Maximum Flow Problems III.

Review: \( G = (V, E); (s, t)\)-cut \( \delta(A) \); edge capacity \( u : E \to \mathbb{R}_{\geq 0} \); (s, t)-flow \( x : E \to \mathbb{R}_{\geq 0} \)

Goal: **Maximum Flow Problem**

Maximize \( f_x(t) = \sum_{w \in V} x_{wt} - \sum_{w \in V} x_{tw} \)

subject to \( f_x(v) = \sum_{w \in V} x_{vw} - \sum_{w \in V} x_{vw} = 0 \quad \forall v \in V \setminus \{s, t\} \)

\[ 0 \leq x_e \leq u_e \quad \forall e \in E \]

1 Cuts

Recall \( \delta(A) = \{vw \mid v \in A, w \in \overline{A}\} \), and \((s, t)\)-cut \( \delta(A) \) if \( s \in A \) and \( t \in \overline{A} = V \setminus A \)

**Theorem 1.** Every \((s, t)\)-cut \( \delta(A) \) and every \((s, t)\)-flow \( x \) satisfy:

\[
\sum_{e \in \delta(A)} x_e - \sum_{e \in \delta(A)} x_e = f_x(t)
\]

flow across the cut \( \delta(A) \)
denoted by \( x(\delta(A)) \)

**Proof.** \( x \) is a flow \( \Rightarrow f_x(v) = 0 \) for all \( v \in V \setminus \{s, t\} \).

Summing up over all \( v \in \overline{A} \setminus \{t\} \):

\[
\sum_{v \in \overline{A}} f_x(v) = f_x(t) + \sum_{v \in \overline{A} \setminus \{t\}} f_x(v) = f_x(t)
\]

Recall: \( f_x(v) = \sum_{w \in V} x_{vw} - \sum_{w \in V} x_{wv} \)

Consider \( e = vw \in E \)

\[ +x_{vw} \text{ contribution to } f_x(w) \]

\[ -x_{vw} \text{ contribution to } f_x(v) \]

Contribution of \( vw \) to the left-hand-side (LHS)

\[
\begin{align*}
 v \in A \quad w & \in A \quad \text{none because LHS sums-up } f_x(v) \text{ for } v \in \overline{A} \\
 v \in A \quad w & \in \overline{A} \quad +x_{vw} \text{ from } f_x(w) \\
 v \in \overline{A} \quad w & \in A \quad -x_{vw} \text{ from } f_x(v) \\
 v \in \overline{A} \quad w & \in \overline{A} \quad +x_{vw} + -x_{vw} = 0
\end{align*}
\]

\( \square \)

**Corollary 1.** Every \((s, t)\)-cut \( \delta(A) \) and every feasible \((s, t)\)-flow \( x \) satisfy:

\[ f_x(t) \leq \sum_{e \in \delta(A)} u_e \]
Proof. $x$ is a feasible flow $\Rightarrow 0 \leq x_e \leq u_e$ for $\forall e \in E$. Thus, by Theorem 1 $\Rightarrow$

$$f_x(t) = \sum_{e \in \delta(A)} x_e - \sum_{e \in \delta(A)} x_e \leq \sum_{e \in \delta(A)} u_e$$

capacity of the cut $\delta(A)$ denoted by $u(\delta(A))$ □

i.e., the value of a feasible flow is at most the capacity of a cut.

Theorem 2. (Max-Flow Min-Cut Theorem) [Ford-Fulkerson 1956], [Kotzig 1956]
The maximum value of a feasible $(s,t)$-flow is equal to the minimum capacity of an $(s,t)$-cut. If all capacities are integral, then there exists an integral maximum feasible flow.

2 Flow augmentation

Let $x$ be a feasible $(s,t)$-flow of value $k$ (for instance, $x = 0$ is a feasible flow of value 0)

For an $st$-path $P = (v_0, \ldots, v_m)$ define $x$-width of $P$ as $$\min_{i \in \{1 \ldots m\}} u_{v_{i-1}v_i} - x_{v_{i-1}v_i}$$

If $P$ is a path of $x$-width $\varepsilon > 0$, then $\forall i$ increase $x_{v_{i-1}v_i}$ by $\varepsilon$ $\Rightarrow$ a feasible flow of value $k + \varepsilon$

(... just like in the proof of the flow-paths theorem...)

we may get stuck before reaching the maximum flow $\Rightarrow$ need to allow more general paths

**Idea:** use also backward edges

**directed path** = path (as defined before)

**undirected path** = a sequence $(v_0, \ldots, v_m)$ where $v_i$ distinct and for all $i \in \{1 \ldots m\}$ either $v_{i-1}v_i \in E$ (“forward” edge) or $v_iv_{i-1} \in E$ (“backward” edge)

$x$-width of an undirected path $(v_0, \ldots, v_m) = \min_{i \in \{1 \ldots m\}} \left\{ \begin{array}{ll}
    u_{v_{i-1}v_i} - x_{v_{i-1}v_i} & \text{if } v_{i-1}v_i \in E \\
    x_{v_{i-1}v_i} & \text{if } viv_{i-1} \in E
  \end{array} \right.$

$x$-increasing path = undirected path of positive $x$-width

$x$-augmenting path = $x$-increasing path from $s$ to $t$

If $P = (v_0, \ldots, v_m)$ is an $x$-augmenting path of width $\varepsilon > 0$, then $\forall i \in \{1 \ldots m\}$

- if $v_{i-1}v_i \in E$, increase $x_{v_{i-1}v_i}$ by $\varepsilon$,
- if $viv_{i-1} \in E$, decrease $x_{v_{i-1}v_i}$ by $\varepsilon$.

$\Rightarrow$ a feasible flow of value $k + \varepsilon$

No $x$-augmenting path $\Rightarrow$ maximum flow (we now prove)

Proof of Max-Flow Min-Cut Theorem. Let $x$ be a feasible flow of maximum value. Let $U = \{z \mid \exists$ an $x$-increasing path from $s$ to $z\}$. Note that $s \in U$.

If $t \in U$, then $\exists$ an $x$-augmenting path $\Rightarrow x$ is not maximum flow, a contradiction.
So \( s \in U \) and \( t \in \overline{U} \Rightarrow \delta(U) \) is an \((s,t)\)-cut. Moreover,

- every \( e = vw \in \delta(U) \) satisfies \( u_e - x_e = 0 \), otherwise \( w \in U \).
- every \( e = vw \in \delta(\overline{U}) \) satisfies \( x_e = 0 \), otherwise \( v \in U \).

The value of \( x \) is equal to the capacity of \( \delta(U) \). By Corollary 1, the value of a feasible \((s,t)\)-flow is at most the capacity of an \((s,t)\)-cut \( \Rightarrow \delta(U) \) is a minimum cut.

Integral capacities \( \Rightarrow \) integral widths of augmenting paths \( \Rightarrow \) integral flow.

3 Closing remarks

Notice the similarity of the above proof with that of the theorem about cuts and the existence of an \(st\)-path. This is no coincidence, as we shall see, and this correspondence will allow us to reduce the problem of finding augmenting paths to simple \((s,t)\)-connectivity question on an auxiliary graph.

**Advance note:** similar situation occurs with the minimum-cost flow problem which reduces to maximum flow and iterations of shortest path question in an auxiliary graph with general weights (Bellman-Ford); this phenomenon is more generally captured by the so-called Primal-Dual method and is related to Linear Programming (LP) formulations of these problems...