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Stability for Lagrangian relative equilibria of three-point-mass systems

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Abstract

In the present paper we apply geometric methods, and in particular the reduced energy–momentum (REM) method, to the analysis of stability of planar rotationally invariant relative equilibria of three-point-mass systems. We analyse two examples in detail: equilateral relative equilibria for the three-body problem, and isosceles triatomic molecules. We discuss some open problems to which the method is applicable, including roto-translational motion in the full three-body problem.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

A relative equilibrium for a mechanical system with symmetry is a solution of the equations of motion that is also the orbit of a one-parameter symmetry group. For a closed system of particles described in centre-of-mass coordinates, a relative equilibrium is a solution in which the whole system rotates with a constant angular velocity about a fixed axis through the centre of mass. Similarly to equilibria for non-symmetric systems, relative equilibria lie at the core of the qualitative analysis of the phase space and can be used as base points for perturbation theory. In molecular chemistry, understanding relative equilibria can lead to the explanation and prediction of spectra (see [KRT99]).

The present paper concerns 'three-point-mass systems', i.e. generalized three-body problems with arbitrary potentials (possibly different for each pair of particles). For such systems, the simplest non-collinear relative equilibria are those that are confined to a fixed plane. We choose to call such relative equilibria *Lagrangian* due to the famous Lagrangian

equilateral solution of the Newtonian three-body problem. We analyse the nonlinear and spectral stability of these relative equilibria.

Our study continues the theme developed by Marsden and Lin in [LM92], in which the geometrical view of mechanics, and aspects of reduction theory, are presented to a broad intended audience of scientists interested in molecular dynamics. In the present paper we apply geometric methods, and in particular the reduced energy–momentum (REM) method, to the analysis of stability of relative equilibria. The REM method, introduced in 1991 by Simo, Lewis and Marsden [SLM91] is the fundamental method for proving nonlinear stability for simple mechanical systems with symmetry. It has been successfully applied to elasticity [SLM91] and rotating liquid drops [Lew93]. The REM is an 'energetics' method that significantly reduces computational effort using the symmetries of the problem. It is specific to 'simple mechanical systems', which are those with Hamiltonian of the form 'kinetic plus potential'. The REM reduces the stability computation to a test of positive definiteness of the second variation of an amended potential, restricted to a subspace of configuration variations. The method relies on a splitting of the space of phase space variations that has the additional property of bringing the linearized equations of motion into a normal form. The latter can be used to test spectral stability.

We employ these methods to analyse the existence and stability (both nonlinear and spectral) of equilateral and isosceles Lagrangian relative equilibria. We then apply these methods to two particular problems: equilateral relative equilibria for equal masses with homogeneous interactions; and isosceles relative equilibria for triatomic molecules of the form AB_2, i.e. two identical atoms. For the latter problem we produce stability diagrams, for molecules of type H_3^+, H_2D^+ and D_2H^+. Our diagrams agree with those in Kozin *et al* [KRT99, KRT00], but our methods involve less computational effort.

The structure of the paper is as follows. We begin by briefly presenting the theory of relative equilibria from the point of view of geometric mechanics. We then review the reduced energy–momentum (REM) method, and the associated method for testing for spectral stability. Section 3 is dedicated to the three-point-mass problem and the class of Lagrangian relative equilibria. In a general setting, we apply the theory of nonlinear and spectral stability presented previously. In section 4 we test stability for equilateral and isosceles relative equilibria for motion in various potentials. We also point out some possible future applications, in particular the stability of the Lagrangian relative equilibria in the roto-translational motion of the full three-body problem. We summarize our results in the conclusion.

2. Relative equilibria of simple mechanical systems

In this section we outline the basic theory of relative equilibria of simple mechanical systems with symmetry, including the locked inertia tensor and the augmented and amended potentials. We then present the reduced energy–momentum method for proving nonlinear stability of relative equilibria, and show how the method also leads to one way of testing spectral stability. The exposition follows [Ma92], to which we refer the reader for proofs and details.

2.1. Simple mechanical systems with symmetry

A simple mechanical system on configuration manifold Q is one with a Lagrangian $L : TQ \to \mathbb{R}$ of the form $L(q, \dot{q}) = K(q, \dot{q}) - V(q)$ ('kinetic minus potential'), where $K(q, \dot{q}) = \frac{1}{2}\|\dot{q}\|_q^2$ for some given Riemannian metric $\langle\langle , \rangle\rangle$ on Q. (We will also use the notation $v_q = (q, \dot{q})$.) This system is said to have symmetry if there is a Lie group G that acts properly on Q by isometries that leave the potential function V invariant. This implies that the Lagrangian is

invariant under the natural lift of this action to TQ. We assume that G acts on the left, and that the actions are free.

The *Legendre transform* for such a system is the diffeomorphism $\mathbb{F}L : TQ \to T^*Q$ defined by

$$\langle \mathbb{F}L(v_q), w_q \rangle = \langle\langle v_q, w_q \rangle\rangle,$$

for all $v_q, w_q \in T_qQ$, where $\langle , \rangle$ is the natural pairing of T_qQ and T_q^*Q. The Lagrangian system on TQ corresponds, via the Legendre transform, to a Hamiltonian system on T^*Q with Hamiltonian of the form $H(p_q) = K(p_q) + V(q)$, where $K(p_q) := K((\mathbb{F}L)^{-1}(p_q))$. The tangent-lifted action of G on TQ corresponds, via the Legendre transform, to the cotangent-lifted action of G on T^*Q.

Let $\mathfrak{g}$ be the Lie algebra of G. We denote the adjoint action of $g \in G$ on $\xi \in \mathfrak{g}$ by $\mathrm{Ad}_g\xi$, and the coadjoint action of g on $\mu \in \mathfrak{g}^*$ by $\mathrm{Ad}^*_{g^{-1}}\mu$. The corresponding infinitesimal actions are denoted ad and ad*.

The *momentum map* corresponding to the action of G on T^*Q is the map $J : T^*Q \to \mathfrak{g}^*$ given by the formula

$$\langle J(p_q), \eta \rangle = \langle p_q, \eta_Q(q) \rangle, \tag{2.1}$$

where η_Q is the infinitesimal generator of the action on Q corresponding to $\eta \in \mathfrak{g}$, and the pairing $\langle , \rangle$ on the left-hand side is the natural pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$. This map is Ad*-equivariant, i.e., equivariant with respect to the coadjoint action on $\mathfrak{g}^*$. Noether's theorem guarantees that J is conserved along solutions of the Hamiltonian vector field. If $G = SO(3)$ then J is the *angular momentum*.

The *locked inertia tensor* is defined to be the map $\mathbb{I} : Q \to \mathcal{L}(\mathfrak{g}, \mathfrak{g}^*)$, given by

$$\langle \eta, \mathbb{I}(q)\xi \rangle := \langle\langle \xi_Q(q), \eta_Q(q) \rangle\rangle. \tag{2.2}$$

Note that $\mathbb{I}(q)$ is always positive definite (since we are assuming that G acts freely). It is easily verified that, for all $\xi, \eta \in \mathfrak{g}$,

$$J(\mathbb{F}L(\xi_Q(q))) = \mathbb{I}(q)\xi. \tag{2.3}$$

Given a $\mu \in \mathfrak{g}^*$, there is a *mechanical connection one-form* α_μ on Q defined via

$$\langle \alpha_\mu(q), v_q \rangle = \langle \mu, \mathbb{I}^{-1}(q)J(\mathbb{F}L(v_q)) \rangle,$$

for all $v_q \in T_qQ$. It is clear from equation (2.3) that $\langle \alpha_\mu(q), \xi_Q(q) \rangle = \langle \mu, \xi \rangle$, for all $q \in Q$ and $\xi \in \mathfrak{g}$, which implies that $J(\alpha_\mu(q)) = \mu$ for all $q \in Q$. The mechanical connection one-form is used in the exposition of the reduced energy–momentum method, but is in fact not needed in the application of the method.

In any G-symmetric dynamical system on a general phase space P, a point $z \in P$ is a *relative equilibrium* if there is a $\xi \in \mathfrak{g}$ such that the solution curve in P passing through z is given by the one-parameter family

$$t \mapsto \exp(t\xi) \cdot z.$$

Since we are assuming the group action to be free, ξ is unique. Observe that if z is a relative equilibrium, then every point on the solution curve through z is also a relative equilibrium, with the same ξ. If $G = SO(3)$, then ξ is the *angular velocity* of the relative equilibrium. In general, ξ is called the *velocity* of the relative equilibrium.

For simple mechanical systems (and indeed any Hamiltonian system with a momentum map), a point z is a relative equilibrium with velocity ξ if and only if z is a critical point of the *augmented Hamiltonian*,

$$H_\xi(z) := H(z) - \langle J(z), \xi \rangle. \tag{2.4}$$

(See [Ma92] for proofs of this and other results in this section).

Remark 2.1. If $(\exp t\xi_e)\cdot z_e$ is a relative equilibrium of a Hamiltonian system with momentum map J, then its momentum has a constant value $\mu_e := J(z_e)$. By equivariance of the momentum map, $\mathrm{Ad}^*_{\exp t\xi_e}\mu_e = \mu_e$ for all t, and hence $\mathrm{ad}^*_{\xi_e}\mu_e = 0$, or equivalently, $\xi_e \in \mathfrak{g}_{\mu_e}$.

Consider a simple mechanical system, with kinetic energy K and potential V. For any $\mu \in \mathfrak{g}^*$, define the *amended kinetic energy*

$$K_\mu(p_q) := \tfrac{1}{2}\|p_q - \alpha_\mu(q)\|^2,$$

and the *amended potential*

$$V_\mu(q) := V(q) + \tfrac{1}{2}\langle \mu, \mathbb{I}(q)^{-1}\mu\rangle. \tag{2.5}$$

It can be shown that, for any $\mu \in \mathfrak{g}^*$ and any $p_q \in J^{-1}(\mu)$,

$$H(p_q) = K_\mu(p_q) + V_\mu(q). \tag{2.6}$$

It follows that, for any $\mu \in \mathfrak{g}^*$, $p_q \in J^{-1}(\mu)$ and $\xi \in \mathfrak{g}$,

$$H_\xi(p_q) = H(p_q) - \langle \mu, \xi\rangle = K_\mu(p_q) + V_\mu(q) - \langle \mu, \xi\rangle. \tag{2.7}$$

Further, any point $p_q \in T^*_q Q$ with momentum μ is a relative equilibrium if and only if q is a critical point of V_μ and $p_q = \alpha_\mu(q)$ (see [Ma92]). Note that the latter condition, $p_q = \alpha_\mu(q)$, is equivalent to p_q being a critical point of K_μ.

There is a similar criterion involving the *augmented potential*,

$$V_\xi(q) := V(q) - \tfrac{1}{2}\langle \mathbb{I}(q)\xi, \xi\rangle. \tag{2.8}$$

As shown in [Ma92], a point $p_q \in T^*Q$ is a relative equilibrium with velocity ξ if and only if q is a critical point of V_ξ and $p_q = \mathbb{F}L(\xi_Q(q))$.

Remark 2.2. It follows, using equation (2.3), that if p_q is a relative equilibrium with velocity ξ then $J(p_q) = \mathbb{I}(q)\xi$.

