Generalized Boolean Satisfiability
ZAP II: Theory

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Knowledge Representation and Reasoning

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Outline

- Motivation and Intuition
- Group Theory Background
- Augmented Resolution
- Proof lengths of our examples
- DPLL in the new setting
- Experimental results
- Summary
- Discussion

Motivation

	rep. eff.	p-simulation hierarchy	inference	unit propagation	learning
SAT	1	EEE	resolution	watched literals	relevance
cardinality	exp	P?E	not unique	watched literals	relevance
PB	exp	P?E	unique	watched literals	+ strengthening
symmetry	1	EEE*	not in P	same as SAT	same as SAT
QPROP	exp	???	in P using reasons	exp improvement	+ first-order

- None admits a poly-sized proof for clique colouring
- Is there a representation such that:
  1. Exponential savings in size of representation
  2. Poly-size proofs of the 3 problems exist
  3. DPLL is still efficient

Yes!

Intuition – Pigeon Hole

- “no two pigeons are in the same hole”:

$$\neg p_{ik} \vee \neg p_{jk} \quad \forall\ i \neq j,\ k$$

- One axiom represents $\binom{n}{2}n$ clauses
- Generate other clauses by permuting the literals

Intuition – Parity Axioms

$x_1 + x_2 + x_3 \equiv 1 \pmod 2$

equivalent to the 4 clauses

$x_1 \vee x_2 \vee x_3$ $\qquad$ $\neg x_1 \vee x_2 \vee \neg x_3$

$x_1 \vee \neg x_2 \vee \neg x_3$ $\qquad$ $\neg x_1 \vee \neg x_2 \vee x_3$

How can the other three clauses be generated from the first using permutations?

Intuition – Parity Axioms cont.

1. $x_1 \vee x_2 \vee x_3$
2. $x_1 \vee \neg x_2 \vee \neg x_3$
3. $\neg x_1 \vee x_2 \vee \neg x_3$
4. $\neg x_1 \vee \neg x_2 \vee x_3$

Permutations:

1 -> 1: ()

1 -> 2: $(x_2\ \neg x_2)(x_3\ \neg x_3)$

1 -> 3: $(x_1\ \neg x_1)(x_3\ \neg x_3)$

1 -> 4: $(x_1\ \neg x_1)\ (x_2\ \neg x_2)$

Permutation Set Properties

- Set of permutations for the parity clause:

  $S = \{(), (x_2 \neg x_2)(x_3 \neg x_3), (x_1 \neg x_1)(x_3 \neg x_3), (x_1 \neg x_1)\ (x_2 \neg x_2)\}$

- Compose elements of S:

$$(x_2 \neg x_2)(x_3 \neg x_3)\ (x_1 \neg x_1)(x_3 \neg x_3) = (x_1 \neg x_1)(x_2 \neg x_2)$$
$$(x_2 \neg x_2)(x_3 \neg x_3)\ (x_1 \neg x_1)(x_2 \neg x_2) = (x_1 \neg x_1)(x_3 \neg x_3)$$
$$(x_1 \neg x_1)(x_3 \neg x_3)\ (x_1 \neg x_1)(x_2 \neg x_2) = (x_2 \neg x_2)(x_3 \neg x_3)$$

  - S is closed under function composition

$$(x_2 \neg x_2)(x_3 \neg x_3)\ (x_2 \neg x_2)(x_3 \neg x_3) = (x_1)\ (x_2)\ (x_3)\ (x_4) = ()$$

  - Each element of S is its own inverse

Group Theory

- Set of permutations S:
  - closed under the composition operator (which is associative)
  - contains an identity ()
  - every element has an inverse

  $\Rightarrow$ S with composition operator is a <u>group</u>
- Groups:
  - structured set with operator
  - properties have been studied intensively
- If a set of clauses intuitively has structure, its set of permutations will be a group

Group Theory - Definitions

- Sym(n) – the Symmetric group on n elements
  - the group made up of all permutations of {1,…,n}
- W_n – the subgroup of Sym(2n) s.t. $\neg l_1$ is mapped to $\neg l_2$ whenever l_1 is mapped to l_2
- Generators: $\langle x \rangle$ is the group generated by the element x
  - $\langle x \rangle = \{1, x, x^2, x^3, \ldots, x^{m-1}\}$ where m is the smallest number s.t. $x^m = 1$
  - Every group has a set of $\leq \log_2|G|$ generators
    - can find in poly time

Progress

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- ✓ Group Theory Background
- ➢ Augmented Resolution
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Reasoning and Inference

- Reasoning with augmented clauses:
  - lifted form of resolution, watched literals for UP, and clause learning
- Lets start with resolution, since the size of resolution proofs determines the complexity of DPLL

Augmented Clauses

- Augmented clause (c, G)
  - c is the prototypical clause
  - G is a group of permutations
  - $c^G = \{g(c) \mid g \in G\}$ - set of ground clauses represented
  - Note that if some clauses are not related to others, they can still be represented as (c, {I})

Augmented Resolution

- Result of resolving (c_1, G_1) and (c_2, G_2)
  - ideally another augmented clause (c, G)
  - (c, G) should capture all clauses which can be produced by resolving clauses in (c_1, G_1) and (c_2, G_2)
- But this is not possible!

