Automated Deduction by Theory Resolution

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with figures from:

- Jörg Siekmann and Graham Wrightson, An Open Research Problem: Strong Completeness of R. Kowalski's Connection Graph Proof Procedure, *Computational Logic: Logic Programming and Beyond 2002: 231-252*

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The Idea

Example (Ordinary Resolution)

Resolve:

$$a \vee b \vee c$$

$$\neg a \vee d$$

to

$$b \vee c \vee d$$

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Idea

Resolve not just over inverse literals but over sets of inconsistent clauses.

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Example (Theory Resolution, here: theory of partial ordering (ORD))

With theory $(\neg(x < x), (x < y) \wedge (y < z) \supset (x > z))$, resolve

$$(a < b) \vee P$$
$$(b < c) \vee Q$$
$$(c < d) \vee R$$
$$\neg(a < d) \vee S$$

to

$$P \vee Q \vee R \vee S$$

Idea

Resolve not just over inverse literals but over sets of inconsistent clauses.

T-satisfiability

Definition (T-interpretation)

A T-interpretation is an interpretation that satisfies the theory T.

Definition (T-unsatisfiable)

A set of clauses S is T-unsatisfiable if no T-interpretation satisfies S. S is minimally T-unsatisfiable if S is T-unsatisfiable but no proper subset of S is.

Theory Resolution: Definition & Example

Definition (Theory Resolution)

Given: $C_1, C_2, \ldots, C_m, m \geq 1$

decompose to: $K_1 \vee L_1, \ldots, K_m \vee L_m, m \geq 1, K_i$ non-empty clauses

$R_1, \ldots, R_n, n \geq 0,$ unit clauses

suppose: $K_1, \ldots, K_m, R_1, \ldots, R_n$ are T-unsatisfiable

then: $L_1 \vee \cdots \vee L_m \vee \neg R_1 \vee \cdots \vee \neg R_n$ is a T-resolvent

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suppose:	$K_1, \ldots, K_m, R_1, \ldots, R_n$ are T-unsatisfiable
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then:	$L_1 \vee \cdots \vee L_m \vee \neg R_1 \vee \cdots \vee \neg R_n$ is a T-resolvent
	$P \vee Q \vee R \vee S$ is a T-resolvent

Soundness Theorem

Theorem (Soundness)

Let T be a theory, S a set of clauses, and C a T-resolvent of S. Then every T-interpretation that satisfies S also satisfies C.

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Generating all possible resolvents is impractical.

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Solution

Restrict the types of resolution steps allowed but maintain completeness!

Types of Resolvents

Definition (Theory Resolution; Types of Resolvents)

Given:	$C_1, C_2, \ldots, C_m, m \geq 1$
decompose to:	$K_1 \vee L_1, \ldots, K_m \vee L_m, m \geq 1, K_i$ non-empty clauses
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suppose:	$K_1, \ldots, K_m, R_1, \ldots, R_n$ are T-unsatisfiable
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	(total)
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unit *(narrow)*

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	$m = 1$ *(unary)*, $m = 2$ *(binary)* unit *(narrow)*
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- ordinary resolution is total, narrow, and binary
- P is a unary total narrow ORD-resolvent of $(a < a) \vee P$
- $P \vee Q \vee R \vee S$ is a 4-ary total narrow ORD-resolvent of $(a < b) \vee P, (b < c) \vee Q, (c < d) \vee R, \neg(a < d) \vee S$
- $(a < c) \vee P \vee Q$ is a partial narrow ORD-resolvent of $(a < b) \vee P$ and $(b < c) \vee Q$ with condition $R_1 = \neg(a < c)$

Total Wide Theory Resolution

total: unconditioned, i.e. no R's needed to resolve
wide: keys can be non-unit

Theorem

Total wide theory resolution is complete.

Intuition

- given a procedure to find all minimally T-unsatisfiable subsets of a set of clauses only containing predicates in T,
- split the set S of clauses to refute into $S_P \cup S_{\bar{P}}$, P the predicates in T,
- first resolve away all predicates P using T-resolution,
- then, resolve the rest using ordinary resolution which is a special case of total wide resolution.

Narrow Theory Resolution

narrow: keys are unit, resolve over sets of literals – no disjunctions anymore!

