Skolemization, Most General Unifiers, First-Order Resolution

Torsten Hahmann

CSC 384, University of Toronto

March 07, 2011

Skolemization

Conversion of sentences FOL to CNF requires skolemization.

Skolemization: remove existential quantifiers by introducing new function symbols.

How: For each existentially quantified variable introduce a n-place function where n is the number of previously appearing universal quantifiers.

Special case: introducing constants (trivial functions: no previous universal quantifier).

Skolemization - Example 1

- Every philosopher writes at least one book.
  $\forall x[Philo(x) \rightarrow \exists y[Book(y) \land Write(x, y)]]$

Skolemization - Example 1

- Every philosopher writes at least one book.
  $\forall x[Philo(x) \rightarrow \exists y[Book(y) \wedge Write(x, y)]]$
- Eliminate Implication:
  $\forall x[\neg Philo(x) \vee \exists y[Book(y) \wedge Write(x, y)]]$

Skolemization - Example 1

- Every philosopher writes at least one book.
  $\forall x[Philo(x) \rightarrow \exists y[Book(y) \wedge Write(x, y)]]$
- Eliminate Implication:
  $\forall x[\neg Philo(x) \vee \exists y[Book(y) \wedge Write(x, y)]]$
- Skolemize: substitute y by $g(x)$
  $\forall x[\neg Philo(x) \vee [Book(g(x)) \wedge Write(x, g(x))]]$

Skolemization - Example 2

- All students of a philosopher read one of their teacher's books.
  $\forall x \forall y [Philo(x) \wedge StudentOf(y, x) \rightarrow$
  $\exists z [Book(z) \wedge Write(x, z) \wedge Read(y, z)]]$

Skolemization - Example 2

- All students of a philosopher read one of their teacher's books.
  $\forall x \forall y [Philo(x) \land StudentOf(y, x) \rightarrow \exists z [Book(z) \land Write(x, z) \land Read(y, z)]]$
- Eliminate Implication:
  $\forall x \forall y [\neg Philo(x) \lor \neg StudentOf(y, x) \lor \exists z [Book(z) \land Write(x, z) \land Read(y, z)]]$

Skolemization - Example 2

- All students of a philosopher read one of their teacher's books.
  $\forall x \forall y [Philo(x) \wedge StudentOf(y,x) \rightarrow \exists z [Book(z) \wedge Write(x,z) \wedge Read(y,z)]]$
- Eliminate Implication:
  $\forall x \forall y [\neg Philo(x) \vee \neg StudentOf(y,x) \vee \exists z [Book(z) \wedge Write(x,z) \wedge Read(y,z)]]$
- Skolemize: substitute z by $h(x,y)$
  $\forall x \forall y [\neg Philo(x) \vee \neg StudentOf(y,x) \vee [Book(h(x,y)) \wedge Write(x,h(x,y)) \wedge Read(y,h(x,y))]]$

Skolemization - Example 3

- There exists a philosopher with students.
  $\exists x \exists y [Philo(x) \wedge StudentOf(y, x)]$

Skolemization - Example 3

- There exists a philosopher with students.
  $\exists x \exists y [Philo(x) \wedge StudentOf(y, x)]$
- Skolemize: substitute x by a and y by b
  $Philo(a) \wedge StudentOf(b, a)$

Most General Unifier

Least specialized unification of two clauses.

We can compute the MGU using the disagreement set $D_k = \{e_1, e_2\}$: the pair of expressions where two clauses first disagree.

REPEAT UNTIL no more disagreement $\rightarrow$ found MGU.

IF either e_1 or e_2 is a variable V and the other is some term (or a variable) t, then choose $V = t$ as substitution.

Then substitute to obtain S_{k+1} and find disagreement set D_{k+1}.

ELSE unification is not possible.