$(c_1, G) = (a \vee b, (bc))$ -> $\{(a \vee b), (a \vee c)\}$

$(c_2, G) = (\neg a \vee d, (dc))$ -> $\{(\neg a \vee d), (\neg a \vee c)\}$

Resolving ground clauses produces:

$\{(b \vee d), (b \vee c), (c \vee d), (c)\}$

Augmented Resolution – cont.

- Conditions for satisfactory resolution rule:
  - 1. $(c_1, G_1) \wedge (c_2, G_2) \Rightarrow (c, G)$
  - 2. resolve(c_1, c_2) is an instance of (c, G)
  - 3. $G_1 \leq H_1$, $G_2 \leq H_2 \Rightarrow (c, G)$ is a resolvant of (c_1, H_1) and (c_2, H_2)
  - 4., 5. & 6. on pages 501-502

Augmented Resolution Rule

- Resolvant of (c_1, G_1), (c_2, G_2) is any clause $(\text{resolve}(c_1, c_2), G)$ s.t.
  - $G \leq \text{stab}(c_i, H_i) \cap W_n$ where $H_1 \leq G_1$, $H_2 \leq G_2$
- What is $\text{stab}(c_i, H_i)$???
  - subgroup of Sym(2n)
  - $\omega \in \text{stab}(c_i, H_i)$ iff ω behaves the same as some $h_1 \in H_1$ on (c_1, H_1), and behaves the same as some $h_2 \in H_2$ on (c_2, H_2),

Properties of Augmented Resolution

- Time to find resolvant:
  - in NP $\Rightarrow$ worst case exponential in size of augmented clauses
  - but: augmented clauses exponentially smaller than CNF encoding
  - $\therefore$ poly in the CNF encoding
- One augmented resolution step: how many resolutions between ground clauses?

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Proof Lengths

- Representational Efficiency

clause type	Boolean axioms	generators	total size
cardinality	$\binom{m}{k-1}$	2	$m+1$
parity	2^{k-1}	3	$k+5$
QPROP	d^v	$2v$	$v(d+1)$

- Proof Lengths

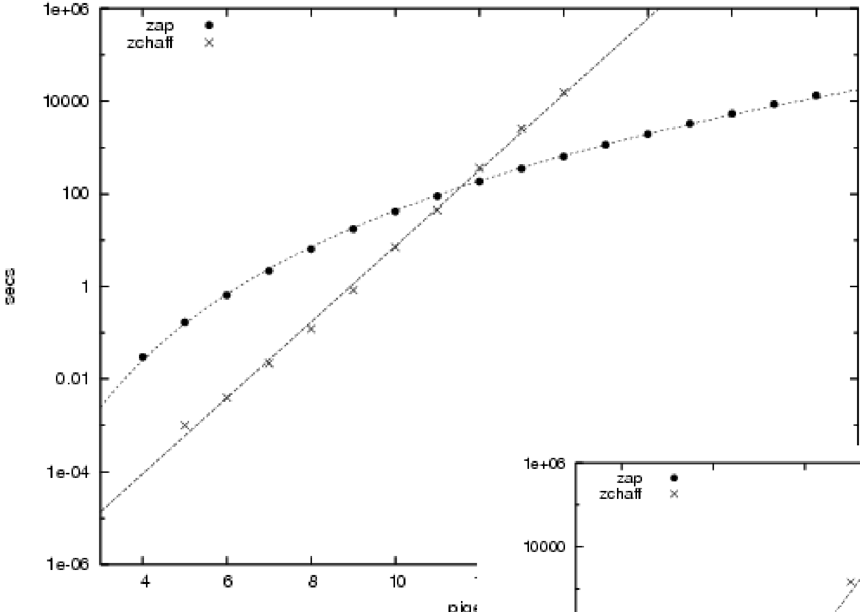
	efficiency of rep'n	proof length	resolution	propagation technique	learning method
SAT	—	EEE	—	watched literals	relevance
cardinality	exponential	P?E	not unique	watched literals	relevance
pseudo-Boolean	exponential	P?E	unique	watched literals	+ strengthening
symmetry	—	EEE*	not believed in P	same as SAT	same as SAT
QPROP	exponential	???	in P using reasons	exp improvement	+ first-order
ZAP	exponential	PPP			

