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Differential Geometry and its Applications 25 (2007) 101–124

DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS

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# A cotangent bundle slice theorem

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Received 2 June 2005; received in revised form 14 October 2005

Available online 21 June 2006

Communicated by P. Michor

**Abstract**

This article concerns cotangent-lifted Lie group actions; our goal is to find local and "semi-global" normal forms for these and associated structures. Our main result is a constructive cotangent bundle slice theorem that extends the Hamiltonian slice theorem of Marle [C.-M. Marle, Modèle d'action hamiltonienne d'un groupe de Lie sur une variété symplectique, Rendiconti del Seminario Matematico, Università e Politecnico, Torino 43 (2) (1985) 227–251] and Guillemin and Sternberg [V. Guillemin, S. Sternberg, A normal form for the moment map, in: S. Sternberg (Ed.), Differential Geometric Methods in Mathematical Physics, in: Mathematical Physics Studies, vol. 6, D. Reidel, 1984]. The result applies to all proper cotangent-lifted actions, around points with fully-isotropic momentum values.

We also present a "tangent-level" commuting reduction result and use it to characterise the symplectic normal space of any cotangent-lifted action. In two special cases, we arrive at splittings of the symplectic normal space. One of these cases is when the configuration isotropy group is contained in the momentum isotropy group; in this case, our splitting generalises that given for free actions by Montgomery et al. [R. Montgomery, J.E. Marsden, T.S. Ratiu, Gauged Lie–Poisson structures, Cont. Math. AMS 128 (1984) 101–114]. The other case includes all relative equilibria of simple mechanical systems. In both of these special cases, the new splitting leads to a refinement of the so-called *reconstruction equations* or *bundle equations* [J.-P. Ortega, Symmetry, reduction, and stability in Hamiltonian systems, PhD thesis, University of California, Santa Cruz, 1998; J.-P. Ortega, T.S. Ratiu, A symplectic slice theorem, Lett. Math. Phys. 59 (1) (2002) 81–93; M. Roberts, C. Wulff, J.S.W. Lamb, Hamiltonian systems near relative equilibria, J. Differential Equations 179 (2) (2002) 562–604]. We also note cotangent-bundle-specific local normal forms for symplectic reduced spaces.


*MSC:* 53D20; 37J15; 70H33; 53D05

*Keywords:* Slice theorem; Marle normal form; Symplectic normal space; Cotangent lift; Reconstruction equations

## 1. Introduction

This article concerns cotangent-lifted actions of a Lie group $G$ on a cotangent bundle $T^*Q$. We are motivated in part by the role of such actions as groups of symmetries of Hamiltonian systems with cotangent bundle phase spaces. Nonetheless, this article is primarily geometric.

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0926-2245/$ – see front matter 
doi:10.1016/j.difgeo.2006.05.003

When $G$ acts freely and properly, it is well known that one can *reduce* $T^*Q$ by the $G$ action to give a lower-dimensional symplectic manifold (see Theorem 2). The reduced manifold inherits some cotangent-bundle structure [1,12,14], and it sometimes actually *is* a cotangent bundle (see for example Theorem 3). In Hamiltonian systems, the solutions of the original system project to solutions of a new Hamiltonian system on the reduced phase space. The problem of *singular reduction* is to generalise this picture to arbitrary proper group actions, not necessarily free. This problem has been addressed with success in the symplectic category [3,23,31] and more recently for the special case of cotangent bundles [5,26]. But the symplectic reduced spaces are in general not smooth, and our understanding of the inherited cotangent bundle structure is far from complete.

A different but related approach is to ask: to what degree can we factor out the symmetry while not losing smoothness? Slice theorems are one kind of answer to this question. For any proper action of $G$ on a manifold $M$, the slice theorem of Palais (Theorem 4) says that every point $z \in M$ has a neighbourhood that is $G$-equivariantly diffeomorphic to a twisted space $G \times_{G_z} S$, where $G_z$ is the isotropy group of the point $z$, and the $G$ action on $G \times_{G_z} S$ is $g' \cdot [g, s] = [g'g, s]$. This local model of the action of $G$ on $M$ is actually "semi-global" in the sense that it is global "in the $G$ direction" but local "in the transverse direction". For symplectic actions, the Hamiltonian slice theorem of Marle [11] and Guillemin and Sternberg [7] (Theorem 5) gives a model space of the form $G \times_{G_z} S$ and a $G$-equivariant *symplectic* diffeomorphism. This theorem is a fundamental tool in the study of Hamiltonian systems with symmetry: it has found applications to singular reduction [3,19,31], and to many dynamical questions involving stability, bifurcation and persistence in the neighbourhood of relative equilibria and relative periodic orbits [9,16,17,20,21,24,27,28,32]. The main aim of the present article is to extend the Hamiltonian slice theorem in the context of cotangent bundles. We succeed in doing so around points with fully isotropic momentum values (Theorem 31). Our new result extends that of Marle, Guillemin and Sternberg in three ways. First, it involves a new cotangent-bundle-specific splitting of the symplectic normal space. Second, it is constructive, up to a Riemannian exponential map. In particular, we do not use the Constant Rank Embedding Theorem or Darboux's Theorem. Third, our construction has a uniqueness property (see Lemma 28).