MGU - Example 1

Find the MGU of $p(f(a), g(X))$ and $p(Y, Y)$:

- $S_0 = \{p(f(a), g(X)) \; ; \; p(Y, Y)\}$

MGU - Example 1

Find the MGU of $p(f(a), g(X))$ and $p(Y, Y)$:

- $S_0 = \{p(f(a), g(X))\ ;\ p(Y, Y)\}$
- $D_0 = \{f(a), Y\}$
- $\sigma = \{Y = f(a)\}$

MGU - Example 1

Find the MGU of $p(f(a), g(X))$ and $p(Y, Y)$:

- $S_0 = \{p(f(a), g(X))\ ;\ p(Y, Y)\}$
- $D_0 = \{f(a), Y\}$
- $\sigma = \{Y = f(a)\}$
- $S_1 = \{p(f(a), g(X))\ ;\ p(f(a), f(a))\}$

MGU - Example 1

Find the MGU of $p(f(a), g(X))$ and $p(Y, Y)$:

- $S_0 = \{p(f(a), g(X))\ ;\ p(Y, Y)\}$
- $D_0 = \{f(a), Y\}$
- $\sigma = \{Y = f(a)\}$
- $S_1 = \{p(f(a), g(X))\ ;\ p(f(a), f(a))\}$
- $D_1 = \{g(X), f(a)\}$

MGU - Example 1

Find the MGU of $p(f(a), g(X))$ and $p(Y, Y)$:

- $S_0 = \{p(f(a), g(X)) \ ; \ p(Y, Y)\}$
- $D_0 = \{f(a), Y\}$
- $\sigma = \{Y = f(a)\}$
- $S_1 = \{p(f(a), g(X)) \ ; \ p(f(a), f(a))\}$
- $D_1 = \{g(X), f(a)\}$
- no unification possible!

MGU - Example 2

- $S_0 = \{p(a, X, h(g(Z))) \ ; \ p(Z, h(Y), h(Y))\}$

MGU - Example 2

- $S_0 = \{p(a, X, h(g(Z)))\ ;\ p(Z, h(Y), h(Y))\}$
- $D_0 = \{a, Z\}$

MGU - Example 2

- $S_0 = \{p(a, X, h(g(Z)))\ ;\ p(Z, h(Y), h(Y))\}$
- $D_0 = \{a, Z\}$
- $\sigma = \{Z = a\}$

MGU - Example 2

- $S_0 = \{p(a, X, h(g(Z)))\ ;\ p(Z, h(Y), h(Y))\}$
- $D_0 = \{a, Z\}$
- $\sigma = \{Z = a\}$
- $S_1 = \{p(a, X, h(g(a)))\ ;\ p(a, h(Y), h(Y))\}$

MGU - Example 2

- $S_0 = \{p(a, X, h(g(Z)))\ ;\ p(Z, h(Y), h(Y))\}$
- $D_0 = \{a, Z\}$
- $\sigma = \{Z = a\}$
- $S_1 = \{p(a, X, h(g(a)))\ ;\ p(a, h(Y), h(Y))\}$
- $D_1 = \{X, h(Y)\}$

MGU - Example 2

- $S_0 = \{p(a, X, h(g(Z)))\ ;\ p(Z, h(Y), h(Y))\}$
- $D_0 = \{a, Z\}$
- $\sigma = \{Z = a\}$
- $S_1 = \{p(a, X, h(g(a)))\ ;\ p(a, h(Y), h(Y))\}$
- $D_1 = \{X, h(Y)\}$
- $\sigma = \{Z = a, X = h(Y)\}$

MGU - Example 2

- $S_0 = \{p(a, X, h(g(Z)))\ ;\ p(Z, h(Y), h(Y))\}$
- $D_0 = \{a, Z\}$
- $\sigma = \{Z = a\}$
- $S_1 = \{p(a, X, h(g(a)))\ ;\ p(a, h(Y), h(Y))\}$
- $D_1 = \{X, h(Y)\}$
- $\sigma = \{Z = a, X = h(Y)\}$
- $S_2 = \{p(a, h(Y), h(g(a)))\ ;\ p(a, h(Y), h(Y))\}$