2.2. Stability criteria for relative equilibria

This section describes the reduced energy–momentum method for showing nonlinear stability of relative equilibria, and an associated method for computing their spectral stability.

We begin with the more general energy–momentum method, introduced by Marsden *et al* in [MaSiLePo89] and developed by Patrick in [Pat92]. This applies not only to simple mechanical systems, but to any G-symmetric Hamiltonian system on a symplectic manifold, with Ad^*-equivariant momentum map J. Consider a relative equilibrium z_e with velocity ξ_e and momentum μ_e, and let G_{μ_e} be the isotropy group of μ_e with respect to the coadjoint action on $\mathfrak{g}^*$.

The idea of the energy–momentum method is to use the augmented Hamiltonian H_{ξ_e} defined equation (2.4), as a Liapunov function. (Recall that $\delta H_{\xi_e}(z_e) = 0$.) However, since H_{ξ_e} is constant on the G_{μ_e} orbit of z_e, the kernel of $\delta^2 H_{\xi_e}(z_e)$ will include at least $\mathfrak{g}_{\mu_e}\cdot z_e$ (the tangent to the G_{μ_e} orbit through z_e). Consideration of the Liapunov method on the symplectic reduced space leads to the following definition.

Definition 2.3. *If $\delta^2 H_{\xi_e}(z_e)$ is definite on some subspace of* $\ker DJ(z_e)$ *transverse to the G_{μ_e} orbit through z_e, then z_e is said to be formally stable.*

Applying the Liapunov method in the symplectic reduced space, one can show that if z_e is formally stable then the corresponding equilibrium $[z_e]_{G_{\mu_e}}$ in the symplectic reduced space is

Liapunov stable, in which case we say that the relative equilibrium z_e is *leafwise stable*. The following definition of stability for z_e, introduced by Patrick [Pat92], is stronger because it also requires that z_e be stable with respect to perturbations that change the momentum value. This is now the accepted definition of nonlinear stability of relative equilibria for symmetric Hamiltonian systems.

Definition 2.4. *A relative equilibrium z_e, with momentum μ_e, is G_{μ_e}-stable (or stable modulo G_{μ_e}) if and only if, for every G_{μ_e}-invariant neighbourhood V of z_e there exists a neighbourhood $U \subseteq V$ of z_e that is invariant under the Hamiltonian flow.*

Theorem 2.5 (energy–momentum method) [Pat92]**.** *In the above context, if $\mathfrak{g}$ admits an inner product invariant under the adjoint action of G_{μ_e} then formal stability of z_e implies G_{μ_e}-stability.*

Remark 2.6. If G is compact, then G_μ is compact as well, in which case the inner product required in the theorem can be constructed by averaging.

The reduced energy–momentum method (REM) for simple mechanical systems refines the energy–momentum method by providing an alternative, computationally cheaper way of checking formal stability. Our presentation of this method follows [Ma92], and all results stated here and in the next subsection appear in one or both of [Ma92] and [SLM91].

Consider a simple mechanical system, with notation as in the previous section. Let $z_e = (q_e, p_e)$ be a relative equilibrium with velocity ξ_e and momentum μ_e. The essence of the REM is to split $\ker DJ(z_e)$ into four subspaces with respect to which $\delta^2 H_{\xi_e}(z_e)$ is block-diagonal, and to relate this splitting of $\ker DJ(z_e)$ to a splitting of $T_{q_e}Q$.

In order to guarantee that these splittings exist, we use the *Arnold form*, which is a bilinear form on $\mathfrak{g}_{\mu_e}^\perp$, the orthogonal complement to $\mathfrak{g}_{\mu_e}$ with respect to $\mathbb{I}(q_e)$. The Arnold form is $\mathcal{A}_{\mu_e} : \mathfrak{g}_{\mu_e}^\perp \times \mathfrak{g}_{\mu_e}^\perp \to \mathbb{R}$, given by

$$\mathcal{A}_{\mu_e}(\eta, \zeta) = \left\langle \mathrm{ad}_\eta^* \mu_e, \chi_{(q_e,\mu_e)}(\zeta) \right\rangle = \left\langle \mu_e, \mathrm{ad}_\eta \chi_{(q_e,\mu_e)}(\zeta) \right\rangle, \tag{2.9}$$

where

$$\chi_{(q,\mu)}(\zeta) = \mathbb{I}(q)^{-1} \,\mathrm{ad}_\zeta^* \mu + \mathrm{ad}_\zeta \mathbb{I}(q)^{-1} \mu.$$

We now define

$$\begin{aligned} \mathcal{V}_{\mathrm{RIG}} &:= \mathfrak{g}_{\mu_e}^\perp \cdot q_e, \\ \mathcal{V}_{\mathrm{INT}} &:= \left\{ \delta q \in \left(\mathfrak{g}_{\mu_e} \cdot q_e\right)^\perp \,|\, \langle \eta, (D\mathbb{I}(q_e)\delta q)\xi_e \rangle = 0, \text{ for all } \eta \in \mathfrak{g}_{\mu_e}^\perp \right\}, \end{aligned} \tag{2.10}$$

where the orthogonal complement $\left(\mathfrak{g}_{\mu_e} \cdot q_e\right)^\perp$ is taken with respect to the given Riemannian metric. The names of the subspaces are inspired by rotationally invariant systems, in which $\mathcal{V}_{\mathrm{RIG}}$ contains 'rigid' variations, i.e. infinitesimal rotations, while $\mathcal{V}_{\mathrm{INT}}$ contains 'internal' variations, meaning ones that are not purely rigid but instead have a shape-changing component. Curiously, $\mathcal{V}_{\mathrm{INT}}$ need not be contained in $(\mathfrak{g} \cdot q_e)^\perp$.

It is shown in [Ma92] that, when the Arnold form is nondegenerate, the following is a splitting of $T_{q_e}Q$:

$$T_{q_e}Q = \mathfrak{g}_{\mu_e} \cdot q_e \oplus \mathcal{V}_{\mathrm{RIG}} \oplus \mathcal{V}_{\mathrm{INT}}.$$

There is an associated four-way splitting

$$\ker DJ(z_e) = \mathfrak{g}_{\mu_e} \cdot z_e \oplus \mathcal{S}_{\mathrm{RIG}} \oplus \mathcal{W}_{\mathrm{INT}} \oplus \mathcal{W}_{\mathrm{INT}}^*,$$

with components defined as follows,

$$\mathcal{S}_{\mathrm{RIG}} = T_{q_e}\alpha_{\mu_e} \cdot \mathcal{V}_{\mathrm{RIG}}, \qquad \mathcal{W}_{\mathrm{INT}} = T_{q_e}\alpha_{\mu_e} \cdot \mathcal{V}_{\mathrm{INT}}, \qquad \mathcal{W}_{\mathrm{INT}}^* = \{\mathrm{vert}(\gamma) \mid \gamma \in [\mathfrak{g} \cdot q]^\circ\},$$

where $\mathrm{vert}(\gamma) \in T_{(q_e,p_e)}T^*Q$ is defined by $\mathrm{vert}(\gamma) = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}(z_e + s\gamma)$. The Hessian $\delta^2 H_{\xi_e}(z_e)$ block-diagonalizes with respect to this splitting (see [Ma92]). One of the blocks is always zero—the one corresponding to the subspace $\mathfrak{g}_{\mu_e} \cdot z_e$. Thus it is the three remaining subspaces that are important:

$$\mathcal{S}_{\mathrm{RIG}} \oplus \mathcal{W}_{\mathrm{INT}} \oplus \mathcal{W}^*_{\mathrm{INT}}. \tag{2.11}$$

All of these subspaces are contained in $\ker DJ(z_e)$. Recall from the previous subsection that z_e is a critical point of K_{μ_e} and of V_{μ_e}. (In the latter case, it is more precise to say that z_e is a critical point of the composition $V_{\mu_e} \circ \pi$, where $\pi(p_q) = q$, but we will abuse notation and omit π.) Also recall that for all $p_q \in J^{-1}(\mu_e)$,

$$H_{\xi_e}(p_q) + \langle \mu_e, \xi_e \rangle = K_{\mu_e}(p_q) + V_{\mu_e}(q).$$

Thus, for variations in $\mathcal{S}_{\mathrm{RIG}} \oplus \mathcal{W}_{\mathrm{INT}} \oplus \mathcal{W}^*_{\mathrm{INT}}$, we have

$$\delta^2 H_{\xi_e}(z_e) = \delta^2 K_{\mu_e}(q_e) + \delta^2 V_{\mu_e}(q_e).$$

In the diagonal blocks corresponding to $\mathcal{S}_{\mathrm{RIG}}$ and $\mathcal{W}_{\mathrm{INT}}$, it can be shown that only the $\delta^2 V_{\mu_e}(q_e)$ term is non-zero, and since V_{μ_e} is a function of q only, it suffices to consider $\delta^2 V_{\mu_e}(q_e)$ on $\mathcal{V}_{\mathrm{RIG}}$ and $\mathcal{V}_{\mathrm{INT}}$. By the definition $\mathcal{V}_{\mathrm{RIG}} = \mathfrak{g}_{\mu_e}^{\perp} \cdot q_e$, the 2-form $\delta^2 V_{\mu_e}(q_e)$ on $\mathcal{V}_{\mathrm{RIG}}$ can be expressed in terms of a 2-form on $\mathfrak{g}_{\mu_e}^{\perp}$. It can be shown that the latter equals the Arnold form, defined in equation (2.9). For variations $\delta_1 q, \delta_2 q \in \mathcal{V}_{\mathrm{INT}}$, a straightforward calculation leads to the following useful identity:

$$\delta^2 V_{\mu_e}(q_e)(\delta_1 q, \delta_2 q) = \delta^2 V_{\xi_e}(q_e)(\delta_1 q, \delta_2 q) + \langle \mathbb{I}^{-1}(q_e) D\mathbb{I}(q_e)(\delta_1 q)\xi_e,\, D\mathbb{I}(q_e)(\delta_2 q)\xi_e \rangle \tag{2.12}$$

where V_{ξ_e} is the augmented potential defined in (3.9).

In the block corresponding to $\mathcal{W}^*_{\mathrm{INT}}$, since $\delta q = 0$ for all variations, we have $\delta^2 H_{\xi_e}(z_e) = \delta^2 K_{\mu_e}(z_e)$, and in fact one can also check that $\delta^2 K_{\mu_e}(z_e) = \delta^2 K(z_e)$ on $\mathcal{W}^*_{\mathrm{INT}}$. Thus $\delta^2 H_{\xi}(q_e, \mathbf{p}_e)$ has the following structure, with respect to the splitting in equation (2.11),

$$\delta^2 H_{\xi}(q_e, p_e) = \begin{bmatrix} (\text{Arnold form}) & 0 & 0 \\ 0 & \delta^2 V_{\mu_e}(q_e) & 0 \\ 0 & 0 & \delta^2 K(q_e, p_e) \end{bmatrix}. \tag{2.13}$$

Note that the $\delta^2 K$ block is always positive definite.