The article is organised as follows. We begin in Section 2 with some background material, including symplectic reduction and slice theorems. In Section 3 we summarise regular and singular commuting symplectic reduction, and introduce a new "tangent level" commuting reduction result that deals with symplectic normal spaces (Theorem 10). In Section 4 we analyse the symplectic normal space for a cotangent-lifted action of $G$ on $T^*Q$, by applying Theorem 10 to $T^*(G \times A)$, where $A$ is a linear slice for the $G$ action on $Q$. In two special cases, we arrive at splittings of $N_s$. We note consequences of these results for singular reduction and for the reconstruction equations (bundle equations). In Section 5, we prove the Cotangent bundle slice theorem, Theorem 31, using two alternative methods. The first method is more "brute-force" and gives an explicit formula in coordinates; the second method is to re-arrange the problem so that a cotangent-lift can be used. We end with a simple example, $SO(3)$ acting on $T^*\mathbf{R}^3$.

Most of the results in this article first appeared in the author's PhD thesis [30].

## 2. Preliminaries

We summarise relevant basic facts on Lie group symmetries, symplectic reduction and slice theorems. This material is well-known; good general references are [1,4,6,23]. All manifolds in this article are real, smooth and finite-dimensional, and all actions are smooth left actions. Gothic letters will always denote Lie algebras of the Lie groups with corresponding Latin letters.

**Lie group actions.** Let $G$ be a Lie group, with Lie algebra $\mathfrak{g}$. If $G$ acts on a manifold $M$, then the action of any $g \in G$ on $z \in M$ will be denoted by $g \cdot z$. For every $\xi \in \mathfrak{g}$, the *infinitesimal generator* of $\xi$ is the vector field $\xi_M$ defined by $\xi_M(z) = \frac{d}{dt} \exp(t\xi) \cdot z|_{t=0}$. We will also write $\xi_M(z)$ as $\xi \cdot z$, and refer to the map $(\xi, z) \mapsto \xi \cdot z$ as the *infinitesimal action* of $\mathfrak{g}$ on $M$. The *orbit* of $z \in M$ is denoted $G \cdot z$, and we write $\mathfrak{g} \cdot z = \{\xi \cdot z \colon \xi \in \mathfrak{g}\}$. The *isotropy subgroup* of a point $z \in M$ is $G_z := \{g \in G \mid g \cdot z = z\}$. An action is *free* if all of the isotropy subgroups $G_z$ are trivial.

The action is *proper* if the map $(g, z) \mapsto (z, g \cdot z)$ is proper (i.e. the preimage of every compact set is compact). The following property is equivalent, for actions on a finite-dimensional Hausdorff second-countable manifolds: given any convergent sequences $\{z_i\}$ and $\{g_i \cdot z_i\}$, the sequence $\{g_i\}$ has a convergent subsequence. For a proper action, all isotropy subgroups are compact. If $G$ acts properly and freely on $M$, then $M/G$ has a unique smooth structure such that $\pi_G : M \to M/G$ is a submersion (in fact, $\pi_G$ is a principal bundle). One useful consequence is that for every $z \in M$, we have $\ker T_z\pi_G = T_z(G \cdot z) = \mathfrak{g} \cdot z$.

Given a $G$ action $\Phi : G \times V \to V$ on a vector space $V$, the *inverse dual* (or *inverse transpose* or *contragredient*) action of $G$ on $V^*$ is defined by $g \cdot \alpha = (\Phi_{g^{-1}})^*(\alpha)$ for all $g \in G$ and $\alpha \in V^*$, which is equivalent to $\langle g \cdot \alpha, v\rangle = \langle \alpha, g^{-1} \cdot v\rangle$ for all $v \in V$.

The adjoint action of $G$ on $\mathfrak{g}$ is denoted by Ad, and the infinitesimal adjoint action by ad. The *coadjoint* action of $G$ on $\mathfrak{g}^*$ is the inverse dual to the adjoint action, given by $g \cdot \nu = \mathrm{Ad}^*_{g^{-1}} \nu := (\mathrm{Ad}_{g^{-1}})^*\nu$. The infinitesimal coadjoint action is given by $\xi \cdot \nu = -\mathrm{ad}^*_\xi \nu$. For any $\mu \in \mathfrak{g}^*$, the notation $G_\mu$ will always denote the isotropy subgroup of $G$ with respect to the coadjoint action.

**Momentum maps.** Suppose $G$ acts symplectically on a symplectic manifold $(M, \omega)$. Recall that any function $F : M \to \mathbf{R}$ defines a Hamiltonian vector field $X_F$ by $i_{X_F}\omega = dF$, in other words $\omega(X_F(z), v) = dF(v)$ for every $v \in T^*_z M$. A *momentum map* is a function $J : M \to \mathfrak{g}^*$ satisfying $X_{J_\xi} = \xi_M$ for every $\xi \in \mathfrak{g}$, where $J_\xi : M \to \mathbf{R}$ is defined by $J_\xi(z) = \langle J(z), \xi\rangle$. If the $G$ action has an $\mathrm{Ad}^*$-equivariant momentum map $J$, then it is called *globally Hamiltonian*. Note that if $J(z) = \mu$ and $G_\mu$ is the isotropy group of $\mu$ with respect to the coadjoint action and $J$ is $\mathrm{Ad}^*$-equivariant then $G_z \subset G_\mu$.

The Kostant–Kirillov–Souriau (KKS) symplectic forms on any coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ are given by

$$\omega^\pm_{\mathcal{O}}(\nu)(\xi \cdot \nu, \eta \cdot \nu) = \pm\langle \nu, [\xi, \eta]\rangle. \tag{1}$$

The momentum map of the coadjoint action of $G$ on $\mathcal{O}$ with respect to $\omega^\pm_{\mathcal{O}}$ is $J_{\mathcal{O}}(\nu) = \pm\nu$.