MGU - Example 2

- $S_0 = \{p(a, X, h(g(Z)))\ ;\ p(Z, h(Y), h(Y))\}$
- $D_0 = \{a, Z\}$
- $\sigma = \{Z = a\}$
- $S_1 = \{p(a, X, h(g(a)))\ ;\ p(a, h(Y), h(Y))\}$
- $D_1 = \{X, h(Y)\}$
- $\sigma = \{Z = a, X = h(Y)\}$
- $S_2 = \{p(a, h(Y), h(g(a)))\ ;\ p(a, h(Y), h(Y))\}$
- $D_2 = \{g(a), Y\}$

MGU - Example 2

- $S_0 = \{p(a, X, h(g(Z)))\ ;\ p(Z, h(Y), h(Y))\}$
- $D_0 = \{a, Z\}$
- $\sigma = \{Z = a\}$
- $S_1 = \{p(a, X, h(g(a)))\ ;\ p(a, h(Y), h(Y))\}$
- $D_1 = \{X, h(Y)\}$
- $\sigma = \{Z = a, X = h(Y)\}$
- $S_2 = \{p(a, h(Y), h(g(a)))\ ;\ p(a, h(Y), h(Y))\}$
- $D_2 = \{g(a), Y\}$
- $\sigma = \{Z = a, X = h(Y), Y = g(a)\}$

MGU - Example 2

- $S_0 = \{p(a, X, h(g(Z)))\ ;\ p(Z, h(Y), h(Y))\}$
- $D_0 = \{a, Z\}$
- $\sigma = \{Z = a\}$
- $S_1 = \{p(a, X, h(g(a)))\ ;\ p(a, h(Y), h(Y))\}$
- $D_1 = \{X, h(Y)\}$
- $\sigma = \{Z = a, X = h(Y)\}$
- $S_2 = \{p(a, h(Y), h(g(a)))\ ;\ p(a, h(Y), h(Y))\}$
- $D_2 = \{g(a), Y\}$
- $\sigma = \{Z = a, X = h(Y), Y = g(a)\}$
- $S_3 = \{p(a, h(g(a)), h(g(a)))\ ;\ p(a, h(g(a)), h(g(a)))\}$

MGU - Example 2

- $S_0 = \{p(a, X, h(g(Z)))\ ;\ p(Z, h(Y), h(Y))\}$
- $D_0 = \{a, Z\}$
- $\sigma = \{Z = a\}$
- $S_1 = \{p(a, X, h(g(a)))\ ;\ p(a, h(Y), h(Y))\}$
- $D_1 = \{X, h(Y)\}$
- $\sigma = \{Z = a, X = h(Y)\}$
- $S_2 = \{p(a, h(Y), h(g(a)))\ ;\ p(a, h(Y), h(Y))\}$
- $D_2 = \{g(a), Y\}$
- $\sigma = \{Z = a, X = h(Y), Y = g(a)\}$
- $S_3 = \{p(a, h(g(a)), h(g(a)))\ ;\ p(a, h(g(a)), h(g(a)))\}$
- No disagreement
  $\Rightarrow \sigma = \{Z = a, X = h(Y), Y = g(a)\}$ is MGU

MGU - Example 3

- $S_0 = \{p(X, X)\ ;\ p(Y, f(Y))\}$

MGU - Example 3

- $S_0 = \{p(X, X)\ ;\ p(Y, f(Y))\}$
- $D_0 = \{X, Y\}$

MGU - Example 3

- $S_0 = \{p(X, X)\ ;\ p(Y, f(Y))\}$
- $D_0 = \{X, Y\}$
- $\sigma = \{X = Y\}$

MGU - Example 3

- $S_0 = \{p(X, X) \ ; \ p(Y, f(Y))\}$
- $D_0 = \{X, Y\}$
- $\sigma = \{X = Y\}$
- $S_1 = \{p(Y, Y) \ ; \ p(Y, f(Y))\}$

MGU - Example 3

- $S_0 = \{p(X, X) \; ; \; p(Y, f(Y))\}$
- $D_0 = \{X, Y\}$
- $\sigma = \{X = Y\}$
- $S_1 = \{p(Y, Y) \; ; \; p(Y, f(Y))\}$
- $D_1 = \{Y, f(Y)\}$

MGU - Example 3

- $S_0 = \{p(X, X) \ ; \ p(Y, f(Y))\}$
- $D_0 = \{X, Y\}$
- $\sigma = \{X = Y\}$
- $S_1 = \{p(Y, Y) \ ; \ p(Y, f(Y))\}$
- $D_1 = \{Y, f(Y)\}$
- no unification possible!

