An Ordered Theory Resolution Calculus

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Outline

- Introduction
- An Example
- Definitions of Orderings and Theory Resolution
- The Ordered Theory Resolution (OTR)
- Rigid E-unification & OTR
- Conclusions

Introduction

- Ordinary Resolution

  - blind search through the space of all possible proofs.
- Resolution with Ordering Strategies

  - carefully designed Ordering improves resolution efficiency.
- Stickel's Theory Resolution

  - incorporates special purpose reasoner.
- Ordered Theory Resolution Calculus

  - a combination of the two above methods.

  - it lifts theory resolution from ground case to the non-ground case.

  -conditions are given for proving Soundness/Completeness

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An Example: Theory and KB

- The theory *T* :

$$\forall X : boy(X) \rightarrow person(X)$$
$$\forall X : girl(X) \rightarrow person(X)$$
$$\forall X,Y : person(X) \wedge child(X,Y) \rightarrow person(Y)$$

- The Knowledge Base *S* :

$$boy(a)$$
$$child(a,b)$$
$$child(b,c)$$
$$sex(a, male)$$

An Example: KB and Order

$$sex(b, female)$$
$$sex(c, male)$$
$$child(X,Y) \rightarrow descendant(X,Y)$$
$$child(X,Y) \wedge descendant(Y,Z) \rightarrow descendant(X,Z)$$
$$person(X) \wedge person(Y) \wedge sex(X,Z) \wedge sex(Y,Z) \rightarrow samesex(X,Y)$$
$$descendant(X,Y) \wedge samesex(X,Y) \rightarrow$$

- Order

  - For terms: *a > b > c*.

  - For predicate symbols:

  samesex > sex > person > child > descendant

An Example: Order

- Literals with different predicate symbols: use the ordering on their predicate symbols (e.g. *child(X,Y) > descendant(X,Y)*).

- Literals with same predicate but different sign: the positive is greater than the negative
  (e.g. *descendant(X,Y) > ~descendant(X,Y)*).

- Literals with same predicate and sign without variable: use the term order (e.g. *person(a) > person(b)*).

- Literals with same predicate and sign with variable: incomparable (e.g. *person(X)* and *person(Y)*).

An Example: OTR

C1: [~<u>person(a),</u> ~person(c), ~descendant(a,c)]

C2: [<u>boy(a)</u>]

C3: [~<u>person(c),</u> ~descendant(a,c)]

C4: [<u>boy(a)</u>]

C5: [<u>child(a, b)</u>]

C6: [<u>child(b, c)</u>]

C7: [~<u>descendant(a,c)</u>]

An Example: Infinite Resolution and Order

(7) [C1, descendant(X,Y)]

(8) [C2, ~descendant(Y1,Z1), descendant(X1,Z1)]

sub:{X/Y1, Y/Z1}, resolve upon (7) and (8)

(9) [C3, descendant(X2,Z2)]

sub:{X2/Y1, Z2/Z1}, resolve upon (8) and (9)

(10) [C4, descendant(X3,Z3)]

sub{X3/Y1, Z3/Z1}, resolve upon (8) and (10)

(11) [C5, descendant(X4,Z4)]

... ...

Observation: With order restriction: *child > descendant* this infinite resolution path can be avoided.

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Definitions: Preliminaries

- **Clause** a set of literals {L1, …, Ln}, often written as [L1, …, Ln].
- **Theory T** a satisfiable set of clauses.
- **(Herbrand)-Interpretation** any total function from the set of ground atoms to {True, False}.
- **T-Interpretation** an interpretation that satisfies the theory T.
- **M is Satisfiable** there exists an interpretation that assigns true to all ground instances of every clause in the set M.

Definitions: Ordering

Propositional Case, an example:

(1) a
(2) b
(3) [~a, b]
(4) [~b, a]
(5) [~a, ~b]

order1: a> b
order2: b> a

both resolution orders yield the same result.

Observation: On the propositional level, we may eliminate all atoms in any order. And we may fix an ordering on atoms, and always eliminate the highest atom. Unfortunately, …

Definitions: Ordering

for FOL, this is not true, but the following ordering strategy has been proven sound and complete at the ground level.

- **Ordering** Let > be a partial ordering on terms and literals.

  - The order > is said to be *stable*, if for all substitution σ, if X > Y, then $X\sigma > Y\sigma$.

  - The order is said to be *total* on ground terms and literals, if for any X an Y, X >Y, or Y > X, or X=Y.
- **Biggest literal** A literal L in M is the biggest literal in M iff for all L' in M and L'$\neq$M it holds that L > L'.
- **Maximal literal** A literal L in M is maximal in M iff there does not exist a L' in M such that L' > L .

Definitions: Ordering

- Resolution with the Ordering Strategy

  - a refinement of ordinary resolution.

  - factors of clauses C are computed by unifying maximal atoms in C only.

  - binary resolvents of clauses C and C' are computed by unifying complementary literals that are maximal in their respective clauses.

