Question #1: Vertex cover in a bipartite graph

A bipartite graph is a graph $G = (U \cup W, E)$ where the vertex set is composed of two parts U and W and every edge has one end point in U and the other in W. That is there is no edge with both its endpoints in U or W. For example the graph seen below (Figure 1) is a bipartite graph. A vertex cover of a graph G is a subset of its vertices X that *touches* all the edges of G, i.e. each edge of G has at least one of its end points in X. For example, the set of vertices coloured red in Figure 1 is a vertex cover.

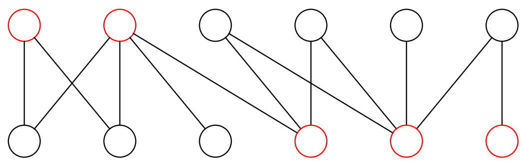


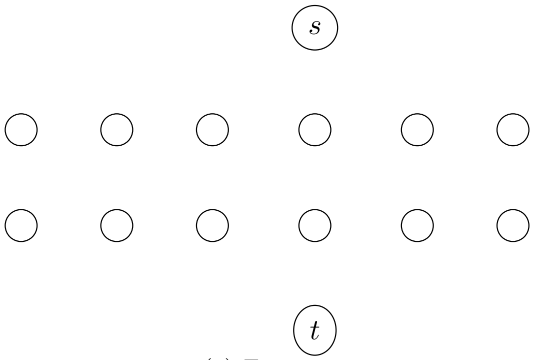
Figure 1: A bipartite graph $G = (U \cup W, E)$ with parts U and W. Red vertices are a vertex cover.

Of course finding a very big vertex cover is easy, e.g. the set of all vertices of G is obviously a vertex cover. Given a bipartite graph $G = (U \cup W, E)$ find the *minimum* vertex cover of G, i.e. a vertex cover of G of minimum possible size.

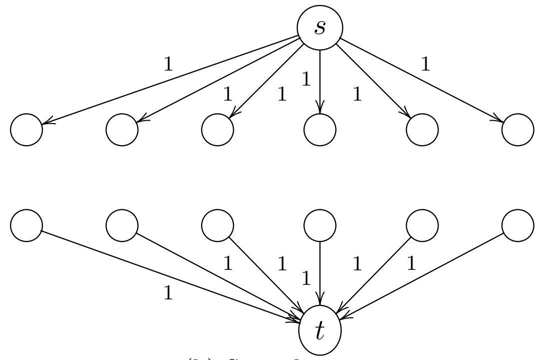
Solution We will show how to do this by reducing it to the Min Cut problem. We will do this step by step like the problem discussed in class. Notice that this is how you should be thinking about the problem *not* how you would write it should this be an assignment/exam question. (In that case you would just give the construction of the flow network and state and prove something like Theorem 1.)
The "choices" we need to make in order to identify a vertex cover X is whether each vertex is in X or not. The choices involved in identifying a cut in a flow network is whether each vertex in the flow network is on the source's side or the sink's. So it makes sense to have one vertex corresponding to each vertex of our bipartite graph in our flow network. That way, at least in a superficial way, each cut corresponds to a possible vertex cover. We also need to have a source and a sink in our flow network. If we do all these for the graph in Figure 1 we will have the network in Figure 2(a). Next, ultimately we want each cut $(S, \overline{S})$ of the resulting flow network to correspond to a vertex cover X of the original graph and that the capacity cut be the same as the size of the vertex cover. It is tempting to put an edge from each vertex to t with capacity 1 and say that the correspondence is all the vertices on the source side of the cut will be the vertices in the cut, but this is not going to work. You would realize this after playing with this for a while or you can argue that unless there are some edges going out of s both the min-cut and maximum flow of the flow network will be 0, completely unrelated to the original graph.
Instead we will treat the vertices in the two parts of the graph, U and W, separately. We will have an edge going from s to each vertex in U and an edge going from each vertex in W to t; all these edges will have capacity 1. This way a capacity of a cut $(S, \overline{S})$ will be the number of vertices in U which are on the sink's side (i.e. are in $\overline{S}$) plus the number of vertices in W that are on the source's side (i.e. are in S). Given such a cut the corresponding (supposed) vertex cover will be