Total

- like ordinary resolution but resolving over more than just two literals
- demanding on the decision procedure of the theory

Partial

- less demanding: just name a condition for T-unsatisfiability of literals
- key selection and corresponding condition matter
- may resolve upon unrelated clauses
- introduced conditions may not be refutable

Example

- we could resolve $(a < b) \lor P$ and $(c < d) \lor R$ with condition $(b < c) \land (d < a)$ but that's useless
- keys should be suitably related
- size and amount of introduced residues should be minimized

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Key Selection

Incremental Approach (Partial Narrow Theory Resolution)

consider:

$$L_{11} \vee L_{12} \cdots \vee L_{1m}$$
$$L_{21} \vee L_{22} \cdots \vee L_{2m}$$
$$\vdots$$
$$L_{n1} \vee L_{n2} \cdots \vee L_{nm}$$

- Naively: consider all combinations $S = \{L_{1i_1}, L_{2i_2}, \ldots, L_{ni_n}\}$
- Instead can work incrementally upon keys $K \subseteq S$ if:
  - there is a set of literals $R = \{R_1, \ldots, R_k\}$ (a condition) such that $K \cup \{\neg R_1, \ldots, \neg R_k\}$ is minimally T-unsatisfiable, and
  - $(S - K) \cup \{R_1 \vee \cdots \vee R_k\}$ is minimally T-unsatisfiable
- called *key selection criterion*
- preserves completeness

Theory Matings

Definition (Path)

A path through clauses $C_1, \ldots, C_m$ is a set of literals $K_i \in C_i$.

Theorem (Mating)

A set of clauses is unsatisfiable iff every path though it contains a complementary pair of literals.

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cf. total narrow theory resolution

Examples of Theory Resolution

Hyperresolution

- From the *electron clauses* $K_i \vee L_i$ with K_i a literal and L_i a clause, and the *nucleus clause* $\neg K_1 \vee \cdots \vee \neg K_m \vee R$ derive $L_1 \vee \cdots \vee L_m \vee R$.
- corresponds to:
  - partial, narrow theory resolution, where
  - $\neg K_1 \vee \cdots \vee \neg K_m \vee R$ is a consequence of the theory

Procedural Attachment

- Expressions can be "evaluated" to produce new ones, e.g. $2 < 3 \longrightarrow true$
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  - unary theory resolution,
  - can be extended to attach procedures to sets of literals e.g. $a < b, b < c \rightarrow a < c$

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Paramodulation/Equality/Unification

- $P(b)$ is a *paramodulant* of $P(a)$ and $a = b$
- the empty clause is a *E-resolvent* of $P(a), \neg P(b)$, and $a = b$
- $a \neq b$ is a *RUE-resolvent* of $P(a)$ and $\neg P(b)$
- corresponds to:
  - some form of binary-partial/total theory resolution, with equality theory

Resolution for Modal Logic of Belief

- recognize unsatisfiability of- and resolve over modal belief literals, e.g.

$$\Box p \vee A, \Box(p \supset q) \vee B, \neg\Box q \vee C \quad \rightarrow \quad A \vee B \vee C$$

- corresponds to:
  - wide total theory resolution with theory of modal belief

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Taxonomic Reasoning in KRYPTON

	1.	$Boy(x) \supset Person(x)$
	2.	$[Boy(x) \wedge Sex(x,y)] \supset Male(y)$
	3.	$Girl(x) \supset Person(x)$
	4.	$[Girl(x) \wedge Sex(x,y)] \supset Female(y)$
	5.	$[NoSon(x) \wedge Child(x,y)] \supset Girl(y)$
	6.	$[NoDaughter(x) \wedge Child(x,y)] \supset Boy(y)$
	7.	$Person(x) \supset Sex(x, sk1(x))$
	8.	$Male(x) \equiv \neg Female(x)$
hypothesis	9.	$NoSon(Chris)$
hypothesis	10.	$NoDaughter(Chris)$
negated conclusion	11.	$Child(Chris, sk2)$
resolve 11 and 5, simplify by 9	12.	$Girl(sk2)$
resolve 12 and 3	13.	$Person(sk2)$
resolve 7 and 13	14.	$Sex(sk2, sk1(sk2))$
resolve 11 and 6, simplify by 10	15.	$Boy(sk2)$
resolve 15 and 2	17.	$Sex(sk2, x) \supset Male(x)$
resolve 14 and 17	18.	$Male(sk1(sk2))$
resolve 8 and 18	19.	$\neg Female(sk1(sk2))$
resolve 12 and 4	20.	$Sex(sk2, x) \supset Female(x)$
resolve 14 and 20, simplify by 19	21.	$\square$