**Lifted actions on (Co-)tangent bundles.** Every cotangent bundle $T^*Q$ has a canonical symplectic form, given in given local coordinates by $\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$. The space $Q$ is called the *configuration space* or *base space*. The *tangent lift* of any action $\Phi : G \times Q \to Q$ is the action of $G$ on $TQ$ given by $g \cdot v = T\Phi_g(v)$. The *cotangent lift* is the action of $G$ on $T^*Q$ given by $g \cdot z = (T\Phi_{g^{-1}})^*z$. The tangent or cotangent lift of any proper (resp. free) action is proper (resp. free). Every cotangent-lifted action is symplectic with respect to the canonical symplectic form, and has an $\mathrm{Ad}^*$-equivariant momentum map given by $\langle J(\alpha_q), \xi\rangle = \langle \alpha_q, \xi \cdot q\rangle$. (When we refer to "the" momentum map for such an action, this is the one we mean.) If $z \in T^*_q Q$ then it is easy to see that $G_z \subset G_q$. If $\mu = J(z)$ then, by equivariance of $J$, we have $G_z \subset G_\mu$. There exist simple examples in which $G_z$ is a proper subset of $G_q \cap G_\mu$. The inequality of the different isotropy subgroups $G_q$, $G_z$ and $G_\mu$ is a key difficulty in the theory of cotangent-lifted actions.

If $G$ acts *linearly* on a vector space $V$, then the cotangent-lifted action on $T^*V \cong V \times V^*$ is given by $g \cdot (a, \alpha) = (g \cdot a, g \cdot \alpha)$, where the action on the second component is the inverse dual action. The infinitesimal action of $G$ on $V^*$ is $\langle \eta \cdot \alpha, a\rangle = \langle \alpha, -\eta \cdot a\rangle$. Note that if we identify $V^{**}$ with $V$ then the inverse dual of the inverse dual of an action is the original action.

We introduce the diamond notation of Holm et al. [8], adding an optional subscript to specify the Lie algebra of the symmetry group or some linear subspace of it. For every $a \in V$, $\alpha \in V^*$, and any subspace $\mathfrak{l}$ of $\mathfrak{g}$, we define $a \diamond_{\mathfrak{l}} \alpha \in \mathfrak{l}^*$ by

$$\langle a \diamond_{\mathfrak{l}} \alpha, \xi\rangle = \langle \alpha, \xi \cdot a\rangle \quad \text{for all } \xi \in \mathfrak{l}. \tag{2}$$

If the subscript is omitted, then $a \diamond \alpha = a \diamond_{\mathfrak{g}} \alpha$, where $\mathfrak{g}$ is the Lie algebra of the whole symmetry group under discussion. Thus, $a \diamond_{\mathfrak{l}} \alpha = a \diamond \alpha|_{\mathfrak{l}}$, the restriction of $a \diamond \alpha$ to $\mathfrak{l}$. Since we will have occasion to use the isomorphism $T^*V \cong V \times V^* \cong V^{**} \times V^* \cong T^*V^*$, so we point out that $\alpha \diamond a = -a \diamond_{\mathfrak{g}} \alpha$ for any $a \in V$ and $\alpha \in V^*$, by definition of the inverse dual action. The momentum map for the cotangent-lifted $G$ action on $T^*V \cong V \times V^*$ is the map $J_V : V \to \mathfrak{g}^*$ given by $J_V(a, \alpha) = a \diamond_{\mathfrak{g}} \alpha$.

**Symplectic reduction and the symplectic normal space.** Let $G$ act properly and symplectically on $(M, \omega)$, with $\mathrm{Ad}^*$-equivariant momentum map $J : M \to \mathfrak{g}^*$. Let $z \in M$ and $\mu = J(z)$. In the following lemma, the superscript $\omega$ denotes symplectic complement.

**Lemma 1** *(Reduction lemma).* $(\mathfrak{g} \cdot z)^\omega = \ker dJ(z)$ *and* $\ker dJ(z) \cap \mathfrak{g} \cdot z = \mathfrak{g}_\mu \cdot z$.

This lemma is fundamental, and is at the heart of the following reduction theorem, which is a slight simplification of the result of Marsden and Weinstein [15].

**Theorem 2** *(Regular "point" symplectic reduction). In the above context, assume also that $G$ acts freely. Then the* reduced space *$J^{-1}(\mu)/G_\mu$ has a symplectic form $\omega_\mu$ uniquely defined by $\pi_\mu^*\omega_\mu = i_\mu^*\omega$, where $\pi_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$ and $i_\mu : J^{-1}(\mu) \to M$ is inclusion.*

Since $\ker T_z\pi_\mu = \mathfrak{g}_\mu \cdot z$, it follows that $T_z(J^{-1}(\mu)/G_\mu)$ is isomorphic to

$$N_s(z) := \ker dJ(z)/\mathfrak{g}_\mu \cdot z,$$

which is called the *symplectic normal space* at $z$ and will often be denoted by just $N_s$. The restriction of $\omega(z)$ to $\ker dJ(z)$ has kernel $\mathfrak{g}_\mu \cdot z$, by the Reduction lemma (Lemma 1), so it descends to a reduced symplectic bilinear form on $N_s(z)$. The push-forward of this form via isomorphism with $T_z(J^{-1}(\mu)/G_\mu)$ is in fact $\omega_\mu(z)$, where $\omega_\mu$ is the reduced symplectic form defined in the above theorem. A key point is that $N_s$ and its symplectic form are well-defined even for non-free actions. For this reason, $N_s$ plays a major role in the rest of this article. Since $G_z \subset G_\mu$, it is not hard to show that the tangent-lifted action of $G_z$ on $T_zM$ leaves $\ker dJ(z)$ invariant and descends to a symplectic action on $N_s$ given by

$$h \cdot (v + \mathfrak{g}_\mu \cdot z) = h \cdot v + \mathfrak{g}_\mu \cdot z. \tag{3}$$

The symplectic reduced spaces for lifted actions on cotangent bundles have extra structure. The following theorem is a special case of a result by Satzer (see [1]).

