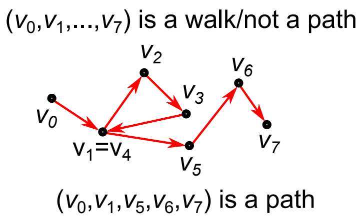
Maximum Flow Problems

Directed Graph (Network) $G = (V, E)$

i.e., edges in E are ordered pairs from $V \times V$



Walk = sequence $(v_0, v_1, \ldots, v_m)$
where $v_{i-1}v_i \in E$ for all $i \in \{1 \ldots m\}$

Path = a walk $(v_0, v_1, \ldots, v_m)$ where all v_i are distinct

st-path = a path $v_0, \ldots, v_m$ with $v_0 = s$ and $v_m = t$

(s, t)-connectivity = $\exists$ an st-path ?

For $A \subseteq V$, the cut $\delta(A)$ is the set of edges $\delta(A) = \{vw \in E \mid v \in A, w \in V \setminus A\}$

Denote $\overline{A} = V \setminus A$

$$\delta(A) \text{ and } \delta(\overline{A}) \text{ are not the same } \begin{cases} \delta(A) \text{ edges going out of } A \\ \delta(\overline{A}) \text{ edges coming into } A \end{cases}$$

A cut is proper if $\emptyset \neq A \neq V$.

G is connected if for all $s, t \in V$, there $\exists$ an st-path.

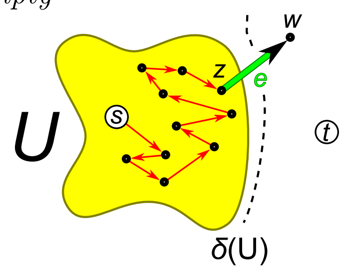
Theorem 1. *G is connected $\iff$ $\forall A \subseteq V$, $\emptyset \neq A \neq V$, we have $\delta(A) \neq \emptyset$*
"all proper cuts are non-empty"

(s, t)-cut $\delta(A)$ if $s \in A$ and $t \in \overline{A}$.

Theorem 2. *$\exists$ an st-path $\iff$ $\forall A \subseteq V$, $s \in A$, $t \notin A$, we have $\delta(A) \neq \emptyset$*
"all (s, t)-cuts are non-empty"

Proof. ($\Rightarrow$) If $\exists$ an st-path $P \Rightarrow$ every (s, t)-cut $\delta(A)$ contains at least one edge of P. So $\delta(A) \neq \emptyset$.

($\Leftarrow$) Let $U = \{z \in V \mid \exists \text{ an } sz\text{-path}\}$. If $t \in U$, then $\exists$ an st-path. If $t \notin U$, then we have $s \in U$, $t \notin U$, and $\delta(U) = \emptyset$. Indeed, if $zw \in \delta(U)$, then $w \in U$, a contradiction. So, $\delta(U)$ is an empty (s, t)-cut.



(s, t)-edge-connectivity = maximum number of edge disjoint st-paths

(s, t)-vertex-connectivity = maximum number of internally vertex disjoint st-paths
(i.e., vertex disjoint except for sharing s and t)

1 Edge capacities

Edge capacity $u : E \to \mathbb{R}_{\geq 0}$ $\rightsquigarrow$ capacitated graph/network $G = (V, E, u)$

important note: *capacity $\neq$ cost*

cost	$\sim$	length, reliability, cost (lease, toll), revenue	can be negative
capacity	$\sim$	thickness (pipe, cable), maximum throughput	always non-negative

Question: Given a capacitated network and two nodes s, t
what is the largest collection $(P_1, \ldots, P_k)$ of st-paths (not necessarily distinct) such that for each edge $e \in E$, the number of paths P_i containing e is at most u_e (capacity of e)?

Answer: ***Maximum flow***