(z) \right\}$,
(iii) Each edge $e = S \in E(\mathcal{H})$ is assigned the label $label(e) = \{ z \mid S \in \mathcal{J}_k(z) \}$.

The converse is a consequence of the following claim.

Claim 2. If $a(G) \leq k$, then in every induced subhypergraph $\mathcal{H}'$ of $\mathcal{H}$, there exists a vertex $z \in V(\mathcal{H}')$ such that $z \notin label(e)$ for all $e \in E(\mathcal{H}')$. In particular, a vertex in $V(\mathcal{H}')$ of largest comp value has this property.

In other words, we claim that $\mathcal{H}'$ contains a vertex z that does not appear in the label of any edge of $\mathcal{H}'$. Note that for a set $X \subseteq V(\mathcal{H})$, we say that $\mathcal{H}'$ is the subhypergraph of $\mathcal{H}$ induced on X if $V(\mathcal{H}') = X$ and $E(\mathcal{H}') = \{ S \in E(\mathcal{H}) \mid S \in X^k \}$.

To prove Claim 2, we let z be a vertex of $\mathcal{H}'$ with lexicographically largest value of $comp(z)$ among the vertices of $\mathcal{H}'$. We show that no edge of $\mathcal{H}'$ has z in its label.

Suppose otherwise, and let $e = S$ be an edge of $\mathcal{H}'$ with $z \in label(e)$. Since $\mathcal{H}'$ is an induced subgraph of $\mathcal{H}$, we have $e \in E(\mathcal{H})$, and thus $S \in \mathcal{J}_k(z)$. In particular, every vertex in S is a vertex of $G - N[z]$. Let C_1, $\ldots$, C_t be the connected components of $G - N[z]$ where $|C_1| \geq |C_2| \geq \ldots \geq |C_t|$. Recall that each $x \in S$ is a vertex in $G - N[z]$, namely, each $x \in S$ is a vertex in one of the sets $C_1, \ldots, C_t$. Since S is non-empty, let us therefore choose largest index i such that $C_i \cap S \neq \emptyset$.

Suppose first that $S \subseteq C_i$. Then $S \cup \{z\}$ is an asteroidal set of G. To see this, we need to verify that for every $u \in S \cup \{z\}$, the vertices in $S \cup \{z\} \setminus \{u\}$ belong to the same connected component of $G - N[u]$. If $u = z$, then this is clear, since $S \subseteq C_i$. For $u \neq z$, note that $z \in I(x, u)$ and $z \in I(y, u)$ for all $x, y \in S \setminus \{u\}$, since $S \in \mathcal{J}_k(z)$. Thus there exists an x-z path in $G - N[u]$, and also a y-z path in $G - N[u]$. Using these paths we can move from x to z to y in $G - N[u]$. This shows that all vertices in $S \cup \{z\} \setminus \{u\}$ belong to the same connected component of $G - N[u]$. Thus $S \cup \{z\}$ is indeed an asteroidal set of G. However, $|S \cup \{z\}| = k+1$ and we assume $a(G) \leq k$.

Thus we deduce $S \not\subseteq C_i$. In particular, there exists $y \in S \setminus C_i$ and also $x \in S \cap C_i$, since $S \cap C_i \neq \emptyset$. The maximality of i implies that $y \in C_j$ for some $j < i$. Recall that z was chosen to have lexicographically largest $comp(z)$ among the vertices of $\mathcal{H}'$. Note that $x \in S \subseteq V(\mathcal{H}')$, since $e = S$ is an edge of $\mathcal{H}'$. Thus $comp(x) \preceq comp(z)$ and by Lemma 3, we deduce that C_j is a connected component of $G - N[x]$, since $j < i$. Note that $y \in C_j$ and $z \in I(x,y)$, since $\{x,y\} \subseteq S \in \mathcal{J}_k(z)$. This implies that there exists a y-z path P in $G - N[x]$. Thus $z \in C_j$ which contradicts the fact that C_j is a connected component of $G - N[z]$. This concludes the proof of Claim 2.