**Theorem 3** *(Regular "point" cotangent bundle reduction at zero). Let $G$ act freely and properly by cotangent lifts on $T^*Q$, and let $J$ be the momentum map of the $G$ action (with respect to the canonical symplectic form on $T^*Q$). Let $\pi_G : Q \to Q/G$ be projection. Define the map $\varphi : J^{-1}(0) \to T^*(Q/G)$ by, for every $p \in T_q^*Q$ and $v \in T_qQ$,*

$$\big\langle \varphi(p), T\pi_G(v)\big\rangle = \langle p, v\rangle.$$

*Then $\varphi$ is a $G$-invariant surjective submersion and descends to a symplectomorphism (i.e. symplectic diffeomorphism)*

$$\bar\varphi : J^{-1}(0)/G \to T^*(Q/G),$$

*where the left-hand side has the reduced symplectic form corresponding to the canonical symplectic form on $T^*Q$, and $T^*(Q/G)$ has the canonical symplectic form.*

The map $\varphi$ is a sort of push-forward, though $\pi_G$ is not injective. Note that $\varphi$ is "injective mod $G$", meaning that $\varphi(z_1) = \varphi(z_2)$ if and only if $z_1 = g \cdot z_2$ for some $g \in G$.

**The Witt–Artin decomposition.** Let $G$ be a Lie group acting symplectically and properly on $(M, \omega)$, with $\mathrm{Ad}^*$-equivariant momentum map $J$. Let $z \in M$ and $\mu = J(z)$. Let $T_0 = \mathfrak{g}_\mu \cdot z$; let $T_1 = \mathfrak{q} \cdot z$ for some splitting $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{q}$; and let $N_1$ is a complement to $T_0$ in $\ker dJ(z)$. It can be shown that $T_zM = T_1 \oplus N_1 \oplus (T_1 \oplus N_1)^\omega$ is a decomposition into symplectic subspaces and $T_0$ is Lagrangian in $(T_1 \oplus N_1)^\omega$. Let $N_0$ be a Lagrangian complement to $T_0$ in $(T_1 \oplus N_1)^\omega$. The Witt–Artin decomposition is

$$T_zM = T_0 \oplus T_1 \oplus N_0 \oplus N_1. \tag{4}$$

The decomposition, which is not unique, can be chosen to be $G_z$-invariant. It can be shown that there is a $G_z$-equivariant isomorphism of $N_0$ with $(\mathfrak{g}_\mu/\mathfrak{g}_z)^*$ and a $G_z$-equivariant symplectomorphism of $N_1$ with $N_s$ (the symplectic normal space defined above).

**Slice theorems.** Let $H$ be a Lie subgroup of a Lie group $G$, and $S$ a manifold on which $H$ acts. Consider the following two left actions on $G \times S$:

$$\text{the } \textit{twist} \text{ action of } H\text{:}\quad h\cdot(g,s)=(gh^{-1},h\cdot s),$$
$$\text{the } \textit{left multiplication} \text{ action of } G\text{:}\quad g'\cdot(g,s)=(g'g,s). \tag{5}$$

These actions are easily seen to be free and proper. The *twisted product* $G\times_H S$ is the quotient of $G\times S$ by the twist action. It is a smooth manifold; in fact $G\times_H S\to G/H$ is the fibre bundle associated with the principal bundle $G\to G/H$ via the $H$ action on $S$. The left multiplication action commutes with the twist action and descends to a smooth $G$ action on $G\times_H S$, namely $g'\cdot[g,s]_H=[g'g,s]_H$.

Now consider a $G$ action on a manifold $M$. Let $z\in M$, with isotropy subgroup $H=G_z$. A *tube* for the $G$ action at $z$ is a $G$-equivariant diffeomorphism from some twisted product $G\times_H S$ to an open neighbourhood of $z$ in $M$, that maps $[e,0]_H$ to $z$. The space $N$ may be embedded in $G\times_H S$ as $\{[e,s]_H\colon s\in S\}$; the image of the latter by the tube is called a *slice*. A *slice theorem* (or *tube theorem*) is a theorem guaranteeing the existence of a tube under certain conditions. Palais [25] was the first to prove a slice theorem for proper actions. Many smooth versions of his original theorem are in common use. A proof of the following version appeared in an early version of the present article, but has been moved to the author's website because a modified version now appears in [23]. The proof is similar to one given in [6].

**Theorem 4** *("Palais' slice theorem"). Let $G$ be a Lie group acting properly and smoothly on a manifold $M$, and let $z\in M$. Let $H=G_z$ be the isotropy group of $z$, and let $N$ be any $H$-invariant complement to $\mathfrak{g}\cdot z$. Choose a local $H$-invariant Riemannian metric around $z$ (such a metric always exists), and let $\exp_z$ be the corresponding Riemannian exponential based at $z$. Then there exists an $H$-invariant neighbourhood $S$ of $0$ in $N$ such that the map*

$$\tau: G\times_H S\longrightarrow M,$$
$$[g,s]_H\longmapsto g\cdot\exp_z s$$

*is a tube for the $G$ action at $z$.*

*If in addition, $M$ is a vector space and $G$ acts linearly, then the "$\exp_z s$" in the formula for $\tau$ may be replaced by "$z+s$", and $\tau$ is a tube for any choice of an $H$-invariant neighbourhood $S$ of $0$ for which $\tau$ is injective.*

An $H$-invariant complement to $\mathfrak{g}\cdot z$ is often called a *linear slice* at $z$ for the $G$ action.