Figure 1: Resolution Proof for *Childless* Problem

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Figure 2: KRYPTON Proof for *Childless* Problem

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Built In Axioms	Inputted Formulas	Derived Formulas	Retained Formulas	Successful Unifications	Time (seconds)	Proof Length
none	11	10	20	37	1.1	9
1–8	3	2	5	4	0.4	2

Empirical Results

Schubert's Steamroller Problem: Axioms

1. $Wolf(x) \supset Animal(x)$
2. $Fox(x) \supset Animal(x)$
3. $Bird(x) \supset Animal(x)$
4. $Caterpillar(x) \supset Animal(x)$
5. $Snail(x) \supset Animal(x)$
6. $Grain(x) \supset Plant(x)$
7. $[Snail(x) \wedge Bird(y)] \supset Much\text{-}smaller(x, y)$
8. $[Caterpillar(x) \wedge Bird(y)] \supset Much\text{-}smaller(x, y)$
9. $[Bird(x) \wedge Fox(y)] \supset Much\text{-}smaller(x, y)$
10. $[Fox(x) \wedge Wolf(y)] \supset Much\text{-}smaller(x, y)$
11. $[Wolf(x) \wedge Fox(y)] \supset \neg Likes\text{-}to\text{-}eat(x, y)$
12. $[Wolf(x) \wedge Grain(y)] \supset \neg Likes\text{-}to\text{-}eat(x, y)$
13. $[Bird(x) \wedge Snail(y)] \supset \neg Likes\text{-}to\text{-}eat(x, y)$
14. $[Bird(x) \wedge Caterpillar(y)] \supset Likes\text{-}to\text{-}eat(x, y)$
15. $Caterpillar(x) \supset Plant(h(x))$
16. $Caterpillar(x) \supset Likes\text{-}to\text{-}eat(h(x))$
17. $Snail(x) \supset Plant(i(x))$
18. $Snail(x) \supset Likes\text{-}to\text{-}eat(i(x))$
19. $Wolf(a\text{-}wolf)$
20. $Fox(a\text{-}fox)$
21. $Bird(a\text{-}bird)$
22. $Caterpillar(a\text{-}caterpillar)$
23. $Snail(a\text{-}snail)$
24. $Grain(a\text{-}grain)$
25. $Animal(x) \supset [[Plant(y) \supset Likes\text{-}to\text{-}eat(x, y)] \vee [[Animal(z) \wedge Much\text{-}smaller(z, x) \wedge Plant(w) \wedge Likes\text{-}to\text{-}eat(z, w)] \supset Likes\text{-}to\text{-}eat(x, z)]]$

Negated conclusion for Proof (a):

26. $\neg Animal(x) \vee \neg Animal(y) \vee \neg Grain(z) \vee \neg Likes\text{-}to\text{-}eat(y, z) \vee \neg Likes\text{-}to\text{-}eat(x, y)$

Negated conclusion for Proof (b):

27. $[Animal(x) \wedge Animal(y)] \supset Grain(j(x, y))$
28. $\neg Animal(x) \vee \neg Animal(y) \vee \neg Likes\text{-}to\text{-}eat(y, j(x, y)) \vee \neg Likes\text{-}to\text{-}eat(x, y)$

Wolves, foxes, birds, caterpillars, and snails are animals, and there are some of each of them. Also, there are some grains, and grains are plants. Every animal either likes to eat all plants or all animals much smaller than itself that like to eat some plants. Caterpillars and snails are much smaller than birds, which are much smaller than foxes, which in turn are much smaller than wolves. Wolves do not like to eat foxes or grains, while birds like to eat caterpillars but not snails. Caterpillars and snails like to eat some plants.

Therefore, there is an animal that likes to eat a grain-eating animal.