Theorem 2.7 (reduced energy–momentum method) [SLM91, Ma92]**.** *Let $z_e = (q_e, p_e)$ be a cotangent relative equilibrium and assume that $\mathcal{V}_{\mathrm{INT}} \neq \{0\}$. If $\delta^2 H_{\xi_e}(z_e)$ is definite, then it must be positive definite. Necessary and sufficient conditions for $\delta^2 H_{\xi_e}(z_e)$ to be positive definite are*

1. *The Arnold form is positive definite on $\mathcal{V}_{\mathrm{RIG}}$ and*
2. *$\delta^2 V_{\mu_e}(q_e)$ is positive definite on $\mathcal{V}_{\mathrm{INT}}$.*

Combining this Patrick's version of the general energy–momentum method (theorem 2.5), we obtain

Corollary 2.8. *If the two conditions in theorem 2.7 are satisfied, and if $\mathfrak{g}$ admits an inner product invariant under the adjoint action of G_{μ_e}, then z_e is G_{μ_e}-stable.*

If the conditions are not satisfied, then we cannot conclude anything (directly) about nonlinear stability. In this case, we can look instead at spectral stability by computing the eigenvalues of the linearization of $X_{H_{\xi_e}}$ at z_e. (This is the appropriate vector field to linearize,

not X_H, since $X_{H_{\xi_e}}(z_e) = 0$.) By differentiating the defining relation $X^T_{H_{\xi_e}}\,\Omega = \mathrm{d}H_{\xi_e}$ with respect to z at $z_e = (q_e, p_e)$, and using $X_{H_{\xi_e}}(z_e) = 0$, we obtain the linearization

$$L = \left[\Omega_{(q_e,p_e)}\right]^{-T} \delta^2 H_\xi(q_e, p_e), \tag{2.14}$$

where the superscript '$-T$' denotes inverse transpose. The restriction of L to $\mathcal{S}_{\mathrm{RIG}} \oplus \mathcal{W}_{\mathrm{INT}} \oplus \mathcal{W}^*_{\mathrm{INT}}$ is the linearized vector field of the symplectic reduced system. The form of $\delta^2 H_\xi(q_e, p_e)$ in these 'coordinates' was given in equation (2.13). The symplectic form $\Omega_{(q_e,p_e)}$ in these coordinates has the following block structure:

$$\Omega_{(q_e,p_e)} = \begin{bmatrix} \Omega_{\mu_e} & [\text{internal-rigid coupling}] & 0 \\ -(\text{internal-rigid coupling})^T & S & B \\ 0 & -B^T & 0 \end{bmatrix}. \tag{2.15}$$

If the coordinates on $\mathcal{W}^*_{\mathrm{INT}}$ are chosen to be conjugate to the coordinates used on $\mathcal{W}_{\mathrm{INT}}$ then B will be the identity matrix, as is assumed in [Ma92, SLM91]; but there is no clear computational advantage to doing so. The detailed definitions of the other blocks are as follows (see [Ma92]). For any $\delta q_i \in \mathcal{V}_{\mathrm{INT}}$ and $\eta_i \in \mathfrak{g}^\perp_{\mu_e}$, we define $\delta z_i = T\alpha_{\mu_e} \cdot \delta q_i \in \mathcal{W}_{\mathrm{INT}}$ and $\Delta z_i = T\alpha_{\mu_e} \cdot (\eta_i)_Q(q_e) \in \mathcal{S}_{\mathrm{RIG}}$, and note that all elements of $\mathcal{W}_{\mathrm{INT}}$ and $\mathcal{S}_{\mathrm{RIG}}$ are of this form. Then

$$\left[\Omega_{\mu_e}\right] \quad \Omega_{(q_e,p_e)}(\Delta z_1, \Delta z_2) = -\langle \mu_e, [\eta_1, \eta_2]\rangle; \tag{2.16}$$

$$\text{(internal-rigid coupling)} \quad \Omega_{(q_e,p_e)}(\Delta z, \delta z) = -\langle\!\langle \zeta_Q(q_e), \delta q\rangle\!\rangle, \quad \text{where } \zeta = \mathbb{I}(q_e)^{-1}\,\mathrm{ad}^*_\eta \mu_e; \tag{2.17}$$

$$[\mathrm{S}] \quad \Omega_{(q_e,p_e)}(\delta z_1, \delta z_2) = -\mathrm{d}\alpha_{\mu_e}(\delta q_1, \delta q_2) = -\mathrm{d}\alpha_{\xi_e}(\delta q_1, \delta q_2), \tag{2.18}$$

where $\alpha_{\xi_e}(q) := \mathbb{F}L((\xi_e)_Q(q))$. These formulae can be used to compute L and its eigenvalues.

Remark 2.9. The space $\mathcal{S}_{\mathrm{RIG}}$ is isomorphic to $\mathfrak{g}^\perp_{\mu_e}$, which in turn is isomorphic to the tangent space at μ_e to the coadjoint orbit through μ_e. If these spaces are identified, then Ω_{μ_e} equals the 'minus' Kostant–Kirillov–Souriau symplectic structure.

Remark 2.10. For any $\eta \in \mathfrak{g}$, it is straightforward to check that the term $\mathrm{ad}^*_\eta \mu_e$ in equation (2.17) must be contained in $\mathfrak{g}^\circ_{\mu_e}$. Since $\mathfrak{g}^\perp_{\mu_e}$ is the orthogonal complement of $\mathfrak{g}_{\mu_e}$ with respect to $\mathbb{I}(q_e)$, it follows that $\mathbb{I}(q_e)^{-1}\,\mathrm{ad}^*_\eta \mu_e \in \mathfrak{g}^\perp_{\mu_e}$. Hence, in equation (2.17), we have $\zeta_Q(q_e) \in \mathfrak{g}^\perp_{\mu_e} \cdot q_e = \mathcal{V}_{\mathrm{RIG}}$. The term δq in the same equation is in $\mathcal{V}_{\mathrm{INT}}$. Thus, when $\mathcal{V}_{\mathrm{RIG}}$ and $\mathcal{V}_{\mathrm{INT}}$ are orthogonal, as is the case for the Lagrangian relative equilibria in section 3, the internal-rigid coupling block of $\Omega_{(q_e,p_e)}$ is zero.

When the internal-rigid coupling block is zero, it is simple to compute $\left(\Omega_{(q_e,p_e)}\right)^{-T}$, and hence L. We find

$$L = \left[\Omega_{(q_e,p_e)}\right]^{-T} \delta^2 H_\xi(q_e, p_e) = \begin{bmatrix} \Omega^{-T}_{\mu_e} \mathcal{A}_{\mu_e} & 0 & 0 \\ 0 & 0 & B^{-T}\delta^2 K \\ 0 & -B^{-1}\delta^2 V_{\mu_e} & -B^{-1} S B^{-T} \delta^2 K \end{bmatrix}. \tag{2.19}$$

The eigenvalues of the lower right-hand 2×2 block of this system are the roots of

$$\det\left(\lambda^2 B(\delta^2 K)^{-1} B^T + \lambda S + \delta^2 V_{\mu_e}\right). \tag{2.20}$$

The proof of these results appears in [Ma92] for $B = I$, and generalizes trivially to any B.

Another way of using equation (2.14) is to note that the determinant of a symplectic matrix is always 1, and the eigenvalues of the Hessian $\delta^2 H_\xi(q_e)$ are always real. If the Hessian

has odd index (an odd number of negative eigenvalues), then the determinant of L is negative. Since non-real eigenvalues come in conjugate pairs, and the product of any non-zero conjugate pair is real and positive, L must have an odd number of negative real eigenvalues. Since the eigenvalues of L are invariant under reflection in the imaginary axis, it follows that L has at least one positive real eigenvalue. Therefore the relative equilibrium is spectrally unstable, and hence nonlinearly unstable.

2.3. Rotationally invariant systems

We now revisit some of the theory in this section in the important case of $G = SO(3)$, the rotation group. Since this group is compact, the energy–momentum method in theorem 2.5 applies to any SO(3)-invariant system in which the group action is free and proper. We have $\mathfrak{so}(3) \cong \mathbb{R}^3$, with $\mathrm{ad}_\eta \zeta = \eta \times \zeta$ and $\mathrm{ad}^*_\eta \nu = \nu \times \eta$, for all $\eta, \zeta \in \mathfrak{so}(3)$ and $\nu \in \mathfrak{so}(3)^*$.

Consider a relative equilibrium $z_e = (q_e, p_e)$, with non-zero angular velocity ξ_e and angular momentum $\mu_e = \mathbb{I}(q_e)\xi_e$. Since $\mathrm{ad}^*_\eta \mu_e = \mu_e \times \eta$, for any η, we see that $\mathfrak{g}_{\mu_e}$ is the subspace spanned by μ_e. By remark 2.1, we know that $\xi_e \in \mathfrak{g}_{\mu_e}$, which in the present context implies that $\mu_e = \lambda_e \xi_e$ for some $\lambda_e \in \mathbb{R}$. Since $\mu_e = \mathbb{I}(q_e)\xi_e$, this implies that ξ_e is an eigenvector of $\mathbb{I}(q_e)$. Thus, *all relative equilibria have an angular velocity that is an eigenvector of the moment of inertia tensor*.

The Arnold form can be simplified in this context:

$$\begin{aligned}\mathcal{A}_{\mu_e}(\eta, \zeta) &= \langle \mathrm{ad}^*_\eta \mu_e, (\mathbb{I}(q_e)^{-1} \mathrm{ad}^*_\zeta \mu_e + \mathrm{ad}_\zeta \mathbb{I}(q_e)^{-1} \mu_e) \rangle \\ &= (\mu_e \times \eta) \cdot (\mathbb{I}^{-1}(q_e)(\mu_e \times \zeta) + \zeta \times \xi_e) \\ &= (\mu_e \times \eta)^T (\mathbb{I}^{-1}(q_e) - \lambda_e^{-1} I)(\mu_e \times \zeta).\end{aligned}$$

If $\hat{\mu}_e$ is the matrix such that $\hat{\mu}_e \eta = \mu_e \times \eta$ for all η, then

$$\mathcal{A}_{\mu_e}(\eta, \zeta) = \eta^T \hat{\mu}_e^T \big(\mathbb{I}^{-1}(q_e) - \lambda_e^{-1} I\big) \hat{\mu}_e \zeta. \tag{2.21}$$

Thus the Arnold form is the restriction to $\mathfrak{g}^\perp_{\mu_e} \times \mathfrak{g}^\perp_{\mu_e}$ of $\hat{\mu}_e^T \big(\mathbb{I}^{-1}(q_e) - \lambda_e^{-1} I\big) \hat{\mu}_e$.

Since $\mathbb{I}(q_e)$ is positive definite, it has three positive eigenvalues. Suppose that λ_e has multiplicity one, i.e., the other two eigenvalues are not equal to λ_e. This implies that $\mathfrak{g}_{\mu_e}$ is the entire eigenspace corresponding to λ_e. It follows that $\mathfrak{g}^\perp_{\mu_e}$, which is by definition the 'orthogonal complement' of $\mathfrak{g}_{\mu_e}$ with respect to $\mathbb{I}(q_e)$, is the direct sum of the eigenspace(s) corresponding to the one or two eigenvalues that are not equal to λ_e. The eigenspaces of distinct eigenvalues are orthogonal with respect to the Euclidean inner product (since $\mathbb{I}(q_e)$ is symmetric), and hence $\mathfrak{g}^\perp_{\mu_e}$ is orthogonal to $\mathfrak{g}_{\mu_e}$ with respect to the Euclidean inner product (as well as with respect to $\mathbb{I}(q_e)$).

