Maximum Flow Problems

Directed Graph (Network) \( G = (V, E) \)
i.e., edges in \( E \) are ordered pairs from \( V \times V \)

Walk = sequence \( (v_0, v_1, \ldots, v_m) \)
where \( v_i \) is in \( E \) for all \( i \in \{1, \ldots, m\} \)

Path = a walk \( (v_0, v_1, \ldots, v_m) \) where all \( v_i \) are distinct

\( st \)-path = a path \( v_0, v_1, \ldots, v_m \) with \( v_0 = s \) and \( v_m = t \)

\( (s, t) \)-connectivity = \( \exists \) an \( st \)-path

For \( A \subseteq V \), the cut \( \delta(A) \) is the set of edges \( \delta(A) = \{vw \in E \mid v \in A, w \in V \setminus A\} \)

Denote \( \overline{A} = V \setminus A \)

\( \delta(A) \) and \( \delta(\overline{A}) \) are not the same
- \( \delta(A) \) edges going out of \( A \)
- \( \delta(\overline{A}) \) edges coming into \( A \)

A cut is proper if \( \emptyset \neq A \neq V \).

\( G \) is connected if for all \( s, t \in V \), there \( \exists \) an \( st \)-path.

**Theorem 1.** \( G \) is connected \( \iff \forall A \subseteq V, \emptyset \neq A \neq V, \text{ we have } \delta(A) \neq \emptyset \)

\( (s, t) \)-cut \( \delta(A) \) if \( s \in A \) and \( t \in \overline{A} \).

**Theorem 2.** \( \exists \) an \( st \)-path \( \iff \forall A \subseteq V, s \in A, t \notin A, \text{ we have } \delta(A) \neq \emptyset \)

\( (s, t) \)-edge-connectivity = maximum number of edge disjoint \( st \)-paths

\( (s, t) \)-vertex-connectivity = maximum number of internally vertex disjoint \( st \)-paths

(i.e., vertex disjoint except for sharing \( s \) and \( t \))

1 Edge capacities

Edge capacity \( u : E \to \mathbb{R}_{\geq 0} \)

important note: capacity \( \neq \) cost
cost \( \sim \) length, reliability, cost (lease, toll), revenue can be negative
capacity \( \sim \) thickness (pipe, cable), maximum throughput always non-negative

**Question:** Given a capacitated network and two nodes \( s, t \)
what is the largest collection \( \{P_1, \ldots, P_k\} \) of \( st \)-paths (not necessarily distinct) such that for each edge \( e \in E \), the number of paths \( P_i \) containing \( e \) is at most \( u_e \) (capacity of \( e \))?

**Answer:** Maximum flow