**The Hamiltonian slice theorem.** Now suppose that $G$ acts *symplectically*. We would like the tube $\tau$ in the previous theorem to be symplectic, with respect to some simple or "natural" symplectic form on a space $G\times_H N$. The Hamiltonian slice theorem, also known as the Marle–Guillemin–Sternberg normal form, accomplishes this for globally Hamiltonian actions. It was first proven by Marle [11] and Guillemin and Sternberg [7], for compact groups $G$, and extended to proper actions of arbitrary groups by Bates and Lerman [3]. We note that the assumption that the $G$ action is globally Hamiltonian has been removed by Ortega and Ratiu [22] and Scheerer and Wulff [29]. The Hamiltonian version is sufficiently general for the present article, since all cotangent-lifted actions have an $\mathrm{Ad}^*$-equivariant momentum map. Our presentation follows [19] and [31].

We are assuming that $G$ acts symplectically and properly on $(M,\omega)$, with $\mathrm{Ad}^*$-equivariant momentum map $J$. Let $z\in M$ and $\mu=J(z)$. Let $H=G_z$ and recall that $H\subset G_\mu$. Let $\mathfrak{h}$, $\mathfrak{g}$ and $\mathfrak{g}_\mu$ be the Lie groups of $H$, $G$ and $G_\mu$ respectively. Let $\mathfrak{m}$ be an $H$-invariant complement to $\mathfrak{h}$ in $\mathfrak{g}_\mu$. Let $N_s$ be the *symplectic normal space* at $z$. The $H$ action on $N_s$ (defined above) and the coadjoint action of $H$ on $\mathfrak{m}^*$ define an $H$ action on $\mathfrak{m}^*\times N_s$, allowing us to define the twisted product

$$G\times_H(\mathfrak{m}^*\times N_s).$$

Note that $\mathfrak{m}\cong(\mathfrak{g}_\mu/\mathfrak{h})=(\mathfrak{g}_\mu/\mathfrak{g}_z)$, and recall that $(\mathfrak{g}_\mu/\mathfrak{g}_z)^*\cong N_0$ and $N_s\cong N_1$, where $N_0$ and $N_1$ are components in the Witt–Artin decomposition (Eq. (4)). Since $N_0\oplus N_1$ is a linear slice at $z$ for the $G$ action on $M$, the space $G\times_H(\mathfrak{m}^*\times N_s)$ can be considered to be a special case of the model space $G\times_H N$ in Theorem 4.

We now define a symplectic form on $G\times_H(\mathfrak{m}^*\times N_s)$, beginning with a presymplectic form on $Z:=G\times\mathfrak{g}_\mu^*\times N_s$. First, let $\Omega_c$ be the pull-back to $G\times\mathfrak{g}_\mu^*$ of the canonical form on $T^*G$ by the map $G\times\mathfrak{g}_\mu^*\to T^*G$, $(g,\nu)\mapsto TL_{g^{-1}}^*\nu$. Second, let $\Omega_\mu$ be the pull-back of the KKS symplectic form $\omega^+_{\mathcal{O}_\mu}$ (see Eq. (1)) by the map $G\times\mathfrak{g}_\mu^*\to\mathcal{O}_\mu$, $(g,\nu)\mapsto Ad^*_{g^{-1}}\mu$. Third, let $\omega_{N_s}$ be the reduced symplectic bilinear form on $N_s$ (defined above). The sum $\Omega_Z=\Omega_c+\Omega_\mu+\Omega_{N_s}$ is a presymplectic form on $Z=G\times(\mathfrak{g}_\mu^*\times N_s)$.

Consider the twist action of $H$ on $Z$ corresponding to the coadjoint action of $H$ on $\mathfrak{g}_\mu^*$ and the $H$ action on $N_s$ defined earlier. Since the $H$ action on $N_s$ is linear, it has an $H$-equivariant momentum map $J_{N_s}$. One can check that the twist action of $H$ on $Z$ is globally Hamiltonian with respect to $\Omega_Z$, with momentum map $J_H:(g,\sigma,v)\longmapsto J_{N_s}(v)-\sigma|_{\mathfrak{h}}$. If we identify $\mathfrak{m}^*$ with $\mathfrak{h}^\circ\subset\mathfrak{g}_\mu^*$, then the following map is well-defined,

$$\begin{aligned} &l:G\times\mathfrak{m}^*\times N_s\longrightarrow J_H^{-1}(0)\subset Z,\\ &(g,\sigma,v)\longmapsto\bigl(g,\sigma+J_{N_s}(v),v\bigr). \end{aligned}\tag{6}$$

It is clearly an $H$-equivariant diffeomorphism. It descends to a diffeomorphism $L$ defined by the following commutative diagram,

$$\begin{tikzcd} G\times(\mathfrak{m}^*\times N_s) \arrow[r, "l"] \arrow[d, "\pi_H"] & J_H^{-1}(0)\subset Z=G\times(\mathfrak{g}_\mu^*\times N_s) \arrow[d, "\pi_{Z,H}"] \\ G\times_H(\mathfrak{m}^*\times N_s) \arrow[r, "L", "\cong"'] & J_H^{-1}(0)/H, \end{tikzcd}\tag{7}$$

where $\pi_H$ and $\pi_{Z,H}$ are the obvious projections.