If λ_e is the largest eigenvalue of $\mathbb{I}(q_e)$ (strictly larger than the others, which are still all positive), then λ_e^{-1} is the smallest eigenvalue of $\mathbb{I}^{-1}(q_e)$. It follows that, for any non-zero $\eta \in \mathfrak{g}^\perp_{\mu_e}$, we have $\eta^T \mathbb{I}^{-1}(q_e)\eta > \eta^T \lambda_e^{-1} \eta$, which implies that the Arnold form is positive definite. Similarly, if λ_e is the smallest eigenvalue of $\mathbb{I}(q_e)$, the Arnold form is negative definite. These results are familiar in example of a rigid body, in which the Arnold form *equals* the second variation of the augmented Hamiltonian, so that the Arnold form being either positive or negative definite guarantees nonlinear stability.

3. Lagrangian relative equilibria in three-point-mass systems

3.1. Three-point-mass systems

Consider an isolated system formed by three point masses where the mutual interaction between any two masses i and j has a potential energy of the form $\epsilon_{ij} f_{ij}(r_{ij})$ where ϵ_{ij} is a

constant, r_{ij} is the inter-particle distance, and f_{ij} is a bonding potential function describing the interaction of masses i and j. Of course, the constant ϵ_{ij} could be absorbed into the definition of the function f_{ij}, but we choose to keep it separate so that, in some applications, all of the functions f_{ij} are identical. The constant ϵ_{ij} may depend, for example, on the masses or charges of particles i and j (see section 4).

We choose to describe the system in Jacobi coordinates $(\mathbf{r}, \mathbf{s})$, where $\mathbf{r}$ is the relative vector from the first to the second mass and $\mathbf{s}$ is the position vector of the third mass relative to the centre of the mass of the first two. Let the masses of the particles be m_1, m_2, m_3, and define

$$\alpha_1 := \frac{m_2}{m_1 + m_2}, \qquad \alpha_2 := \frac{m_1}{m_1 + m_2}. \tag{3.1}$$

Then the potential energy is given by

$$V(\mathbf{r}, \mathbf{s}) = \epsilon_{12} f_{12}(|\mathbf{r}|) + \epsilon_{13} f_{13}(|\mathbf{s} + \alpha_1 \mathbf{r}|) + \epsilon_{23} f_{23}(|\mathbf{s} - \alpha_2 \mathbf{r}|). \tag{3.2}$$

After reduction by the translational symmetry of the problem, the Lagrangian of the system may be written as

$$L(\mathbf{r}, \mathbf{s}, \dot{\mathbf{r}}, \dot{\mathbf{s}}) = \tfrac{1}{2} M_1 \dot{\mathbf{r}}^2 + \tfrac{1}{2} M_2 \dot{\mathbf{s}}^2 - V(\mathbf{r}, \mathbf{s}), \tag{3.3}$$

where

$$M_1 := \frac{m_1 m_2}{m_1 + m_2}, \qquad M_2 := \frac{m_3(m_1 + m_2)}{m_1 + m_2 + m_3}, \tag{3.4}$$

and $\mathbf{v}^2$ is the square of the Euclidean norm (for $\mathbf{v} = \dot{\mathbf{r}}$ or $\mathbf{v} = \dot{\mathbf{s}}$). Note that this Lagrangian has the form $L = K - V$, with kinetic energy $K(\dot{\mathbf{r}}, \dot{\mathbf{s}}) = \frac{1}{2} \|(\dot{\mathbf{r}}, \dot{\mathbf{s}})\|^2$, where the norm corresponds to the Riemannian metric with constant diagonal matrix $\mathbb{M} := \operatorname{diag}(M_1, M_1, M_1, M_2, M_2, M_2)$. Thus the three-point-mass system is simple mechanical.

The corresponding Hamilton function is

$$H(\mathbf{r}, \mathbf{s}, \mathbf{p}_r, \mathbf{p}_s) = \frac{1}{2M_1} \mathbf{p}_r^2 + \frac{1}{2M_2} \mathbf{p}_s^2 + V(\mathbf{r}, \mathbf{s}), \tag{3.5}$$

where $\mathbf{p}_r = M_1 \dot{\mathbf{r}}$ and $\mathbf{p}_s = M_2 \dot{\mathbf{s}}$, that is, the momenta corresponding to $\mathbf{r}$ and $\mathbf{s}$, respectively.

Since we are interested only in non-degenerate and non-symmetric relative equilibria, the configuration manifold is given by $(\mathbf{r}, \mathbf{s}) \in Q := (\mathbb{R}^3 \times \mathbb{R}^3) \backslash A$, where A is the set of double and triple collisions or symmetric configurations (i.e. collinear $\mathbf{r}$ and $\mathbf{s}$). The spatial rotation group $SO(3)$ acts naturally on Q, and by (co)tangent lifts on the phase spaces TQ and T^*Q. We make the usual identification of $\mathfrak{so}(3)$ with $\mathbb{R}^3$, so the infinitesimal generator of $\xi \in \mathfrak{so}(3)$ is

$$(\xi_e)_Q(\mathbf{r}, \mathbf{s}) = (\xi_e \times \mathbf{r}, \xi_e \times \mathbf{s}). \tag{3.6}$$

From this and the definition of the momentum map (equation (3.5)), it follows that the momentum map $\mathbf{J} : T^*Q \to \mathfrak{so}(3)^*$ is given by

$$\mathbf{J}(\mathbf{r}, \mathbf{s}, \mathbf{p}_r, \mathbf{p}_s) = \mathbf{r} \times \mathbf{p}_r + \mathbf{s} \times \mathbf{p}_s. \tag{3.7}$$

Since the Hamiltonian (in equation (3.5)) is invariant under the $SO(3)$ action, J is conserved along the flow (this is the well-known conservation of angular momentum).

By direct calculation, the locked inertia tensor can be shown to be

$$\mathbb{I}(\mathbf{r}, \mathbf{s}) = (M_1 \mathbf{r}^2 + M_2 \mathbf{s}^2) I_3 - (M_1 \mathbf{r} \otimes \mathbf{r} + M_2 \mathbf{s} \otimes \mathbf{s}) \tag{3.8}$$

where $(\mathbf{v} \otimes \mathbf{w})(\xi, \eta) := (\mathbf{v} \cdot \xi)(\mathbf{w} \cdot \eta)$ and I_3 is the 3×3 identity matrix. The augmented potential thus takes the form

$$V_{\xi_e}(\mathbf{r}, \mathbf{s}) = V(\mathbf{r}, \mathbf{s}) - \tfrac{1}{2}(M_1 \mathbf{r}^2 + M_2 \mathbf{s}^2) \xi_e{}^2 + \tfrac{1}{2}(M_1 (\mathbf{r} \cdot \xi_e)^2 + M_2 (\mathbf{s} \cdot \xi_e)^2) \tag{3.9}$$

and the relative equilibria are the solutions of

$$\begin{cases} \epsilon_{12} f'_{12}(|\mathbf{r}|)\,\dfrac{\mathbf{r}}{|\mathbf{r}|} + \epsilon_{13} f'_{13}(|\mathbf{s}+\alpha_1\mathbf{r}|)\,\dfrac{\alpha_1(\mathbf{s}+\alpha_1\mathbf{r})}{|\mathbf{s}+\alpha_1\mathbf{r}|} + \epsilon_{23} f'_{23}(|\mathbf{s}-\alpha_2\mathbf{r}|)\,\dfrac{(-\alpha_2)(\mathbf{s}-\alpha_2\mathbf{r})}{|\mathbf{s}-\alpha_2\mathbf{r}|} \\ \quad - M_1\xi_e^2\mathbf{r} + M_1(\mathbf{r}\cdot\xi_e)\xi_e = \mathbf{0} \\ \epsilon_{13} f'_{13}(|\mathbf{s}+\alpha_1\mathbf{r}|)\,\dfrac{\mathbf{s}+\alpha_1\mathbf{r}}{|\mathbf{s}+\alpha_1\mathbf{r}|} + \epsilon_{23} f'_{23}(|\mathbf{s}-\alpha_2\mathbf{r}|)\dfrac{\mathbf{s}-\alpha_2\mathbf{r}}{|\mathbf{s}-\alpha_2\mathbf{r}|} - M_2\xi_e^2\mathbf{s} + M_2(\mathbf{s}\cdot\xi_e)\xi_e = \mathbf{0}. \end{cases} \tag{3.10}$$

Rearranging the terms in (3.10), we obtain

$$\begin{cases} \left(\epsilon_{12}\dfrac{f'_{12}(|\mathbf{r}|)}{|\mathbf{r}|} + \alpha_1^2\epsilon_{13}\dfrac{f'_{13}(|\mathbf{s}+\alpha_1\mathbf{r}|)}{|\mathbf{s}+\alpha_1\mathbf{r}|} + \alpha_2^2\epsilon_{23}\dfrac{f'_{23}(|\mathbf{s}-\alpha_2\mathbf{r}|)}{|\mathbf{s}-\alpha_2\mathbf{r}|} - M_1\xi_e^2\right)\mathbf{r} \\ \quad + \left(\alpha_1\epsilon_{13}\dfrac{f'_{13}(|\mathbf{s}+\alpha_1\mathbf{r}|)}{|\mathbf{s}+\alpha_1\mathbf{r}|} - \alpha_2\epsilon_{23}\dfrac{f'_{23}(|\mathbf{s}-\alpha_2\mathbf{r}|)}{|\mathbf{s}-\alpha_2\mathbf{r}|}\right)\mathbf{s} + M_1(\mathbf{r}\cdot\xi_e)\xi_e = \mathbf{0} \\ \left(\alpha_1\epsilon_{13}\dfrac{f'_{13}(|\mathbf{s}+\alpha_1\mathbf{r}|)}{|\mathbf{s}+\alpha_1\mathbf{r}|} - \alpha_2\epsilon_{23}\dfrac{f'_{23}(|\mathbf{s}-\alpha_2\mathbf{r}|)}{|\mathbf{s}-\alpha_2\mathbf{r}|}\right)\mathbf{r} \\ \quad + \left(\epsilon_{13}\dfrac{f'_{13}(|\mathbf{s}+\alpha_1\mathbf{r}|)}{|\mathbf{s}+\alpha_1\mathbf{r}|} + \epsilon_{23}\dfrac{f'_{23}(|\mathbf{s}-\alpha_2\mathbf{r}|)}{|\mathbf{s}-\alpha_2\mathbf{r}|} - M_2\xi_e^2\right)\mathbf{s} + M_2(\mathbf{s}\cdot\xi_e)\xi_e = \mathbf{0}. \end{cases} \tag{3.11}$$

We conclude from these equations that either ξ_e is coplanar with $\mathbf{r}$ and $\mathbf{s}$, or $\mathbf{r}\cdot\xi_e = \mathbf{s}\cdot\xi_e = 0$. In the latter case, since $\mathbf{r}$ and $\mathbf{s}$ are assumed to be noncollinear, ξ_e must be perpendicular to the plane spanned by $\mathbf{r}$ and $\mathbf{s}$, and it follows from equation (3.6) that the motion remains in a fixed plane. It is this case that we will investigate further.

