

CSC418/2504 Tutorial

3D Transformations

All transformations directly extend to 3D:

- Translation:

$$T(dx, dy, dz) = \begin{pmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Scaling:

$$S(sx, sy, sz) = \begin{pmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Rotation:

$$R_x(A) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos A & -\sin A & 0 \\ 0 & \sin A & \cos A & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_y(A) = \begin{pmatrix} \cos A & 0 & \sin A & 0 \\ 0 & 1 & 0 & 0 \\ -\sin A & 0 & \cos A & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_z(A) = \begin{pmatrix} \cos A & -\sin A & 0 & 0 \\ \sin A & \cos A & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Composing 3D Transformations

Composing 3D transforms works same as 2D: write each transformation matrix in the order the transformation sequence is done

- translation & rotations on same axes are additive, while scaling is multiplicative
- however, note that rotations on different axis are NOT commutative!

General transform:

$$M = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

One useful trick: inverse of the top-left 3x3 submatrix: simply transpose it

$$R = \begin{vmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{vmatrix}$$

$$R^{-1} = (1/\text{Det } R) \begin{vmatrix} r_{11} & -r_{21} & r_{13} \\ -r_{12} & r_{22} & -r_{32} \\ r_{31} & -r_{23} & r_{33} \end{vmatrix}$$

Simple Exercise #1:

Find the inverse of the following transformation matrix:

$$\text{rot}(x, \alpha) \text{ trans}(a, b, c) \text{ rot}(x, -\alpha)$$

Answer:

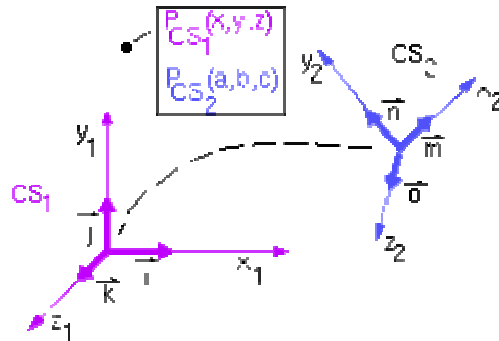
If $M = A.B.C$ then $M^{-1} = C^{-1}.B^{-1}.A^{-1}$, so we have the inverse as

$$\text{rot}(x, \alpha) \text{ trans}(-a, -b, -c) \text{ rot}(x, -\alpha)$$

I might mention that just as in the matrix product $A.B.A^{-1}$ the A and A^{-1} don't cancel out, we may observe geometrically that the above rotation matrices (and similar examples) don't cancel out.

Change of Basis Transform

Suppose we have two coordinate systems, CS 1 and CS 2, having coincident origins, but having different orientations:



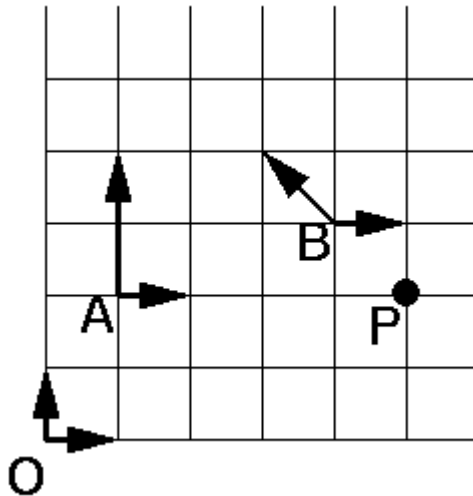
Let i, j, k and m, n, o be unit vectors as shown. These are called basis vectors. The transformation between the two coordinate systems can be obtained as follows:

$$\begin{aligned}
P &= a\vec{m} + b\vec{n} + c\vec{U} \\
&= a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{in } CS_2 \\
&= a \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} + b \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} + c \begin{bmatrix} o_x \\ o_y \\ o_z \end{bmatrix} \quad \text{in } CS_1 \\
\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} &= \begin{bmatrix} m_x & n_x & o_x & 0 \\ m_y & n_y & o_y & 0 \\ m_z & n_z & o_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix}
\end{aligned}$$

The elements of the top-left 3x3 portion of any geometric transformation are really the basis vectors of the local coordinate system expressed in the coordinates of the new coordinate system.