- The correctness of the Ordering Strategy

  - it is sound and complete (full proof can be found on page 253 of Goubault-larrecq and Mackie's 1996 book: Proof Theory and Automated Deduction).

Definitions: Stickel's Original Theory Resolution

- Formal concepts first introduced by Mark Stickel in his 1985 paper.
- Based on grounded First Order Logic.

  - gives many First Order Theory Resolution examples.

  - does not prove the completeness for the first-order case.
- General Theory Resolution is sound and complete.
- Further restrictions on the definitions of Theory Resolution required to make it practical while preserving completeness.

  - total wide TR, narrow TR.

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Ordered Theory Resolution Calculus

- It incorporates order in theory resolution.
- It lifts from ground case to the non-ground case.
- Problem: How to deal with variables ? Concept of Most General Unifiers (MGU) could be used to keep search as general as possible.
- Unifers need not be unique.

- for example,

Theory T: (t1) forall_x. A(x) => B(x),

(t2) forall_ x. A(f(g(x))) => B(x)

Literals to refute: { A(w), -B(z) }

CSR_T = { {w/z}, {w/f(g(z))} }

note: with a theory, substitutions could be non-obvious ones.

OTR Calculus: Substitutions

- Let L be a literal set. L is **T-complementary** iff for all ground substitution ρ the set $L\rho$ is T-unsatisfiable.
- L is **minimal T-complementary** iff L is T-complementary and all subsets of L are not T-complementary.
- L is **(minimal) T-refutable by** σ iff $L\sigma$ is (minimal) T-complementary.
- **Complete and most general set of T-refuting substitutions for L (or short: CSR_T(L))** iff

  - for all σ in CSR_T(L): L is T-refutable by σ (correctness)

  - for all substitution θ such that L is T-refutable by θ : the exists a σ in CSR_T(L) and a substitution σ' such that

  $\theta = \sigma\sigma'|var(\theta)$ (completeness)

OTR Calculus: Inference Rules

DEFINITION 3.3
(OTR-CALCULUS) Let $\mathcal{T}$ be a theory. The inference rules of the ordered theory resolution calculus (OTR-Resolution) are defined as follows:

Ordered Factoring:

$$\frac{C}{C\sigma} \quad \begin{cases} \text{if (1) } \sigma \text{ is a most general (syntactical) unifier for some } \{L_1, \ldots, L_n\} \subseteq C, \\ \text{and (2) } L_1\sigma \text{ is maximal in } C\sigma \end{cases}$$

Ordered theory resolution:

$$\frac{C_1 \quad \ldots \quad C_n}{(C_1\sigma - \{L_1\sigma\}) \cup \ldots \cup (C_n\sigma - \{L_n\sigma\})} \quad \begin{cases} \text{if (1) } \sigma \in CSR_{\mathcal{T}}(\{L_1, \ldots, L_n\}) \text{ for some } L_1 \in C_1, \ldots, L_n \in C_n, \\ \text{and (2) } L_i\sigma \text{ is maximal in } C_i\sigma \text{ (for } i = 1 \ldots n) \end{cases}$$

The inference rules of the ordered theory resolution calculus.

OTR Calculus: Soundness and Completeness

- **Soundness** Let T be a theory and M be a clause set. If there exists an OTR(T) refutation of M then M is T-unsatisfiable.
- **Completeness** Let T be a theory and M be a T-unsatisfiable clause set. There exists and OTR(T)-refutation of M.
- The above completeness holds if a decision procedure can be provided that will yield a correct and complete set CSR_T(L) when given L .
- Theory resolution would not work if finding theory refuting substitutions was semi- or undecidable. In this case, a single resolution step might not terminate.

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Rigid E-Unification & OTR

- **Equality in reasoning** (Axioms of Equality, paramodulation, equational matings and etc).
- **Rigid E-Unifier** Given a finite set $E=\{u1=v1, \ldots, un=vn\}$ of equations and a pair <u, v> of terms, if there exists a substitution θ such that, treating $E\theta$ as a set of ground equations, we have: $u\theta =_{E\theta} v\theta$, then θ is a rigid E-unifier of u and v.
- **Example** Let $E=\{fa=a, ggx=fa\}$, $<u,v> =<gggx, x>$. Then, the substitution $\theta =[x/ga]$ is a rigid E-unifier of u and v. As $\theta(E)=\{fa=a, ggga=fa\}$, and $\theta(gggx) =_{E\theta} \theta(x)$, since $\theta(gggx)=gggga=gfa=ga=\theta(x)$.

Rigid E-Unification & OTR

- **Proposition** Let M be a literal set and θ be a substitution. Then M is E-refutable by θ iff M θ is E-unsatisfiable, where the variables of M θ are treated as constants.
- **Relation to OTR Calculus** If the theory is equality, then, complete set of refuting substitutions in OTR Calculus can be computed using existing rigid E-unification algorithms.

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Conclusions

- Adds ordering restriction to theory resolution.
- Lifts theory resolution to non-ground case.
- Caveats
  - Building ordering instances is tricky
  - Hardest part is coming up with a decision procedure for finding CSR_T(L) for a specific theory instantiation T.