3.2. Lagrangian relative equilibria

We will call a relative equilibrium *Lagrangian* if the three mass points are noncollinear but their motion is confined to a fixed plane which, without loss of generality, we assume to be the xy plane. Let ξ_e be the angular velocity of such a relative equilibrium. From equation (3.6), we see that in order for the motion to remain in the xy plane, ξ_e must be perpendicular to it, i.e. $\xi_e = (0, 0, (\xi_e)_z)$. Making this assumption, the terms $\mathbf{r}\cdot\xi_\mathbf{e}$ and $\mathbf{s}\cdot\xi_\mathbf{e}$ in equation (3.11) vanish, while ξ_e^2 becomes $(\xi_e)_z^2$. Since $\mathbf{r}$ and $\mathbf{s}$ are assumed to not be parallel, one can easily check that the system in equation (3.11) is now equivalent to

$$\begin{cases} (\xi_e)_z^2 = \dfrac{\epsilon_{12}}{M_1}\dfrac{f'_{12}(|\mathbf{r}|)}{|\mathbf{r}|} + \dfrac{\alpha_1^2\epsilon_{13}}{M_1}\dfrac{f'_{13}(|\mathbf{s}+\alpha_1\mathbf{r}|)}{|\mathbf{s}+\alpha_1\mathbf{r}|} + \dfrac{\alpha_2^2\epsilon_{23}}{M_1}\dfrac{f'_{23}(|\mathbf{s}-\alpha_2\mathbf{r}|)}{|\mathbf{s}-\alpha_2\mathbf{r}|} \\ (\xi_e)_z^2 = \dfrac{\epsilon_{13}}{M_2}\dfrac{f'_{13}(|\mathbf{s}+\alpha_1\mathbf{r}|)}{|\mathbf{s}+\alpha_1\mathbf{r}|} + \dfrac{\epsilon_{23}}{M_2}\dfrac{f'_{23}(|\mathbf{s}-\alpha_2\mathbf{r}|)}{|\mathbf{s}-\alpha_2\mathbf{r}|} \\ \alpha_1\epsilon_{13}\dfrac{f'_{13}(|\mathbf{s}+\alpha_1\mathbf{r}|)}{|\mathbf{s}+\alpha_1\mathbf{r}|} = \alpha_2\epsilon_{23}\dfrac{f'_{23}(|\mathbf{s}-\alpha_2\mathbf{r}|)}{|\mathbf{s}-\alpha_2\mathbf{r}|}. \end{cases} \tag{3.12}$$

These are the conditions for the existence of a Lagrangian relative equilibrium in the xy plane. Given a relative equilibrium $(\mathbf{r}_e, \mathbf{s}_e)$, we can choose the coordinate system so that

$$\mathbf{r}_e = (b, 0, 0), \qquad \mathbf{s}_e = (d, h, 0), \tag{3.13}$$

with $b > 0$. The parameters b and h are the 'base' and 'height' of the triangle, respectively. Since the three mass points are assumed to be noncollinear, h is non-zero.

We will focus on equilateral and isosceles triangle configurations.

Equilateral relative equilibria. These are described by

$$\mathbf{r}_e = (b, 0, 0), \qquad \mathbf{s}_e^{\pm} = \left(\frac{b}{2}\left(\frac{m_1 - m_2}{m_1 + m_2}\right), \pm\frac{\sqrt{3}}{2}b, 0\right). \tag{3.14}$$

Note that $|\mathbf{r}| = |\mathbf{s} + \alpha_1\mathbf{r}| = |\mathbf{s} - \alpha_2\mathbf{r}| = b$. For any masses, any constants ϵ_{ij} and any functions f_{ij}, the conditions in equation (3.12) (for the existence of relative equilibria) are equivalent to

$$\epsilon_{ij}\frac{f'_{ij}(b)}{b} = \frac{m_i m_j}{m_1 + m_2 + m_3}(\xi_e)_z^2, \qquad 1 \leqslant i < j \leqslant 3. \tag{3.15}$$

Isosceles relative equilibria. We assume, without loss of generality, that $r_{13} = r_{23} = l$, for some constant l, so that $l = |\mathbf{s}+\alpha_1\mathbf{r}| = |\mathbf{s}-\alpha_2\mathbf{r}| = \sqrt{\frac{b^2}{4}(\alpha_1+\alpha_2)^2 + h^2} = \sqrt{\frac{b^2}{4} + h^2}$. We further assume that $m_1 = m_2 = m$, for some constant m, which implies that $M_1 = \frac{m}{2}$, $M_2 = \frac{2mm_3}{2m+m_3}$, $\alpha_1 = \alpha_2 = \frac{1}{2}$ and $d = 0$. Under these assumptions, and for any constants ϵ_{ij} and any functions f_{ij}, the conditions in equation (3.12) (for the existence of relative equilibria) are equivalent to

$$\frac{mm_3}{2m+m_3}\xi_e^2 = \epsilon_{13}\frac{f'_{13}(l)}{l} = \epsilon_{23}\frac{f'_{23}(l)}{l} = \frac{m_3}{m}\epsilon_{12}\frac{f'_{12}(b)}{b}. \tag{3.16}$$

Remark 3.1. Since the Lagrangian relative equilibria remain in a fixed plane, it is of course possible to model them directly as a planar system, with symmetry group $SO(2)$. Addressing stability in the full three-dimensional space, including stability in directions normal to the plane, requires some analysis that moves beyond the planar model. Our purpose in starting with the more general $SO(3)$-symmetric spatial system is to illustrate methods that will also be useful for analysing the relative equilibria that are not confined to a fixed plane.

3.3. Nonlinear stability

We now apply the reduced energy–momentum method (REM), presented in section 2, to the class of Lagrangian relative equilibria described above. As noted earlier, the three-point-mass system is simple mechanical, with kinetic energy defined in terms of the Riemannian metric with constant diagonal matrix $\mathbb{M} := \mathrm{diag}(M_1, M_1, M_1, M_2, M_2, M_2)$. This is the inner product with respect to which the configuration-space complements in the REM are defined. Variations in configuration space will be denoted $\delta\mathbf{q} := (\delta\mathbf{r}, \delta\mathbf{s}) = ((\delta r_x, \delta r_y, \delta r_z), (\delta s_x, \delta s_y, \delta s_z))$.

We have $\mathfrak{g} = \mathfrak{so}(3) \simeq \mathbb{R}^3$. As in the previous section, let $\mathbf{q}_e = (\mathbf{r}_e, \mathbf{s}_e) = ((b, 0, 0), (d, h, 0))$, with b and h non-zero, and suppose that $\mathbf{q}_e$ is the configuration of a Lagrangian relative equilibrium with angular velocity $\xi_e = (0, 0, (\xi_e)_z)$, which we assume to be non-zero. By applying formula (3.8), we find that the locked inertia tensor at this point is

$$\mathbb{I}(\mathbf{r}_e, \mathbf{s}_e) = \begin{bmatrix} M_2h^2 & -M_2hd & 0 \\ -M_2hd & M_1b^2 + M_2d^2 & 0 \\ 0 & 0 & M_1b^2 + M_2(d^2+h^2) \end{bmatrix}. \tag{3.17}$$

As noted in remark 2.2, the angular momentum of a relative equilibrium is given by $\mu_e = \mathbb{I}(q_e)\xi_e$, so in the present context $\mu_e = \lambda_e\xi_e$, where

$$\lambda_e := M_1b^2 + M_2(d^2 + h^2). \tag{3.18}$$

Note that $\mu_e = (0, 0, (\mu_e)_z)$, where $(\mu_e)_z := \lambda_e(\xi_e)_z$. From this we see that $\mathfrak{g}_{\mu_e} = \{(0, 0, \zeta) \in \mathfrak{so}(3) | \zeta \in \mathbb{R}\}$.

It is clear from equation (3.17) that λ_e is the largest eigenvalue of the locked inertia tensor. By the general argument in section 2.3, we can immediately conclude that the Arnold form is positive definite. Note however that we will compute the Arnold form explicitly in the next subsection, for use in testing spectral stability.

In the REM, complements in $\mathfrak{g}$ are taken with respect to the locked inertia tensor $\mathbb{I}$. From equation (3.17), we see that $\mathfrak{g}_{\mu_e}^{\perp} = \{(\eta_x, \eta_y, 0) | \eta_x, \eta_y \in \mathbb{R}\}$. The space of rigid variations is

$$\begin{aligned}\mathcal{V}_{\mathrm{RIG}} &= \mathfrak{g}_{\mu_e}^{\perp} \cdot \mathbf{q}_e = \{((0,0,-b\eta_y),(0,0,h\eta_x - d\eta_y)) | \eta_x, \eta_y \in \mathbb{R}\} \\ &= \{((0,0,\delta r_z),(0,0,\delta s_z)) | \delta r_z, \delta s_z \in \mathbb{R}\}. \end{aligned} \tag{3.19}$$

The space of internal variations is

$$\mathcal{V}_{\mathrm{INT}} = \left\{(\delta\mathbf{r}, \delta\mathbf{s}) \in \left(\mathfrak{g}_{\mu_e} \cdot \mathbf{q}_e\right)^{\perp} | \langle \eta, (D\mathbb{I}(\mathbf{r}_e, \mathbf{s}_e)(\delta\mathbf{r}, \delta\mathbf{s}))\xi_e \rangle = 0 \text{ for all } \eta = (\eta_x, \eta_y, 0) \in \mathfrak{g}_{\mu_e}^{\perp}\right\}. \tag{3.20}$$

Since $\mathfrak{g}_{\mu_e} \cdot \mathbf{q}_e = \{((0, b\zeta, 0), (-h\zeta, d\zeta, 0)) | \zeta \in \mathbb{R}\}$, we have

$$\left(\mathfrak{g}_{\mu_e} \cdot \mathbf{q}_e\right)^{\perp} = \{(\delta\mathbf{r}, \delta\mathbf{s}) | (M_1 b)\delta r_y - (M_2 h)\delta s_x + (M_2 d)\delta s_y = 0\}. \tag{3.21}$$

From (3.8) we have

$$\begin{aligned} D\mathbb{I}(\mathbf{r}_e, \mathbf{s}_e)(\delta\mathbf{r}, \delta\mathbf{s}) &= 2(M_1\mathbf{r}_e \cdot \delta\mathbf{r} + M_2\mathbf{s}_e \cdot \delta\mathbf{s})I_3 - M_1(\mathbf{r}_e \otimes \delta\mathbf{r} + \delta\mathbf{r} \otimes \mathbf{r}_e) \\ &\quad - M_2(\mathbf{s}_e \otimes \delta\mathbf{s} + \delta\mathbf{s} \otimes \mathbf{s}_e) \end{aligned}$$

and therefore (since $\delta\mathbf{r} \cdot \xi_e = \delta\mathbf{s} \cdot \xi_e = 0$),

$$(D\mathbb{I}(\mathbf{r}_e, \mathbf{s}_e)(\delta\mathbf{r}, \delta\mathbf{s}))\xi_e = 2(M_1\mathbf{r}_e \cdot \delta\mathbf{r} + M_2\mathbf{s}_e \cdot \delta\mathbf{s})\xi_e - M_1(\delta\mathbf{r} \cdot \xi_e)\mathbf{r}_e - M_2(\delta\mathbf{s} \cdot \xi_e)\mathbf{s}_e. \tag{3.22}$$

Since $\eta \cdot \xi_e = 0$ for all $\eta \in \mathfrak{g}_{\mu_e}^{\perp}$, we have

$$\begin{aligned} \langle \eta, (D\mathbb{I}(\mathbf{r}_e, \mathbf{s}_e)(\delta\mathbf{r}, \delta\mathbf{s}))\xi_e \rangle &= (-M_1(\delta\mathbf{r} \cdot \xi_e)\mathbf{r}_e - M_2(\delta\mathbf{s} \cdot \xi_e)\mathbf{s}_e) \cdot \eta \\ &= -(M_1 b(\xi_e)_z \delta r_z + M_2 d(\xi_e)_z \delta s_z)\eta_x - (M_2 h(\xi_e)_z \delta s_z)\eta_y. \end{aligned}$$

For a variation to be in $\mathcal{V}_{\mathrm{INT}}$, this expression must be zero for arbitrary $\eta_x, \eta_y \in \mathbb{R}$. Since $(\xi_e)_z$, M_1, M_2, b, h are all non-zero, this occurs exactly when $\delta r_z = \delta s_z = 0$. Combining this result with equation (3.21), we have

$$\mathcal{V}_{\mathrm{INT}} = \left\{(\delta\mathbf{r}, \delta\mathbf{s}) | \delta r_y - \frac{M_2 h}{M_1 b}\delta s_x + \frac{M_2 d}{M_1 b}\delta s_y = 0 \text{ and } \delta r_z = \delta s_z = 0\right\}. \tag{3.23}$$

Remark 3.2. Comparing this expression with equations 3.19 and 3.21, we see that $\mathcal{V}_{\mathrm{INT}} = (\mathfrak{g} \cdot \mathbf{q}_e)^{\perp}$.