We define the presymplectic form $\omega_Y$ on $G\times_H(\mathfrak{m}^*\times N_s)$ as the pull-back by $L$ of the reduced presymplectic form on $J_H^{-1}(0)/H$ corresponding to $\Omega_Z$. It can be shown that there exists a $G$-invariant neighbourhood $Y$ of $[e,0,0]_H$ in $G\times_H(\mathfrak{m}^*\times N_s)$ in which $\omega_Y$ is symplectic. Finally, note that there is left $G$-action on $Y$ given by $g'\cdot[g,\sigma,v]_H=[g'g,\sigma,v]_H$ It is easy to check that this is symplectic with respect to $\omega_Y$.

**Theorem 5** *(Hamiltonian slice theorem). In the above context, there exists a symplectic tube from $Y\subset G\times_H(\mathfrak{m}^*\times N_s)$ to $M$ that maps $[e,0,0]_H$ to $z$. The momentum map of the $G$ action on $Y$ is*

$$J_Y\bigl([g,\sigma,v]_H\bigr)=\mathrm{Ad}^*_{g^{-1}}\bigl(\mu+\sigma+J_{N_s}(v)\bigr).$$

## 3. Commuting reduction

In this section we consider a manifold with two commuting symplectic actions. We first review regular and singular commuting reduction and then introduce a new "tangent-level version" of commuting reduction, which we will use in the next section in our analysis of the symplectic normal space of a cotangent-lifted action.

We have already seen an example of commuting symplectic actions in the presentation of the Hamiltonian slice theorem: the $G$ and $H$ actions on the manifold $G\times(\mathfrak{g}_\mu^*\times N_s)$ (see Eq. (7)). In this context, commuting reduction leads to a singular local normal form for a symplectic reduced space, Theorem 8. A second example of commuting symplectic actions, key to the rest of this article, will appear in the next section: a bundle $T^*(G\times A)$ with the cotangent lifts of the left multiplication action of $G$ and the twist action of a subgroup $K$ of $G$. Commuting reduction in this context leads to a cotangent-bundle specific local normal form for a symplectic reduced space, Theorem 18."Tangent-level reduction" in this context will be used to characterise the symplectic normal space of a cotangent-lifted action: see Theorem 19 and following results.

Let $G$ and $K$ be Lie groups acting symplectically and properly on a symplectic manifold $M$, with equivariant momentum maps $J_G$ and $J_K$ respectively, and suppose that the actions commute. Let $\mu\in\mathfrak{g}^*$ and $\nu\in\mathfrak{k}^*$. The idea of commuting reduction is to first reduce by the $K$ action (say) and then reduce the $K$-reduced space by the induced $G$ action; and then switch the order, reducing first by $G$ and then by $K$. Under very general conditions, the two doubly-reduced spaces are isomorphic. We first state the "regular version" of commuting reduction, due to Marsden and Weinstein [15]; the key assumption here is that all of the group actions are free.

**Theorem 6** *(Regular commuting reduction). In the above context, suppose that $G$ and $K$ act freely and $J_K$ is $G$-invariant and $J_G$ is $K$-invariant. Then $G$ induces a symplectic action on $M_\nu:=J_K^{-1}(\nu)/K_\nu$ with equivariant momentum map $J_{\bar G}$ determined by $J_{\bar G}\circ\pi_{K_\nu}=J_G$ (where $\pi_{K_\nu}:M\to M/K_\nu$ is projection, and both sides of the equation are restricted to $J_K^{-1}(\nu)$). If the reduced $G$ action is free, then the reduced space for this action at $\mu$ is symplectomorphic to the reduction of $M$ at $(\mu,\nu)$ by the product action of $G\times K$.*

Note that applying this theorem a second time, with the roles of $G$ and $K$ reversed, shows that the reduced space at $\nu$ for the action of $K$ on $J_G^{-1}(\mu)/G_\mu$ is symplectomorphic to the reduced space at $\mu$ for the action of $G$ on $M_\nu$.

Sjamaar and Lerman [31], working with reduction at zero of compact group actions, showed that a similar result holds even if the actions are not free. In this case, the reduced spaces need not be smooth manifolds, but are Poisson varieties. In the general case, for proper actions and arbitrary momentum values, we need to add the hypotheses that $G_\mu$ and $K_\nu$ are compact and that $\mathcal{O}_\mu$ and $\mathcal{O}_\nu$ are locally closed, the latter for reasons discussed in [13].

**Theorem 7** *(Singular commuting reduction). In the above context, suppose that $J_K$ is $G$-invariant, $J_G$ is $K$-invariant, $G_\mu$ and $K_\nu$ are compact and the coadjoint orbits $\mathcal{O}_\mu$ and $\mathcal{O}_\nu$ are locally closed. Then $G$ induces a Poisson action on $M_\nu = J_K^{-1}(\nu)/K_\nu$, with equivariant momentum map $J_{\bar{G}}$ determined by $J_{\bar{G}} \circ \pi_{K_\nu} = J_G$. The reduced space for the action of $G$ on $M_\nu$ at $\mu$ is Poisson diffeomorphic to the reduction of $M$ at $(\mu, \nu)$ by the product action of $G \times K$.*

It follows that the reduced space at $\nu$ for the action of $K$ on $J_G^{-1}(\mu)/G_\mu$ is Poisson diffeomorphic to the reduced space at $\mu$ for the action of $G$ on $M_\nu$.

The Hamiltonian Slice Theorem (Theorem 5), together with singular commuting reduction, applied to the $G$ and $H$ actions on $G \times (\mathfrak{g}_\mu^* \times N_s)$ (see Eq. (7)), can be used to deduce the following local normal form for a symplectic reduced space (when $G_\mu$ is compact). The result was first published by Sjamaar and Lerman [31] for $\mu = 0$; the general case is due to Bates and Lerman [3]. The proof given in [3] does not use a commuting reduction theorem and does not require $G_\mu$ compact.

