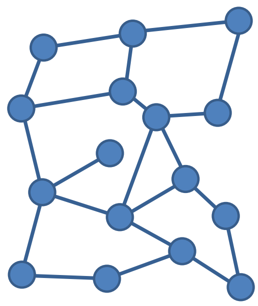
CS 137 - Graph Theory - Lecture 2
February 14, 2012

(further reading Rosen K. H.: *Discrete Mathematics and its Applications*, 5th ed., chapters 8.2, 8.4, 8.5)

1.1. Summary

- Handshaking Lemma
- Paths and cycles in graphs
- Connectivity, Eulerian graphs



1.2. Handshaking Lemma

Let G be a graph and let $\{v_1, \ldots, v_n\}$ be the vertex set of G such that

$$deg(v_1) \geq deg(v_2) \geq \ldots \geq deg(v_n).$$

Then the sequence

$$(deg(v_1), deg(v_2), \ldots, deg(v_n))$$

is called the *degree sequence of* G.

For example, the degree sequence of the graph on the right is $(4,4,4,3,3,3,3,3,2,2,2,2,2,2,2,1)$.

Lemma 1. *Let G be a graph. Then* $\sum_{u \in V(G)} \deg(u) = 2|E(G)|$.

Proof. Every edge $\{u, v\}$ contributes $+2$ to the left-hand side, namely $+1$ to $deg(u)$ and $+1$ to $deg(v)$. □

Exercise: How many edges in a graph G with the degree sequence?
- $(3,3,3,3,3,3)$
- $(3,3,3,3,3)$
- $(3,2,1,0)$

Corollary 2. *In every graph G, the number of vertices of odd degree is even.*

Corollary 3. *Every graph has at least two vertices with the same degree.*

1.3. Paths and cycles in graphs

A *walk* is a sequence $u_1, u_2, \ldots, u_t$ of vertices of G such that $\{u_i, u_{i+1}\} \in E(G)$ for each $i \in \{1, \ldots, t-1\}$. Vertices u_1 and u_t are the *endpoints* of the walk while the vertices $u_2, \ldots, u_{t-1}$ are *internal vertices* of the walk.

The *length* of a walk is the number of edges on the walk, i.e. the walk $u_1, \ldots, u_t$ has length $t-1$.

Note: a walk can consist of only a single vertex (has zero length).

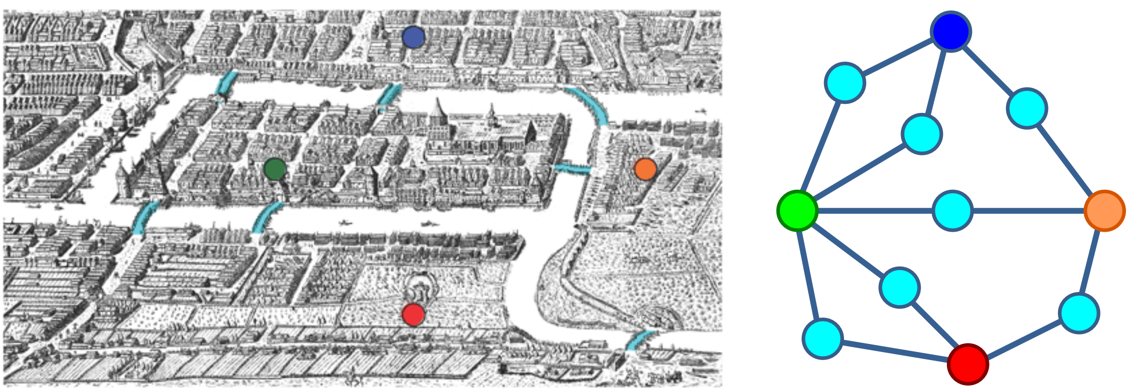
A *closed walk* is a walk whose endpoints are the same.

A *path* is a walk that does not repeat vertices.

A *cycle* is a closed walk of length ≥ 3 that does not repeat vertices (besides its endpoints)

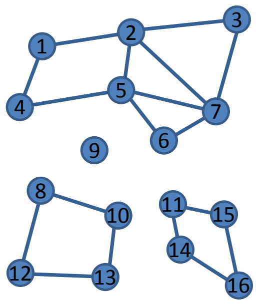
Lemma 4. *If every vertex in G has degree at least two, then G contains a cycle.*

Proof. Take a longest path $u_1, \ldots, u_t$ in G. Since $deg(u_1) \geq 2$, the vertex u_1 has a neighbour x different from u_2. By the maximality of the path, $x = u_i$ for some $i \in \{3, \ldots, t\}$. Thus $u_1, \ldots, u_i, u_1$ is a cycle in G. □



Bridges of Königsberg (1736) - can one travel across all bridges without crossing the same bridge twice?

1.4. Connectivity, Eulerian graphs



A graph G is *connected* if there exists a path between any two vertices of G. Otherwise, G is *disconnected* and it decomposes into *connected components*.

Example: the graph on the right has 4 components whose vertex sets are respectively $\{1,2,3,4,5,6,7\}$, $\{9\}$, $\{8,10,12,13\}$, and $\{11,14,15,16\}$

A *trail* is a walk that does not repeat edges.

A *closed trail* or a *circuit* is a closed walk that does not repeat edges.

An *Euler circuit* is a circuit of G that goes through all edges of G.

Example: each of the connected components of the graph on the right has an Euler circuit, namely:
- $1,2,3,7,2,5,7,6,5,4,1$ is an Euler circuit of the topmost component,
- 9 is an Euler circuit of the middle component,
- $8,10,13,12,8$ is an Euler circuit of the bottom-left component, and
- $11,15,16,14,11$ is an Euler circuit of the bottom-right component.

Notes:
- a single-vertex walk is always a circuit.
- in the above graph the degree of every vertex is even.

We now prove that the latter point, in fact, is both necessary and sufficient for the existence of Euler circuits.

Lemma 5. *Every connected component of G has an Euler circuit if and only if each vertex in G has even degree.*

Proof. ($\Rightarrow$) An Euler circuit contributes +2 to the degree of a vertex each time it passes through that vertex. Thus each vertex in G must have even degree.

($\Leftarrow$) We proceed by induction on the number of edges in G. Consider a connected component K of G. If K has only one vertex v, then v is an Euler circuit of this component. Thus assume that K has more than one vertex. Since the degrees of all vertices in this component are even, this means that each vertex in K has degree at least two. By Lemma 4, K contains a cycle C. Let G' be the graph resulting from G by deleting the edges of the cycle C. Note that every vertex in G' has even degree. Thus, by induction, every connected component G' has an Euler circuit. Observe that K consists of the cycle C and the union of some connected components of G'. It is not difficult to see that combining the Euler circuits of these components together with a circuit that traverses all edges of the cycle C yields an Euler circuit of K. □

Notes:
- the above process actually produces a decomposition of G into edge-disjoint cycles.
- we have the following theorem as a corollary.

A graph is *Eulerian* if it has an Euler circuit.

A connected component is *trivial* if it consits of one vertex (such a vertex is also called an *isolated vertex*).

Theorem 6. *A graph G is Eulerian if and only if every vertex in G has even degree, and G contains at most one non-trivial connected component.*

A *Hamiltonian path/cycle* is a path/cycle of G that goes through all vertices of G.

Related optimisation problems:
- Chinese postman problem: "find a closed walk of least cost that goes through all edges"
- Travelling salesman problem: "find a Hamiltonian cycle of least cost"