Full example problem

Given the following sentences, answer the question 'What is connected to the Galbraith building?' using resolution with answer extraction:

Connected is a binary symmetric relation.

An object X is part of another object Y iff everything X is connected to, Y is also connected to.

Room GB221 is part of Galbraith building.

Room GB221 is connected to itself.

Full example problem - Representation in FOL

(a) Represent these sentences in first order logic.

Connected is a binary symmetric relation.

An object X is part of another object Y iff everything X is connected to, Y is also connected to.

Room GB221 is part of Galbraith building.

Room GB221 is connected to itself.

Full example problem - Representation in FOL

- Connected is a symmetric relation.
  $(\forall X, Y)\ connected(X, Y) \supset connected(Y, X)$

Full example problem - Representation in FOL

- An object X is part of another object Y iff everything X is connected to, Y is also connected to.
  $(\forall X, Y)\,(part(X, Y) \equiv ((\forall Z)\,connected(Z, X) \supset connected(Z, Y)))$

Full example problem - Representation in FOL

- Room GB221 is part of Galbraith building.
  $part(gb221, galbraith)$

Full example problem - Representation in FOL

- Room GB221 is connected to itself.
  $connected(gb221, gb221)$

Full example problem - CNF Conversion

(b) Convert the formulas to clausal form. Indicate any Skolem functions or constants used.

$(\forall X, Y)\ connected(X, Y) \supset connected(Y, X)$

$(\forall X, Y)\ part(X, Y) \equiv (\forall Z)\ connected(Z, X) \supset connected(Z, Y)$

$part(gb221, galbraith)$

$connected(gb221, gb221)$

Full example problem - CNF Conversion

- $(\forall X, Y)\ connected(X, Y) \supset connected(Y, X)$
  $[-connected(X, Y), connected(Y, X)]$

Full example problem - CNF Conversion

- $(\forall X, Y)\ part(X, Y) \equiv (\forall Z)\ connected(Z, X) \supset connected(Z, Y)$
  $\rightarrow$: $[-part(X, Y), -connected(Z, X), connected(Z, Y)]$
  $\leftarrow$: $[part(X, Y), connected(f(X, Y), X)]$
  $[part(X, Y), -connected(g(X, Y), Y)]$

Full example problem - CNF Conversion

- $part(gb221, galbraith)$
  $[part(gb221, galbraith)]$

Full example problem - CNF Conversion

- $connected(gb221, gb221)$
  $[connected(gb221, gb221)]$

Full example problem - Goal

(c) Convert the negation of the statement 'What is connected to the Galbraith building?' to clause form (using an answer literal).

- FOL: $(\exists X)\ connected(galbraith, X)$

Full example problem - Goal

(c) Convert the negation of the statement 'What is connected to the Galbraith building?' to clause form (using an answer literal).

- FOL: $(\exists X)\ connected(galbraith, X)$
- negate goal!! $(\neg\exists X)\ connected(galbraith, X)$

Full example problem - Goal

(c) Convert the negation of the statement ‘What is connected to the Galbraith building?’ to clause form (using an answer literal).

- FOL: $(\exists X)\ connected(galbraith, X)$
- negate goal!! $(\neg\exists X)\ connected(galbraith, X)$
- CNF with answer literal: $(\forall X)\ \neg connected(galbraith, X)$
  $[-connected(galbraith, X), ans(X)]$

Full example problem - Resolution

(d) Answer the question using resolution and answer extraction. Use the notation developed in class: every new clause must be labeled by the resolution step that was used to generate it. For example, a clause labeled R[4c, 1d]x = a, y = f(b) means that it was generated by resolving literal c of clause 4 against literal d of clause 1, using the MGU x = a, y = f(b).