Schubert's Steamroller Problem: Axioms

1. $Wolf(x) \supset Animal(x)$
2. $Fox(x) \supset Animal(x)$
3. $Bird(x) \supset Animal(x)$
4. $Caterpillar(x) \supset Animal(x)$
5. $Snail(x) \supset Animal(x)$
6. $Grain(x) \supset Plant(x)$
7. $[Snail(x) \land Bird(y)] \supset Much\text{-}smaller(x, y)$
8. $[Caterpillar(x) \land Bird(y)] \supset Much\text{-}smaller(x, y)$
9. $[Bird(x) \land Fox(y)] \supset Much\text{-}smaller(x, y)$
10. $[Fox(x) \land Wolf(y)] \supset Much\text{-}smaller(x, y)$
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14. $[Bird(x) \land Caterpillar(y)] \supset Likes\text{-}to\text{-}eat(x, y)$
15. $Caterpillar(x) \supset Plant(h(x))$
16. $Caterpillar(x) \supset Likes\text{-}to\text{-}eat(h(x))$
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19. $Wolf(a\text{-}wolf)$
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22. $Caterpillar(a\text{-}caterpillar)$
23. $Snail(a\text{-}snail)$
24. $Grain(a\text{-}grain)$
25. $Animal(x) \supset [[Plant(y) \supset Likes\text{-}to\text{-}eat(x, y)] \lor$
$[[Animal(z) \land Much\text{-}smaller(z, x) \land Plant(w) \land Likes\text{-}to\text{-}eat(z, w)] \supset Likes\text{-}to\text{-}eat(x, z)]]$

Negated conclusion for Proof (a):

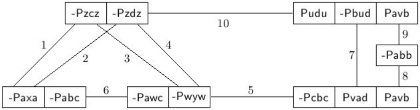
26. $\neg Animal(x) \lor \neg Animal(y) \lor \neg Grain(z) \lor \neg Likes\text{-}to\text{-}eat(y, z) \lor \neg Likes\text{-}to\text{-}eat(x, y)$

Negated conclusion for Proof (b):

27. $[Animal(x) \land Animal(y)] \supset Grain(j(x, y))$
28. $\neg Animal(x) \lor \neg Animal(y) \lor \neg Likes\text{-}to\text{-}eat(y, j(x, y)) \lor \neg Likes\text{-}to\text{-}eat(x, y)$

Connection-Graph Resolution

$$S = \{\{-P(z,c,z), -P(z,d,z)\}, \{P(a,x,a), -P(a,b,c)\}, \{P(a,w,c), P(w,y,w)\}, \{P(u,d,u), -P(b,u,d), P(u,b,b)\}, \{-P(a,b,b)\}, \{-P(c,b,c), P(v,a,d), P(a,v,b)\}\}$$



- Heuristic to decide among resolutions
- Theory resolution here: add more links!

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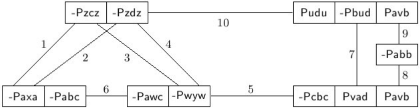

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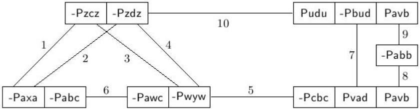

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Empirical Results

Built In Axioms	Inputted Formulas	Derived Formulas	Retained Formulas	Successful Unifications	Time (seconds)	Proof Length
none	26	4,518	72	245,820	5,694	61
1–6	20	2,751	119	57,100	1,261	27
1–10	16	257	23	3,602	88	32
1–13	13	215	37	3,353	73	32
1–14	12	115	23	2,494	54	32
1–18	8	310	20	11,130	224	17
none	27	?	?	?	?	?
1–6	21	7,037	30	172,338	11,027	34
1–10	17	243	23	5,793	140	21
1–13	14	216	20	5,836	143	21
1–14	13	151	19	5,840	124	21
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- no statements about preprocessing time (to build in the axioms)

Summary

- Total theory resolution is possible if there exists a decision procedure for deciding T-unsatisfiability of arbitrary sets of clauses over predicates in the theory.
- Partial narrow theory resolution is possible if there exists a decision procedure for deciding T-unsatisfiability of sets of literals under some condition.

Summary

- Total theory resolution is possible if there exists a decision procedure for deciding T-unsatisfiability of arbitrary sets of clauses over predicates in the theory.
- Partial narrow theory resolution is possible if there exists a decision procedure for deciding T-unsatisfiability of sets of literals under some condition.

Theory Resolution is ..

- .. a nice generalization of many existing approaches,
- .. formalizes a way for integrating domain knowledge into resolution
- .. can reduce proof lengths and time

Critique

- key selection criterion for partial narrow TR seems very strong
- no results for restrictions, *"which should I use?"*
- no empirical results on satisfiable cases
- moves part of problem into decision procedure of theory or in the presented empirical problem into the preprocessing
- hard to gage the practical impact