**Theorem 8.** *Let $G$ act properly on the symplectic manifold $(M, \omega)$ with equivariant momentum map $J$. Let $z \in M$ and $H = G_z$ and $\mu = J(z)$, and let $N_s$ be the symplectic normal space to $\mathfrak{g} \cdot z$. Assume that the coadjoint orbit $\mathcal{O}_\mu$ is locally closed. Then there is a local Poisson diffeomorphism between the reduced space $J^{-1}(\mu)/G_\mu$ and the reduced space at $0$ for the $H$ action on $N_s$.*

In the case of cotangent-lifted actions, our analysis of the symplectic normal space, in the next section, together with the above theorem comprise a cotangent-bundle-specific local normal form for symplectic reduced spaces, as we note later in Remark 25.

We now introduce another approach to singular commuting reduction, assuming that the original actions are free but not assuming that the quotient action on the once-reduced space is free. Recall that, in the case of a free action, the symplectic normal space "is" the tangent space to the reduced space. This observation suggests studying symplectic normal spaces in place of the possibly singular doubly-reduced spaces.

Since symplectic normal spaces are quotients, the following lemma and notation will be useful; the lemma is easily checked.

**Lemma 9.** *Let $\omega_A$ and $\omega_B$ be bilinear forms on vector spaces $A$ and $B$, respectively. Suppose $f : A \to B$ satisfies $f^*\omega_B = \omega_A$. Then the quotient map $\bar{f} : A/\ker(\omega_A) \to B/\ker(\omega_B)$ is well-defined and injective. If $f$ is surjective, then $\bar{f}$ is bijective. If $\omega_A$ and $\omega_B$ are presymplectic (i.e. skew-symmetric) then $\bar{f}$ is symplectic. Also, if $\bar{g} : B/\ker(\sigma_B) \to C/\ker(\sigma_C)$ is defined similarly then $\overline{f \circ g} = \bar{f} \circ \bar{g}$.*

**Theorem 10** *("Tangent-level" commuting reduction). Let $G$ and $K$ be free, symplectic, commuting actions on a symplectic manifold $(M, \omega)$, with momentum maps $J_G$ and $J_K$ respectively. Then the product action of $G \times K$ has momentum map given by $J_{G\times K}(x) = (J_G(x), J_K(x))$. Let $x \in M$ and $(\mu, \nu) = J_{G\times K}(x)$. The symplectic normal space at $x$ for the product action is*

$$N_s(x) = \ker T_x J_{G\times K}/(\mathfrak{g}_\mu \cdot x + \mathfrak{k}_\nu \cdot x).$$

*Suppose further that $G$ acts properly and that $J_G$ is $\mathrm{Ad}^*$-equivariant and that $J_G^{-1}(\mu)$ is $K$-invariant. Then the quotient action of $K$ on $J_G^{-1}(\mu)/G_\mu$ is symplectic with respect to the reduced symplectic form, and its momentum map $J_{\bar{K}}$ satisfies $J_{\bar{K}} \circ \pi_{G_\mu} = J_K|_{J_G^{-1}(\mu)}$ (where $\pi_{G_\mu} : J_G^{-1}(\mu) \to J_G^{-1}(\mu)/G_\mu$ is projection). The map $(g, k) \longmapsto k$ is a Lie group isomorphism from $(G \times K)_x$ to $K_{[x]_{G_\mu}}$ (where $[x]_{G_\mu} = G_\mu \cdot x$). We identify these two groups and call them $H$. Let $N_s([x]_{G_\mu})$ be the symplectic normal space at $[x]_{G_\mu}$ for the $K$ action on $J_G^{-1}(\mu)/G_\mu$. Let $H$ act on each* *symplectic normal space, as in Eq. (3). Then the following is an $H$-equivariant vector space symplectomorphism,*

$$\begin{aligned}&\overline{T_x\pi_{G_\mu}} : N_s(x) \longrightarrow N_s\big([x]_{G_\mu}\big),\\ &v + (\mathfrak{g}_\mu \cdot x + \mathfrak{k}_\nu \cdot x) \longmapsto T\pi_{G_\mu}(v) + \big(\mathfrak{k}_\nu \cdot [x]_{G_\mu}\big).\end{aligned}$$

**Proof.** It is easily verified that the product action has the given momentum map. Since $(G \times K)_{(\mu,\nu)} = G_\mu \times K_\nu$ and the actions commute, we have $(\mathfrak{g} \oplus \mathfrak{k})_{(\mu,\nu)} \cdot x = (\mathfrak{g}_\mu \oplus \mathfrak{k}_\nu) \cdot x = (\mathfrak{g}_\mu \cdot x + \mathfrak{k}_\nu \cdot x)$, so the symplectic normal space at $x$ is $N_s(x) = \ker T_x J_{G\times K}/(\mathfrak{g}_\mu \cdot x + \mathfrak{k}_\nu \cdot x)$. The claims about the quotient action of $K$ on $J_G^{-1}(\mu)/G_\mu$ are part of regular commuting reduction (Theorem 6), and in any case are easy to prove by "diagram-chasing".