We now construct the required ordering σ of $V(G)$ using the following algorithm.

Input: a graph $G = (V, E)$
Output: ordering $\sigma = \sigma(1), \sigma(2), \ldots, \sigma(n)$ of the vertices of G

construct the unrelated pairs hypergraph $\mathcal{H}$
initialize $\mathcal{H}' \leftarrow \mathcal{H}$
for $i = |V|$ *downto* 1 **do**
  pick a vertex v of $\mathcal{H}'$ that has the largest comp value among all vertices of $\mathcal{H}'$ $(\star)$
  $\sigma(i) \leftarrow v$
  remove v and all hyperedges of $\mathcal{H}'$ containing v from $\mathcal{H}'$
end for

Clearly, the ordering σ is well-defined. Let us now verify that σ is a k-asteroidal ORDER. Let $z \in V(G)$, let $S \in \mathcal{J}_k(z)$, and consider the edge $e = S \in E(\mathcal{H})$. By the definition of $\mathcal{H}$, we have $z \in label(e)$. Consider the hypergraph $\mathcal{H}'$ in the algorithm when z is chosen by rule $(\star)$. By Claim 2, this means that z is not in the label of any edge of $\mathcal{H}'$. Therefore, e is not an edge of $\mathcal{H}'$. Since $\mathcal{H}'$ is an induced subhypergraph of $\mathcal{H}$, there must exist $x \in S$ with $x \notin V(\mathcal{H}')$. That is, x was removed earlier in the algorithm implying $z <_\sigma x$.

This proves that σ is a k-asteroidal ORDER of G which concludes the proof. $\square$

Corollary 7. *A graph G has asteroidal number at most k if and only if an arbitrary LEXCOMP ORDER of G is a k-asteroidal ORDER of G.*

PROOF. This follows immediately from Claim 2. $\square$

4. Discussion

In this section we examine the previous results from two perspectives. Firstly we observe how our results produce a hierarchy of VOCs that captures all graphs, and secondly we ask whether the AT-free ORDER can be achieved by traditional graph searches.

4.1. Hierarchy of VOCs

First, we note that it is easy to show that a PO ORDER is an AT-free ORDER, and thus so are a COCOMP ORDER, an I ORDER and a UI ORDER. Indeed, suppose that σ is a PO ORDER of a graph $G = (V, E)$, but it is not an AT-free ORDER of G. Thus there are vertices z and x, y such that $z \in I(x,y)$ and $\{x,y\} <_\sigma z$. By symmetry, we may assume that $x <_\sigma y <_\sigma z$. Since $z \in I(x,y)$, there exists an x-z path P in $G - N[y]$. Namely, we have $N[y] \cap P = \emptyset$. However, note that $xz \notin E$. Thus the vertices $x <_\sigma y <_\sigma z$ violate the definition of a PO ORDER for the path P. So no such vertices exist which shows that σ is indeed an AT-free ORDER. In a similar manner, one can show that every k-asteroidal ORDER is also a $(k+1)$-asteroidal ORDER. Thus starting with UI ORDER, we have an infinite hierarchy of vertex orderings:

$$\text{UI} \subset \text{I} \subset \text{COCOMP} \subset \text{PO} \subset \text{AT-free} \subset \text{3-asteroidal} \subset \ldots \subset k\text{-asteroidal} \subset \ldots$$

Note that every graph admits at least one of these orderings (since every graph has a specific asteroidal number).