By theorem 2.7 (the REM), the Lagrangian relative equilibrium is nonlinearly stable whenever the second variation of the amended potential V_{μ_e}, restricted to the space $\mathcal{V}_{\mathrm{INT}}$, is positive definite. We now compute $\delta^2 V_{\mu_e}$, restricted to $\mathcal{V}_{\mathrm{INT}}$, which we parametrize by $\delta r_x, \delta s_x, \delta s_y$. We recall equation (2.12):

$$\delta^2 V_{\mu_e}(\mathbf{q}_e)(\delta_1\mathbf{q}, \delta_2\mathbf{q}) = \delta^2 V_{\xi_e}(\mathbf{q}_e)(\delta_1\mathbf{q}, \delta_2\mathbf{q}) + \langle \mathbb{I}^{-1}(\mathbf{q}_e) D\mathbb{I}(\mathbf{q}_e)(\delta_1\mathbf{q})\xi_e, D\mathbb{I}(\mathbf{q}_e)(\delta_2\mathbf{q})\xi_e \rangle.$$

Since $\delta r_z = \delta s_z = 0$ for all variations in $\mathcal{V}_{\mathrm{INT}}$, it follows that $\delta\mathbf{r} \cdot \xi_e = \delta\mathbf{s} \cdot \xi_e = 0$, so equation (3.22) becomes

$$(D\mathbb{I}(\mathbf{q}_e)(\delta\mathbf{q}))\xi_e = 2[M_1 b\delta r_x + M_2 d\delta s_x + M_2 h\delta s_y]\xi_e. \tag{3.24}$$

From equations (3.17) and (3.18), we have

$$\mathbb{I}^{-1}(\mathbf{q}_e) = \begin{bmatrix} \frac{M_1 b^2 + M_2 d^2}{M_1 M_2 h^2 b^2} & \frac{d}{M_1 h b^2} & 0 \\ \frac{d}{M_1 h b^2} & \frac{1}{M_1 b^2} & 0 \\ 0 & 0 & \frac{1}{\lambda_e} \end{bmatrix}. \tag{3.25}$$

It follows that the matrix of $\langle \mathbb{I}^{-1}(\mathbf{q}_e) D\mathbb{I}(\mathbf{q}_e)(\delta\mathbf{q})_1\xi_e,\, D\mathbb{I}(\mathbf{q}_e)(\delta\mathbf{q})_2\xi_e\rangle$, in coordinates δr_x, δs_x, δs_y, is

$$\frac{4(\xi_e)_z^2}{\lambda_e}\begin{bmatrix} M_1^2b^2 & M_1M_2bd & M_1M_2bh \\ M_1M_2bd & M_2^2d^2 & M_2^2dh \\ M_1M_2bh & M_2^2dh & M_2^2h^2 \end{bmatrix}.$$

The augmentation term in V_ξ is $-\frac{1}{2}\xi_e^T\mathbb{I}(\mathbf{q})\xi_e = -\frac{1}{2}\langle\langle \xi_e\cdot\mathbf{q}, \xi_e\cdot\mathbf{q}\rangle\rangle$ which, since the group action is linear, has second variation,

$$\begin{aligned}-\langle\langle \xi_e\cdot(\delta\mathbf{q})_1, \xi_e\cdot(\delta\mathbf{q})_2\rangle\rangle &= -M_1(\xi_e)^2((\delta r_x)_1(\delta r_x)_2 + (\delta r_y)_1(\delta r_y)_2)\\ &\quad - M_2(\xi_e)^2((\delta s_x)_1(\delta s_x)_2 + (\delta s_y)_1(\delta s_y)_2).\end{aligned}$$

Using the coordinate transformation $\delta r_y = \frac{M_2h}{M_1b}\delta s_x - \frac{M_2d}{M_1b}\delta s_y$, taken from the definition of $\mathcal{V}_{\text{INT}}$, we calculate that the restriction of the second variation of $-\frac{1}{2}\xi_e^T\mathbb{I}(\mathbf{q})\xi_e$ to $\mathcal{V}_{\text{INT}}$, with respect to the coordinates δr_x, δs_x, δs_y, has matrix,

$$-(\xi_e)_z^2\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 + \frac{M_2^2h^2}{M_1b^2} & -\frac{M_2^2dh}{M_1b^2} \\ 0 & -\frac{M_2^2dh}{M_1b^2} & M_2 + \frac{M_2^2d^2}{M_1b^2} \end{bmatrix}.$$

Finally, we use the same coordinate transformation to express $\delta^2V(\mathbf{q}_e)$ in coordinates δr_x, δs_x, δs_y, and combine the result with the previous two displayed equations. Denoting $\alpha := \frac{M_2h}{M_1b}$, $\beta := -\frac{M_2d}{M_1b}$, and $\gamma := \frac{4}{\lambda_e} = \frac{4}{M_1b^2+M_2(d^2+h^2)}$, we have

$$\begin{aligned}&[\delta^2V_{\mu_e}]|_{\mathcal{V}_{\text{INT}}}\\ &= \begin{bmatrix} V_{r_xr_x} & \alpha V_{r_xr_y} + V_{r_xs_x} & \beta V_{r_xr_y} + V_{r_xs_y} \\ \alpha V_{r_xr_y} + V_{r_xs_x} & \alpha^2V_{r_yr_y} + 2\alpha V_{r_ys_x} + V_{s_xs_x} & \alpha\beta V_{r_yr_y} + \beta V_{r_ys_x} + \alpha V_{r_ys_y} + V_{s_xs_y} \\ \beta V_{r_xr_y} + V_{r_xs_y} & \alpha\beta V_{r_yr_y} + \beta V_{r_ys_x} + \alpha V_{r_ys_y} + V_{s_xs_y} & \beta^2V_{r_yr_y} + 2\beta V_{r_ys_y} + V_{s_ys_y} \end{bmatrix}\\ &+ (\xi_e)_z^2\begin{bmatrix} M_1(\gamma M_1b^2 - 1) & \gamma M_1M_2bd & \gamma M_1M_2bh \\ \gamma M_1M_2bd & -\alpha^2M_1 + M_2(\gamma M_2d^2 - 1) & -\alpha\beta M_1 + \gamma M_2^2dh \\ \gamma M_1M_2bh & -\alpha\beta M_1 + \gamma M_2^2dh & -\beta^2M_1 + M_2(\gamma M_2h^2 - 1) \end{bmatrix}.\end{aligned}$$

3.4. Spectral stability

As described in section 3, when the stability matrix is indefinite, one applies spectral analysis to the reduced linearized system. In order to compute this linearized system using equation (2.14), we need to calculate the following second variation, from equation (2.13),

$$\delta^2H_\xi(\mathbf{q}_e, \mathbf{p}_e) = \begin{bmatrix} \mathcal{A}_{\mu_e} & 0 & 0 \\ 0 & \delta^2V_{\mu_e}(\mathbf{q}_e) & 0 \\ 0 & 0 & \delta^2K(\mathbf{q}_e, \mathbf{p}_e) \end{bmatrix}.$$

(In writing the upper block as the Arnold form $\mathcal{A}_{\mu_e}$, we have identified $\mathcal{V}_{\text{RIG}}$ with $\mathfrak{g}_{\mu_e}^\perp$). We have already computed $\delta^2V_{\mu_e}(\mathbf{q}_{\mu_e})$ in the previous section.

Recall from section 2.3 that, since $G = SO(3)$, the Arnold form $\mathcal{A}_{\mu_e}$ is the restriction to $\mathfrak{g}_{\mu_e}^\perp \times \mathfrak{g}_{\mu_e}^\perp$ of $\hat{\mu}_e^T\big(\mathbb{I}^{-1}(\mathbf{q}_e) - \lambda_e^{-1}I\big)\hat{\mu}_e$. The matrix $\hat{\mu}_e$, defined by $\hat{\mu}_e\eta = \mu_e\times\eta$ for all η, is in our case

$$\hat{\mu}_e = (\mu_e)_z\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathbb{I}_{xy}^{-1}(\mathbf{q}_e)$ be the upper left block of the inverse inertia tensor, given in equation (3.25). Since $\hat{\mu}_e$ and $\mathbb{I}^{-1}$ are both block diagonal, the restriction of $\hat{\mu}_e^T\left(\mathbb{I}^{-1}(\mathbf{q}_e) - \lambda_e^{-1} I\right)\hat{\mu}_e$ to $\mathfrak{g}_{\mu_e}^{\perp} \times \mathfrak{g}_{\mu_e}^{\perp}$ is easy to calculate, giving

$$\begin{aligned}\mathcal{A}_{\mu_e} &= (\mu_e)_z^2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \left(\mathbb{I}_{xy}(q_e)^{-1} - \lambda_e^{-1} I\right) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= (\mu_e)_z^2 \left(\begin{bmatrix} \frac{1}{M_1 b^2} & -\frac{d}{M_1 h b^2} \\ -\frac{d}{M_1 h b^2} & \frac{M_1 b^2 + M_2 d^2}{M_1 M_2 h^2 b^2} \end{bmatrix} - \lambda_e^{-1} I \right).\end{aligned}$$

We now calculate $\delta^2 K(z_e)$. In Cartesian coordinates $(\delta \mathbf{r}_e, \delta \mathbf{s}_e)$, it has matrix

$$\mathbb{M}^{-1} = \operatorname{diag}\left(\frac{1}{M_1}, \frac{1}{M_1}, \frac{1}{M_1}, \frac{1}{M_2}, \frac{1}{M_2}, \frac{1}{M_2}\right). \tag{3.26}$$