Our clauses:

1. $[-connected(R, S), connected(S, R)]$
2. $[-part(T, U), -connected(V, T), connected(V, U)]$
3. $[part(W, X), connected(f(W, X), W)]$
4. $[part(Y, Z), -connected(g(Y, Z), Z)]$
5. $[part(gb221, galbraith)]$
6. $[connected(gb221, gb221)]$
7. $[-connected(galbraith, A), ans(A)]$

Full example problem - Resolution

1. $[-connected(R, S), connected(S, R)]$
2. $[-part(T, U), -connected(V, T), connected(V, U)]$
3. $[part(W, X), connected(f(W, X), W)]$
4. $[part(Y, Z), -connected(g(Y, Z), Z)]$
5. $[part(gb221, galbraith)]$
6. $[connected(gb221, gb221)]$
7. $[-connected(galbraith, A), ans(A)]$

Full example problem - Resolution

1. $[-connected(R, S), connected(S, R)]$
2. $[-part(T, U), -connected(V, T), connected(V, U)]$
3. $[part(W, X), connected(f(W, X), W)]$
4. $[part(Y, Z), -connected(g(Y, Z), Z)]$
5. $[part(gb221, galbraith)]$
6. $[connected(gb221, gb221)]$
7. $[-connected(galbraith, A), ans(A)]$
8. R[7a, 1b] $\{S = galbraith, R = U\}$
   $[-connected(A, galbraith), ans(A)]$

Full example problem - Resolution

1. $[-connected(R, S), connected(S, R)]$
2. $[-part(T, U), -connected(V, T), connected(V, U)]$
3. $[part(W, X), connected(f(W, X), W)]$
4. $[part(Y, Z), -connected(g(Y, Z), Z)]$
5. $[part(gb221, galbraith)]$
6. $[connected(gb221, gb221)]$
7. $[-connected(galbraith, A), ans(A)]$
8. R[7a, 1b] $\{S = galbraith, R = U\}$
   $[-connected(A, galbraith), ans(A)]$
9. R[8a, 2c] $\{V = A, U = galbraith\}$
   $[-part(T, galbraith), -connected(A, T), ans(A)]$

Full example problem - Resolution

1. $[-connected(R, S), connected(S, R)]$
2. $[-part(T, U), -connected(V, T), connected(V, U)]$
3. $[part(W, X), connected(f(W, X), W)]$
4. $[part(Y, Z), -connected(g(Y, Z), Z)]$
5. $[part(gb221, galbraith)]$
6. $[connected(gb221, gb221)]$
7. $[-connected(galbraith, A), ans(A)]$
8. R[7a, 1b] $\{S = galbraith, R = U\}$
   $[-connected(A, galbraith), ans(A)]$
9. R[8a, 2c] $\{V = A, U = galbraith\}$
   $[-part(T, galbraith), -connected(A, T), ans(A)]$
10. R[9a, 5] $\{T = gb221\}$
    $[-connected(A, gb221), ans(A)]$

Full example problem - Resolution

1. $[-connected(R, S), connected(S, R)]$
2. $[-part(T, U), -connected(V, T), connected(V, U)]$
3. $[part(W, X), connected(f(W, X), W)]$
4. $[part(Y, Z), -connected(g(Y, Z), Z)]$
5. $[part(gb221, galbraith)]$
6. $[connected(gb221, gb221)]$
7. $[-connected(galbraith, A), ans(A)]$

8. R[7a, 1b] $\{S = galbraith, R = U\}$
   $[-connected(A, galbraith), ans(A)]$
9. R[8a, 2c] $\{V = A, U = galbraith\}$
   $[-part(T, galbraith), -connected(A, T), ans(A)]$
10. R[9a, 5] $\{T = gb221\}$
    $[-connected(A, gb221), ans(A)]$
11. R[10a, 6] $\{A = gb221\}$
    $[ans(gb221)]$