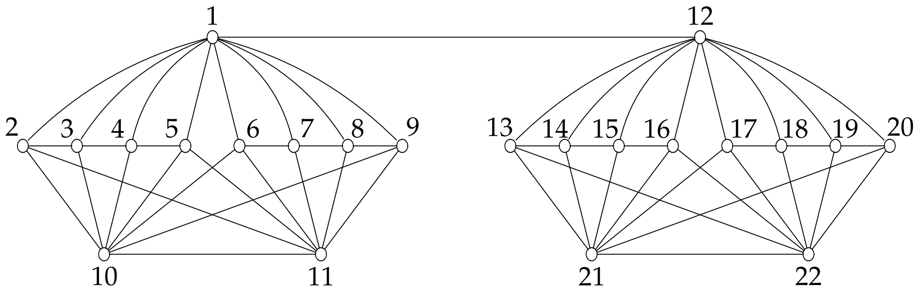


Figure 7: "Two cupcakes": no AT-free order is also an LBFS.

4.2. Graph searches and AT-free ORDERs

As seen in the two previous corollaries, the LEXCOMP ORDER plays a key role in finding constructive proofs of the VOCs of AT-free graphs and graphs of bounded asteroidal number. Although a LEXCOMP ORDER is a graph search in the sense that every vertex is visited, it is not necessarily a Generic Search ORDER as illustrated by a C_4 where we start with two nonadjacent vertices. This raises the question of whether every AT-free graph admits an AT-free ORDER that is a particular traditional graph search (as described in Section 2.2).

Recall first that an MNS of a chordal graph is always a PEO. Similarly, an LBFS of an AT-free graph is always an AEO. It would be useful if a similar statement could be shown for AT-free ORDER. However, as we will see shortly, this is not the case. From another perspective, we mentioned earlier that every cocomparability graph admits a COCOMP ORDER that is also an LDFS. In particular, a simple LDFS$^+$ of a COCOMP ORDER yields such an order. This property turned out to be immensely useful in recent algorithmic results on cocomparability graphs [4, 6, 24]. It again would seem useful if this carried over to AT-free ORDERs. Alas, this is not the case. Not only does the $+$ technique fail (or several sweeps of $+$ searches), but actually there are AT-free graphs where **no** LBFS, LDFS, or even DFS is also an AT-free ORDER. We now discuss examples of such graphs.

To see that there are AT-free graphs where no DFS can be an AT-free ORDER, consider two C_5s that share a vertex (the "centre vertex"). Suppose that σ is a DFS and also an AT-free ORDER of this graph. The first vertex in σ other than the centre belongs to one of the two C_5s. In the other C_5, there are two neighbours of the centre; the rightmost of the two (in σ) violates the definition of an AT-free ORDER.

We now discuss the graph shown in Figure 7 and show that no LBFS of it can be an AT-free ORDER. The graph has 22 vertices and consists of two identical parts "cupcakes" on vertices $\{1,\ldots,11\}$ and $\{12,\ldots,22\}$. Each part admits further symmetries: vertices $\{2,3,4,5,10\}$ and $\{6,7,8,9,11\}$ can exchange labels, vertices $2,3$ can exchange labels with $5,4$ respectively, and similarly $6,7$ can exchange labels with $9,8$ respectively. We shall use these symmetries in our analysis below.

We now show that this graph does not admit an LBFS that is also an AT-free ORDER. Assume that σ is such an order, where by symmetry $12 <_\sigma 1$. Since σ is an LBFS, it follows that $1 <_\sigma \{2,\ldots,9\} <_\sigma \{10,11\}$. In particular, either $\{2,3,4,5\} <_\sigma \{6,7,8,9\}$ or $\{6,7,8,9\} <_\sigma \{2,3,4,5\}$. By symmetry, assume the former. Then note that $9 <_\sigma 7$ and $6 <_\sigma 8$, since $9 \in I(2,7)$ and $6 \in I(2,8)$. So either $6 <_\sigma \{7,8,9\}$ or $9 <_\sigma \{6,7,8\}$. Again, by symmetry, we shall assume that $6 <_\sigma \{7,8,9\}$. Then it follows that $7 <_\sigma \{8,9\}$, because σ is an LBFS and $1 <_\sigma \{2,\ldots,9\} <_\sigma \{10,11\}$. But then $7 <_\sigma 9$, a contradiction.