We need to restrict this to $\mathcal{W}_{\text{INT}}^*$, which we identify with $[\mathfrak{g} \cdot \mathbf{q}_e]^\circ$. Since $\zeta_Q((b,0,0),(d,h,0)) = ((0, b\zeta_3, -b\zeta_2), (-h\zeta_3, d\zeta_3, h\zeta_1 - d\zeta_2))$, for all ζ, we have

$$(\mathfrak{g} \cdot \mathbf{q}_e)^\circ = \left\{ \left(\left(\gamma_1, \frac{h}{b}\gamma_2 - \frac{d}{b}\gamma_3, 0 \right), (\gamma_2, \gamma_3, 0) \right) \middle| \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R} \right\}.$$

In coordinates $\gamma_1, \gamma_2, \gamma_3$ on $(\mathfrak{g} \cdot \mathbf{q}_e)^\circ$, direct calculation shows that

$$\delta^2 K(\mathbf{q}_e, \mathbf{p}_e) = \begin{bmatrix} \frac{1}{M_1} & 0 & 0 \\ 0 & \frac{1}{M_2} + \frac{h^2}{M_1 b^2} & -\frac{dh}{b^2 M_1} \\ 0 & -\frac{dh}{b^2 M_1} & \frac{1}{M_2} + \frac{d^2}{M_1 b^2} \end{bmatrix}.$$

We next compute the symplectic form $\Omega(\mathbf{q}_e, \mathbf{p}_e)$ on $\mathcal{S}_{\text{RIG}} \oplus \mathcal{W}_{\text{INT}} \oplus \mathcal{W}_{\text{INT}}^*$, as in equation (2.15). The 'internal-rigid coupling' block in $\Omega(\mathbf{q}_e, \mathbf{p}_e)$ is zero, by remark 2.10, since in our case, $\mathcal{V}_{\text{RIG}}$ is orthogonal to $\mathcal{V}_{\text{INT}}$, as noted in remark 3.2. Thus $\Omega(\mathbf{q}_e, \mathbf{p}_e)$ has the following structure:

$$\Omega_{(\mathbf{q}_e, \mathbf{p}_e)} = \begin{bmatrix} \Omega_{\mu_e} & 0 & 0 \\ 0 & S & B \\ 0 & -B^T & 0 \end{bmatrix}. \tag{3.27}$$

Each block can be computed directly from the canonical symplectic form by the appropriate change of coordinates. Doing this for B gives

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 + \frac{M_2}{M_1}\frac{h^2}{b^2} & -\frac{M_2}{M_1}\frac{hd}{b^2} \\ 0 & -\frac{M_2}{M_1}\frac{dh}{b^2} & 1 + \frac{M_2}{M_1}\frac{d^2}{b^2} \end{bmatrix}.$$

Note that B is symmetric. This is not a general property of B but a consequence of our choice of coordinates.

Rather than apply the same direct method for the other two blocks, we use equations 2.16 and 2.18 for Ω_{μ_e} and S, respectively. Identifying the subspace $\mathcal{S}_{\text{RIG}}$ with $\mathfrak{g}_{\mu_e}^{\perp}$, which in our case is the xy plane, we find that the 'coadjoint orbit' block of Ω_{μ_e} has the following matrix, where $(\mu_e)_z = \lambda_e(\xi_e)_z$,

$$\Omega_{\mu_e} = (\mu_e)_z \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

It remains to compute the block S, defined in equation (2.18) by

$$\Omega_{(\mathbf{q}_e, \mathbf{p}_e)}(\delta z_1, \delta z_2) = -\mathrm{d}\alpha_{\xi_e}(\delta q_1, \delta q_2),$$

for all $\delta z_i = T\alpha_{\mu_e} \cdot \delta q_i \in \mathcal{W}_{\text{INT}}$, where $\alpha_{\xi_e}(q) := \mathbb{F}L((\xi_e)_Q(q))$. In our case,

$$\alpha_{\xi_e}(q) = \mathbb{M}(\xi_e \times q) = Nq,$$

for all q, where N is the following block diagonal matrix

$$N := (\xi_e)_z \operatorname{diag}\left(\begin{bmatrix} 0 & -M_1 \\ M_1 & 0 \end{bmatrix}, 0, \begin{bmatrix} 0 & -M_2 \\ M_2 & 0 \end{bmatrix}, 0\right).$$

Hence $\mathrm{d}\alpha_{\xi_e}(\mathbf{q}_e)(\delta q_1, \delta q_2) = (\delta q_2)^T N \delta q_1 - (\delta q_1)^T N \delta q_2$. Since N is antisymmetric, this equals $-2(\delta q_1)^T N \delta q_2$. Thus $-\mathrm{d}\alpha_{\xi_e}(\mathbf{q}_e)$ has matrix $2N$. The block S is this matrix $2N$, restricted to $\mathcal{V}_{\text{INT}}$. Using the coordinate transformation $\delta r_y = \frac{M_2 h}{M_1 b}\delta s_x - \frac{M_2 d}{M_1 b}\delta s_y$ from equation (3.23), we calculate that S has the following matrix, with respect to the coordinates $\delta r_x, \delta s_x, \delta s_y$ for $\mathcal{V}_{\text{INT}}$,

$$S = 2(\xi_e)_z M_2 \begin{bmatrix} 0 & -\frac{h}{b} & \frac{d}{b} \\ \frac{h}{b} & 0 & -1 \\ -\frac{d}{b} & 1 & 0 \end{bmatrix}.$$

Since the coupling block in $\Omega_{(\mathbf{q}_e,\mathbf{p}_e)}$ is zero, the reduced linearized vector field is given by equation (2.19). In the present case, since B is symmetric, we have

$$L = \left[\Omega_{(\mathbf{q}_e,\mathbf{p}_e)}\right]^{-T} \delta^2 H_\xi(\mathbf{q}_e, \mathbf{p}_e) = \begin{bmatrix} \frac{1}{(\mu_e)_z^2}\Omega_{\mu_e}\mathcal{A}_{\mu_e} & 0 & 0 \\ 0 & 0 & B^{-1}\delta^2 K \\ 0 & -B^{-1}\delta^2 V_{\mu_e} & -B^{-1}SB^{-1}\delta^2 K \end{bmatrix}.$$

A direct calculation shows that the eigenvalues corresponding to the block $\frac{1}{(\mu_e)_z^2}\Omega_{\mu_e}\mathcal{A}_{\mu_e}$ are

$$\lambda_{1,2} = \pm \mathrm{i}(\xi_e)_z.$$

The eigenvalues of the remaining 2×2 block are the roots of

$$\det\left(\lambda^2 B(\delta^2 K)^{-1} B + \lambda S + \delta^2 V_{\mu_e}\right), \tag{3.28}$$

by equation (2.20).

4. Applications

4.1. Stability for the classical three-body problem

The equilateral relative equilibria configurations for the classical Newtonian three-body problem were discovered in 1772 by Lagrange. An example of such a relative equilibrium is the Sun–Jupiter–Trojan asteroid system, which is known to be linearly stable. This result is due to Gascheau [Ga1843] who, in his PhD thesis, deduced that the triangular relative equilibria configurations are linearly stable if and only if the masses satisfy

$$\frac{m_1 m_2 + m_2 m_3 + m_3 m_1}{(m_1 + m_2 + m_3)^2} < \frac{1}{27}. \tag{4.1}$$

The stability of the classical Lagrangian relative equilibria cannot be decided by the reduced energy–momentum method: the second variation of the augmented Hamiltonian has index 2. Therefore, one resorts to spectral stability and, by applying methods presented elsewhere (see, for instance, [RoG02]), deduces results concerning linear stability. Here we outline how to obtain the spectral stability conditions by applying the methodology outlined in the previous section.

Recall from equation (3.14) the equilateral relative equilibria:

$$\mathbf{r}_e = (b, 0, 0), \qquad \mathbf{s}_e^{\pm} = \left(\frac{b}{2} \left(\frac{m_1 - m_2}{m_1 + m_2} \right), \pm \frac{\sqrt{3}}{2} b, 0 \right).$$

Substituting these values into the formulae for $\delta^2 K$, S and B from the previous section (in which $\mathbf{s}_e = (d, h, 0)$) gives

$$\delta^2 K = \begin{bmatrix} \frac{m_1+m_2}{m_2 m_1} & 0 & 0 \\ 0 & \frac{4m_1m_2(m_1+m_2)+10m_1m_2m_3+3m_3(m_1^2+m_2^2)}{4m_1m_2m_3(m_1+m_2)} & -\frac{\sqrt{3}(m_1-m_2)}{4m_1m_2} \\ 0 & \frac{\sqrt{3}(m_1-m_2)}{4m_1m_2} & \frac{4m_1m_2+m_3(m_1+m_2)}{4m_1m_2m_3} \end{bmatrix},$$

$$S = \xi \begin{bmatrix} 0 & -\frac{m_3(m_1+m_2)\sqrt{3}}{m_1+m_2+m_3} & \frac{m_3(m_1-m_2)}{m_1+m_2+m_3} \\ \frac{m_3(m_1+m_2)\sqrt{3}}{m_1+m_2+m_3} & 0 & -\frac{2m_3(m_1+m_2)}{m_1+m_2+m_3} \\ -\frac{m_3(m_1-m_2)}{m_1+m_2+m_3} & \frac{2m_3(m_1+m_2)}{m_1+m_2+m_3} & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 + \frac{3m_3(m_1+m_2)^2}{4m_1m_2(m_1+m_2+m_3)} & -\frac{\sqrt{3}}{4}\frac{m_3(m_1+m_2)(m_1-m_2)}{m_1m_2(m_1+m_2+m_3)} \\ 0 & -\frac{\sqrt{3}}{4}\frac{m_3(m_1+m_2)(m_1-m_2)}{m_1m_2(m_1+m_2+m_3)} & 1 + \frac{m_3(m_1-m_2)^2}{4m_1m_2(m_1+m_2+m_3)} \end{bmatrix}.$$

Assuming the classical Newtonian potential, with gravitational constant $G = 1$, we can take $\epsilon_{ij} = m_i m_j$ and $f(x) = -1/x$ in equation (3.15), yielding $\xi_e^2 = (m_1 + m_2 + m_3)/b^3$. Due to the homogeneity of the problem we can take $\xi_e = 1$ and therefore, by (3.18) and the previous relation, we can write $\mu_e = (m_1m_2 + m_2m_3 + m_1m_3)/(m_1 + m_2 + m_3)^{1/3}$. The next step is to compute $\delta^2 V_{\mu_e}$ and substitute μ_e (not presented here). After building the linearized matrix L, one calculates the characteristic polynomial:

$$(\lambda^2 + 1)^2 \left(\lambda^4 + \lambda^2 + \frac{27}{4} \frac{m_1m_2 + m_2m_3 + m_3m_1}{(m_1 + m_2 + m_3)^2} \right).$$

It is easy to deduce that the spectral stability is achieved if and only if the constant term in the second bracket is less then $1/4$.

4.2. Triatomic molecules

The bifurcations and stability diagrams of relative equilibria for molecular type systems are important as they are used to predict a number of features in the energy spectrum of the molecules (see [KP96]). In this perspective, Kozin *et al* present a comprehensive and complete picture of bifurcation and stability of relative equilibria for the triatomic H_3^+, D_2H^+ and H_2D^+ molecules and compute the associated normal mode frequencies (see [KRT99, KRT00]). Their methodology relies on the classical rotation-vibrational Hamiltonian theory (see [WDC55]) and in essence uses the angular momentum integral to reduce the number of degrees of freedom. For the Lagrangian relative equilibria, the reduction drops the problem to a three degrees of freedom system. Nonlinear and spectral stability are studied via the signature of a 6×6 Hessian matrix and the eigenvalues of the 6×6 reduced linearized system. In this section we reproduce their results for Lagrangian relative equilibria using the theory presented earlier. We emphasize that our nonlinear stability analysis relies on the signature of the 3×3 matrix $[\delta^2 V_\mu]|_{\mathcal{V}_{\text{INT}}}$, and therefore is computationally cheaper.