Modified DPLL

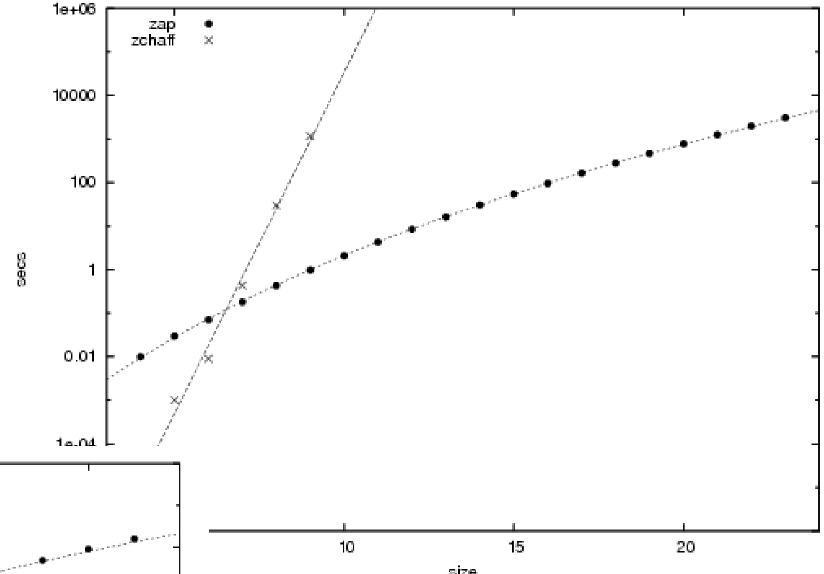
RBL(C, D, P):

```
⟨x, y⟩ ← UNIT-PROPAGATE(C ∪ D, P)
if x = true
   then (c, G) ← y
        if c is empty
           then return FAILURE
           else remove successive elements from P so that c is unit
                D ← D ∪ {c}
                remove from D all augmented clauses without k-relevant instances
                return RBL(C, D, P)
   else P ← y
        if P is a solution to C
           then return P
           else l ← a literal not assigned a value by P
                return RBL(C, ⟨P, (l, true)⟩)
```

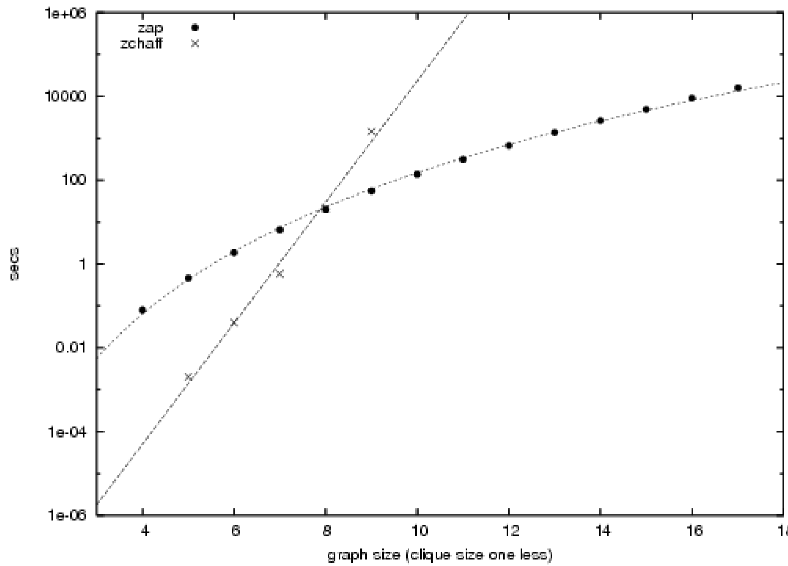
Experimental Results



Pigeon Hole



Parity



Clique Colouring

Summary

- New representation language of augmented clauses
- New inference rule: augmented resolution

	efficiency of rep'n	proof length	resolution	propagation technique	learning method
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symmetry	—	EEE*	not believed in P	same as SAT	same as SAT
QPROP	exponential	???	in P using reasons	exp improvement	+ first-order
ZAP	exponential	PPP	in P using reasons	watched literals, exp improvement	+ first-order + parity + others

Discussion

- How often does group structure occur?
- How hard is it to find existing group structure?
- What about the completeness of augmented resolution?
- How does this representation subsume cardinality, pseudo-boolean, QPROP?