$$X = \{u \in U : u \in \overline{S}\} \cup \{w \in W : w \in S\}.$$

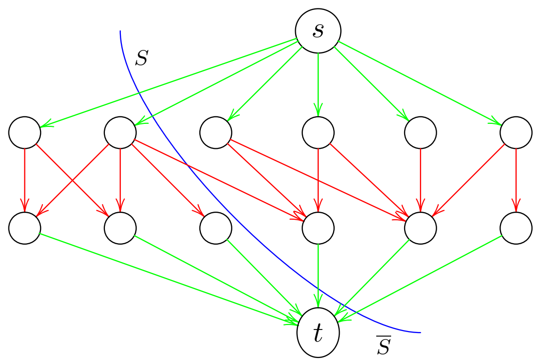
It is clear that $capacity(S, \overline{S}) = |X|$. So far we have the network in Figure 2(b).
Of course, so far we don't have any condition making sure that the set X corresponding to each cut $(S, \overline{S})$ is a vertex cover, i.e. touches each edge. To do this we will add a number of edges to network all of capacity



(a) First step



(b) Second step



(c) Final flow network. Green edges have capacity 1; red edges have capacity ∞.

Figure 2: Constructing our flow network.

∞. Once we add these edges all cuts of *finite capacity* will correspond to valid vertex covers. This will not be too hard. Just add the edges of the original graph oriented from top to bottom (that is from U to W) to the flow network. The result is shown in Figure 2(c). The cut $(S, \overline{S})$ corresponding to the minimum vertex cover in Figure 1 is also shown in Figure 2(c).

Theorem 1. *Let $G = (U \cup W, E)$ be a bipartite graph and G' be the flow network constructed as above from G. The following two are true:*

1. *For every cut $(S, \overline{S})$ of finite capacity (i.e. capacity$(S, \overline{S}) < \infty$) in G' define the set X as,*

$$X = \{u \in U : u \in \overline{S}\} \cup \{w \in W : w \in S\}.$$

The X is a vertex cover of G and $|X| =$ capacity$(S, \overline{S})$.

2. *For every vertex cover X of G, define the cut $(S, \overline{S})$ as,*

$$S = \{u \in U : u \notin X\} \cup \{w \in W : w \in X\} \cup \{s\} \quad \overline{S} = \{u \in U : u \in X\} \cup \{w \in W : w \notin X\} \cup \{t\}$$

The $(S, \overline{S})$ has finite capacity and in fact, capacity$(S, \overline{S}) = |X|$.

Proof. For the first part, consider a cut $(S, \overline{S})$ of finite capacity and consider the X defined in the theorem. Because the cut is finite capacity there are no red edge from S to $\overline{S}$. That would mean that each of the red edges either start inside the U part of $\overline{S}$ or end in the W part of S (or both.) So all the edges of the original graph either have one endpoint in the U part of X or its W part. It is clear that $|X| =$ *capacity*$(S, \overline{S})$.

Now for the second part, consider a vertex cover X. Consider the cut $(S, \overline{S})$ defined in the Theorem. First we will show that it does not have an edge of infinite capacity. Assume the contrary that an edge going from u to w is cut by this cut and has infinite capacity. Then u must be in the U part of S and w must be in the W part of $\overline{S}$. So neither u nor w are in X in the original graph but there is an edge between these two vertices in G. That means that X is not a vertex cover which is a contradiction. Now that we know that only edges of capacity 1 cut by the cut it is clear from the way the cut is defined that its capacity is $|X|$. $\square$

Given Theorem 1 to find the minimum vertex cover in a bipartite graph G one can construct the flow network G', find its min cut and construct the minimum vertex cover from that.