Simple Exercise #2:

Consider the 2D point P and the three coordinate frames O, A, and B below:



Notation:

M_{RS} is a 3x3 homogeneous matrix

$P_R=(x,y,1)$ is a homogeneous point in coordinate frame S such that $P_S = M_{RS} P_R$

a) Find the coordinates of P in the three frames.

In O: $P=(5,2)$

In A: $P=(4,0)$

In B: $P=(0,-1)$

b) Find M_{OA} and M_{OB} .

We first want to find the translation between the coordinate frames to make them coincident.

For M_{OA} , to bring A onto O we have a translation of $(-1, -1)$ in A coordinates.

The basis vectors of O represented in A are: (1, 0) and (0, 0.5)

Therefore:

$$\text{MOA} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 0.5 & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

For M_{OB} , to bring B onto O we have a translation of (-7, -3) in B coordinates.

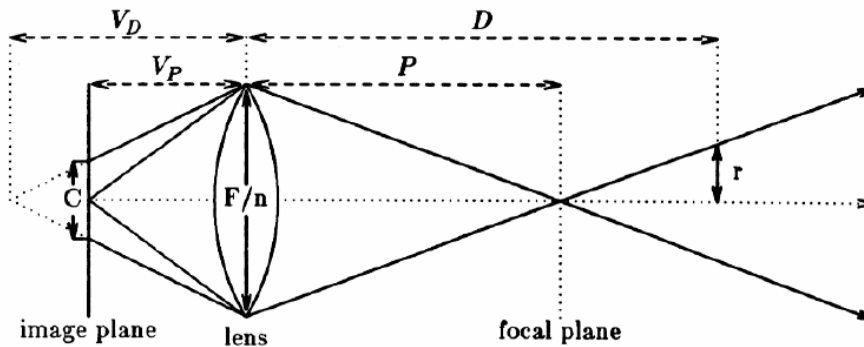
The basis vectors of O represented in B are: (1, 0) and (1, 1)

Therefore:

$$\text{MOB} = \begin{vmatrix} 1 & 1 & -7 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{vmatrix}$$

Test these on P to verify that they give the correct answer.

Real Camera Concepts



$$\begin{aligned} F & - \text{focal length} & V_P & = FP/(P-F) & r & = \frac{1}{2} (F/n) (D-P)/P \\ n & - \text{aperture number} & V_D & = FD/(D-F) & R & = (-V_P/D) r \\ C & - \text{circle of confusion} & C & = (|V_D - V_P|/V_D) (F/n) & R & = \frac{1}{2} C \end{aligned}$$

Aperture: a hole or an opening through which light is admitted. More specifically, the aperture of an optical system is the opening that determines the cone angle of a bundle of rays that come to a focus in the image plane.

Circle of Confusion: an optical spot caused by a cone of light rays from a lens not coming to a perfect focus when imaging a point source.

Depth-of-field: the distance in front of and beyond the subject that appears to be in focus.

Focal length: a measure of how strongly an optical system converges (focuses) or diverges (diffuses) light.

Virtual Pin-hole Camera

A pinhole camera is a camera without a conventional glass lens. An extremely small hole in a very thin material can focus light by confining all rays from a scene through a single point. This camera model is commonly used in computer graphics.

In the simplest case where the projection center is placed at the origin of the world frame and the image plane is the plane $Z=1$, the projection process can be modeled as follows:

$$x = \frac{X}{Z} \quad y = \frac{Y}{Z}$$

For a world point (X,Y,Z) and the corresponding image point (x,y) . Using the homogeneous representation of the points a linear projection equation is obtained:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

This projection is illustrated in the figure below. The optical axis passes through the center of projection C and is orthogonal to the retinal plane R . Its intersection with the retinal plane is defined as the principal point c .