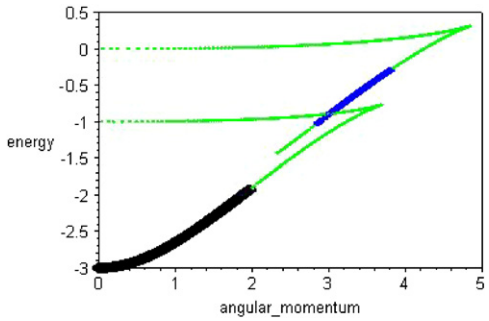

Figure 1. Energy–momentum diagram in the case $\nu = 1/2$ ($m_1 = m_2 = m_3/2$).

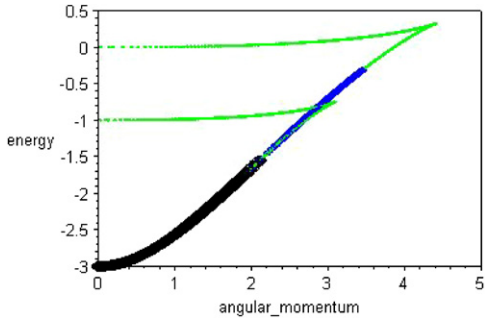

Figure 2. Energy–momentum diagram in the case $\nu = 1$ (equal masses).

Molecular-type systems may be described in a similar manner to a classical n-body problem, but with modified potential. The atom–atom interaction is governed by laws with the following generic features: (1) they become highly repulsive when the interatomic distance approaches zero (i.e. near collision), (2) attain a finite value (the well depth) when the two atoms orbit circular and uniformly around each other and (3) vanish asymptotically at infinity (see, for instance, [LM92]). A commonly used function is the Morse potential $F^M_{D,a,r_0}(r) = D(\mathrm{e}^{-2a(r-r_0)} - 2\,\mathrm{e}^{-a(r-r_0)})$, where the parameters D and r_0 represent the well depth and the equilibrium distance, respectively, and a controls the width of the potential.

Our numerical experiments are based on a generic Morse potential with parameters taken to be unity. Two atom masses are taken to be equal and normalized, that is $m_1 = m_2 = 1$. Denoting the mass ratio $\nu := m_1/m_3$, we present energy–momentum diagrams for $\nu = 1$ (similar to a H_3^+ molecule), $\nu = 1/2$, (H_2D^+) and $\nu = 2$ (D_2H^+) in figures 1–3, respectively. The isosceles relative equilibria $(\mathbf{r}_e, \mathbf{s}_e) = ((b, 0, 0), (0, h, 0))$ are drawn as thick dots if they are nonlinearly stable, medium dots if they are spectrally stable but not nonlinearly stable, and small dots if they are spectrally unstable.

In each of the figures, we observe two families of Lagrangian relative equilibria, each of which has a cusp at its point of maximum energy and maximum momentum. (The two families are most clearly seen in figure 1.) The lower family emanates from the equilibrium equilateral configuration (with zero energy and zero momentum), and is denoted (L). The other family is denoted (H). In figure 2, the two families join, and in figure 3, part of family (L) disappears, since these relative equilibria have become collinear and are thus no longer part of our analysis. The *bending angle* is defined as $2\arctan(b/(2h))$.

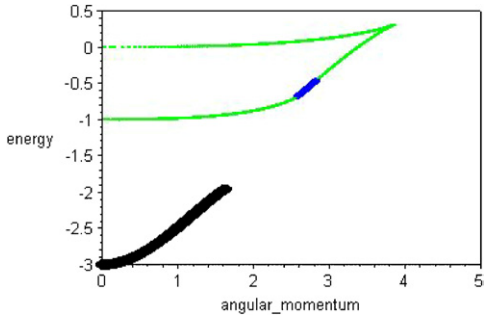

Figure 3. Energy–momentum diagram in the case $\nu = 2$ ($m_1 = m_2 = 2m_3$).

First we describe family (L), beginning at the point with zero energy and zero momentum, which in all cases is nonlinearly stable. In the $\nu = 1/2$ case, as the angular momentum is 'turned on', the bending angle immediately begins to decrease. At momentum value of around 2, the relative equilibrium becomes unstable. The bending angle continues to decrease all along the family, even past the cusp, until the molecule breaks (dissociates), with the distance h tending to infinity, and with angular momentum tending to zero. In the case $\nu = 1$, as angular momentum increases from zero, the equilibrium equilateral shape is initially preserved. Nonlinear stability is lost at momentum value around 2.2, though the relative equilibrium remains spectrally stable. From this point up to a momentum value of approximately 2.8, the two families coincide. The two families split, (H) remaining equilateral and (L) becoming isosceles with decreasing bending angle. Following the family (L) line, the bending angle continues decreasing, past the cusp, until the molecule breaks into two pieces. For $\nu = 2$, as angular momentum is turned on, the shape becomes isosceles by an increase in the bending angle and the family ends when the collinear configuration is reached. For $\nu = 1$, stability weakens to spectral stability when family (L) encounters family (H) and is lost when the equilateral triangle starts bending. For $\nu = 1/2$, nonlinear stability transfers to instability for values of the momenta around 2, while for $\nu = 2$ the relative equilibria family disappears (becoming collinear) before a change in stability takes place.

Family (H) emanates from the fictitious unstable equilateral equilibrium of infinite side length and zero momentum where the energy is at its highest level. This family is predominantly unstable. For $\nu = 1$, as the momentum starts increasing, the family preserves its equilateral shape past the cusp and encounters family (L). This family is unstable except for a region of spectral stability that starts before encountering family (H). For $\nu \neq 1$, as the momentum is turned on the shape is continuously changed by an increase in the bending angle. For $\nu = 1/2$, the angle continues to increase past the cusp until the family ends in a collinear configuration; from the stability standpoint, all relative equilibria are unstable except for a segment of linear stability. For $\nu = 2$, the shape initially experience an increase in the bonding angle, but short while this changes to a decrease that continues until the collinear configuration is reached; all relative equilibria are unstable expect for a region of spectral stability for momenta in between 2.55 to 2.88. Our results coincide qualitatively with those of Kozin *et al*.

4.3. Future applications

Our calculations are the ingredients for a piece of software for the analysis of Lagrangian relative equilibria of a very general class of three particle interactions: those in which the

potential energy is a sum of three two-particle potentials that differ by a multiplicative constant. The applications that we have presented, while not new, verify the accuracy of the calculations and hopefully convince the reader that the REM is an efficient tool.

We foresee many new applications, particularly in celestial mechanics, astronomy and astrophysics, where the standard Newtonian interaction is only one of many potentials studied. Perturbations to the Newtonian potential arise when more accuracy in predictions is desired and therefore a more realistic approach to the modelling of the physical problems is required. Commonly encountered corrections include those due to relativity (see [DMS00, StMi97] and references therein), radiation pressure (see [Kun00] and references therein) and oblateness (see below). To date, these corrections have been studied mainly in the context of the two-body problem or the restricted three-body problem (motion of a massless particle in the field of two mass points, where the two mass points are in a relative equilibrium). One study of relative equilibria for the (unrestricted) three-body problem for non-Newtonian potentials is that of Tkhai [Tkhai95], who studied the linear stability of a three-point-mass system with inverse homogeneous interaction. Santoprete [San06] explored aspects of the same problem for inverse quasi-homogeneous laws.

While the classical three-body problem has been under scrutiny for almost 300 years, little is known about the full three body problem (or the three rigid body problem). A comprehensive study describing the dynamics appears in [Dub74] and references therein. In its full generality, this problem is described by a coupled system with 18 degrees of freedom (3×3 for the positions of the centres of mass of each body + 3×3 for the bodies orientations), with 10 prime integrals (energy, and linear and angular momentum). A more tractable sub-problem concerns roto-translational motion, in which the dynamics decouples into the motion of the centres of mass and the motion of each rigid body. As proven in [Dub84] and [CE85], this is possible when each body is a spinning ellipsoid with revolution axis perpendicular to the equatorial plane of the motion of the centres of masses. In this case, the (decoupled) motion of the centres of mass is modelled by a three-point-mass system with a mutual potential of the form

$$V^{(ij)} = -\frac{Gm_i m_j}{r_{ij}}\left[1 + \sum_{\alpha=1}^{\infty} \frac{F_{ij}^{(\alpha)}(a_i^2 - c_i^2; a_j^2 - c_j^2)}{r_{ij}^{2\alpha}}\right], \qquad 1 \leqslant i < j \leqslant 3,$$

where G is the gravitational constant, and m_i, a_i and c_i are the mass, the major and the minor axes of ellipsoid i, respectively, and $F_{ij}^{(\alpha)}$ are homogeneous polynomials of degree (α) in $\left(a_i^2 - c_i^2\right)$ and $\left(a_j^2 - c_j^2\right)$. This system has relative equilibria in which the centres of mass are in a Lagrangian configuration while the bodies rotate uniformly. To our knowledge, the only known Lagrangian relative equilibria occur when the bodies have the same oblateness:

$$a_1 - c_1 = a_2 - c_2 = a_3 - c_3. \tag{4.2}$$

In this case the configuration is equilateral; see [Dub84] and [CE85]. The general problem of existence and stability of Lagrangian relative equilibria (not necessarily equilateral) in the roto-translational full three-body problem remains open. It is easy to see that the analysis becomes more complicated if other corrections to the mutual interactions are included. Non-trivial and interesting results are to be expected when dealing with systems for which the mass ratio $(m_1m_2 + m_2m_3 + m_3m_1)/(m_1 + m_2 + m_3)^2$ is close to the critical value $1/27$ (see section **??**), in which case small perturbations in the potential become relevant. From a practical standpoint, tri-star systems, 'binary star plus planet' systems and tripartite clustering phenomena provide immediate opportunities to apply the results.

We also mention other possible applications related to certain classical approximations of atomic systems of the form $e^- Ze^-$. Such approximations are interesting mathematically, and have been the subject of stimulating studies (see, for instance [BGY98] and [DPC03]).

5. Conclusions

We have reviewed the reduced energy–momentum method for testing stability in symmetric systems with Hamiltonian of the form 'kinetic plus potential'. We have implemented the method for planar rotationally invariant relative equilibria of three-point-mass systems.

We applied the method to two classes of problem: equilateral relative equilibria for the three-body problem, and isosceles triatomic molecules. For the former problem, we offer a systematic way to obtain a classical stability result with significance in celestial mechanics and astronomy. For the latter problem we produce stability diagrams for molecules of type H_3^+, H_2D^+ and D_2H^+ that agree with those in Kozin *et al* [KRT99, KRT00], but produced with less computational effort. We also mention some open problems to which the method is applicable, in particular the roto-translational motion in the full three-body problem. Our calculations offer a short-cut in the analysis of stability in any of these problems.

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