Although these results are somewhat disappointing, there may be other ways of generating vertex orderings that are AT-free ORDERs. In particular, we conjecture that every AT-free graph has a BFS that is an AT-free ORDER. Furthermore, perhaps there exists a graph search whose **reversal** is an AT-free ORDER. As examples, note that the graph in Figure 7 admits a BFS that is also an AT-free order. It also admits a BFS and a DFS whose reversals are AT-free orders. These are as follows:

both a BFS and an AT-free ORDER:

1 2 5 6 9 3 4 7 8 12 10 11 13 16 17 20 14 15 18 19 21 22

a BFS whose reversal is an AT-free ORDER:
10 11 3 4 2 5 6 9 7 8 1 12 14 15 13 16 17 18 19 20 21 22

a DFS whose reversal is an AT-free ORDER:
11 10 3 4 5 1 2 6 7 8 9 12 14 15 16 22 13 21 17 18 19 20

However, no LBFS, no LDFS, and in fact, no MNS of the graph in Figure 7 is a reversal of an AT-free ORDER. Indeed, suppose that σ is such an order. Note that the last vertex of an AT-free order is always admissible. The graph in Figure 7 has only 4 admissible vertices, namely 10, 11, 21, and 22. Since σ is a reversal of an AT-free order, one of these four vertices is the first vertex in σ. By symmetry, we may assume that 10 is first in σ. Thus the second vertex in σ must be a neighbour of 10. It cannot be 3 or 4, since $\{3,4\} \subseteq I(11,12)$ and neither 11 nor 12 has been visited yet. It also cannot be 2, 5, 6, or 9, since $\{2,9\} \subseteq I(4,7)$ and $\{5,6\} \subseteq I(3,8)$. Thus the second vertex in σ must be 11. Now, because σ is an MNS, the third vertex in σ is one of $2,5,6,9$. But this is impossible, since again $\{2,9\} \subseteq I(4,7)$ and $\{5,6\} \subseteq I(3,8)$.

5. Concluding Remarks

As the main result of this paper, we have described and proved a Vertex Ordering Characterization (VOC) for AT-free graphs (the first VOC for this class) and also VOCs for graphs of bounded asteroidal number in general. Even though our VOC theorem brings us closer to understanding AT-free graphs, further research is still needed to obtain useful structural tools for AT-free graphs with (hopefully) nice algorithmic consequences. It is noteworthy that the LEXCOMP ORDER provides such an AT-free VOC. Perhaps the most important questions resulting from our work are:

Question 1. What structure of AT-free graph is revealed through the LEXCOMP ORDER?

Question 2. Can the LEXCOMP ORDER be used to develop polynomial time algorithms for the open problems on AT-free graphs listed below?

Question 3. Can the LEXCOMP ORDER be of any use in understanding the structure of other families of graphs?

As shown in Section 4.2, we cannot expect an AT-free ORDER to have nice properties such as being an LBFS or LDFS or even a DFS. However, there are still some possibilities left. We summarize them as follows:

Question 4. Does every AT-free graph admit an AT-free ORDER that is also a BFS?

Question 5. Does every AT-free graph admit an AT-free ORDER whose reversal is a DFS?

In fact, we can ask even more generally.

Question 6. Does every AT-free graph admit an AT-free ORDER that is, or whose reversal is, a Generic Search ORDER?

Finally, the following are longstanding open questions on AT-free graphs which have not yet been examined from the structural perspective.

Question 7. What is the complexity of colouring in AT-free graphs?

Question 8. What is the complexity status of all Hamiltonian problems on AT-free graphs?

Note that the complexity of k-colouring in AT-free graphs for every fixed k has already been resolved [20, 29]. However, the general question (for unbounded k) as well as all Hamiltonian problems remain challenging open problems.

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