We will now show that $\theta : (G \times K)_x \to K_{[x]_{G_\mu}}, (g,k) \longmapsto k$, is an isomorphism. To show it's well-defined, let $(g,k) \in (G \times K)_x$, so $k \cdot x = g^{-1} \cdot x$. Since $J_G$ is $\mathrm{Ad}^*$-equivariant and $J_G^{-1}(\mu)$ is $K$-invariant, we have $\mu = J_G(x) = J_G((g,k) \cdot x) = g \cdot \mu$, so $g \in G_\mu$. This implies that $k \in K_{[x]_{G_\mu}}$. So $\theta$ is well-defined. It is clearly smooth, and a homomorphism. For every $k \in K_{[x]_{G_\mu}}$, we have $k \cdot x \in G_\mu x$; since $G$ acts freely, there is a unique element $\gamma(k) \in G_\mu$ such that $k \cdot x = \gamma(k)^{-1} \cdot x$. Clearly $(\gamma(k), k) \cdot x = x$, so the map $k \mapsto (\gamma(k), k)$ is an inverse for $\theta$. The smoothness of $\theta^{-1}$ is a consequence of the implicit function theorem applied to the restricted action $F : G_\mu \times K_{[x]_{G_\mu}} \to G_\mu \cdot x$ given by $F(g,k) = (g,k) \cdot x$. Indeed, note that $(G \times K)_x = F^{-1}(x)$, and that $D_1F(g,k)$ is surjective for every $(g,k) \in G_\mu \times K_{[x]_{G_\mu}}$, since the $G$ action is free. Hence $\theta$ is a Lie group isomorphism. We identify $(G \times K)_x$ with $K_{[x]_{G_\mu}}$ via $\theta$, calling both groups $H$.

Next, observe that

$$\begin{aligned}\ker T_x J_{G\times K} &= \ker T_x J_G \cap \ker T_x J_K = T_x J_G^{-1}(\mu) \cap \ker T_x J_K = \ker T_x\big(J_K|_{J_G^{-1}(\mu)}\big)\\ &= \ker T_x(J_{\bar K} \circ \pi_{G_\mu}) = (T_x\pi_{G_\mu})^{-1}(\ker T_{[x]_{G_\mu}} J_{\bar K}).\end{aligned}$$

Since $T_x\pi_{G_\mu}$ is surjective, this implies that

$$T_x\pi_{G_\mu}(\ker T_x J_{G\times K}) = \ker T_{[x]_{G_\mu}} J_{\bar K}.$$

By definition of the quotient action of $K$, we have $T_x\pi_{G_\mu}(\mathfrak{g}_\mu \cdot x + \mathfrak{k}_\nu \cdot x) = \mathfrak{k}_\nu \cdot [x]_{G_\mu}$. The map $T_x\pi_{G_\mu}$ is a presymplectic surjection, by definition of the reduced symplectic form $\omega_{\mathrm{red}}$ on $J_G^{-1}(\mu)/G_\mu$. By the Reduction lemma (Lemma 1), $\mathfrak{g}_\mu \cdot x + \mathfrak{k}_\nu \cdot x$ is the kernel of the restriction of $\omega$ to $\ker T_x J_{G\times K}$, and $\mathfrak{k}_\nu \cdot [x]_{G_\mu}$ is the kernel of the restriction of $\omega_{\mathrm{red}}$ to $\ker T_{[x]_{G_\mu}} J_{\bar K}$. Hence Lemma 9 implies that $\overline{T_x\pi_{G_\mu}}$, as defined in the statement of the theorem, is a well-defined symplectic isomorphism from $N_s(x)$ to $N_s([x]_{G_\mu})$.

Since we have already shown that $(g,k) \in (G \times K)_x$ implies $g \in G_\mu$, the $H$-equivariance of $\pi_{G_\mu}$ is easily checked. The $H$-equivariance of $T_x\pi_{G_\mu}$, and hence $\overline{T_x\pi_{G_\mu}}$, follows. □

## 4. The symplectic normal space of a cotangent-lifted action

The main result of this section will be a characterisation of the symplectic normal space $N_s$ for a cotangent-lifted action, given in Theorem 19. In two special cases this leads to new splittings of $N_s$, given in Corollaries 20 and 23. Our analysis of the special case $G_q \subset G_\mu$, and much of the general set-up developed in this section, will be used in the cotangent bundle slice theorem, Theorem 31. We also note implications for singular reduction, in Theorem 18 and Remark 25, and the reconstruction equations (bundle equations), in Eqs. (16) and (17).

Let $G$ act properly by cotangent lifts on $T^*Q$. Applying Palais' slice theorem at $q \in Q$, and then cotangent-lifting the resulting diffeomorphism, gives a local symplectic diffeomorphism from $T^*Q$ to $T^*(G \times_{G_q} A)$, where $A$ is a linear slice at $q$. Note that the untwisted product $T^*(G \times A)$ has two obvious commuting actions, namely cotangent lifts of the following actions: left multiplication by $G$, and twist by $G_q$, as defined in Eq. (5). We will apply Theorem 10 ("tangent-level" commuting reduction) to $T^*(G \times A)$. We begin with some basic computations that will be useful in this and the following section.

Let $S$ be any manifold on which $K$ acts (we have in mind $S = U$ or $S = A$, but the following facts are general). Recall from Eq. (5) the following two left actions on $G \times S$, which commute and are both free and proper:

$K$ acts by *twisting*: $\quad k^K \cdot (g,n) = (gk^{-1}, k \cdot n),$

$$G \text{ acts by } \textit{left multiplication}: \quad h^G \cdot (g, n) = (hg, n). \tag{8}$$

Note that, since $K$ is a subset of $G$, there is room for confusion of the two actions, so we have introduced superscripts to identify them. Each of these actions has a corresponding tangent-lifted action on $T(G \times S) \cong TG \times TS$ and cotangent-lifted action on $T^*(G \times S) \cong T^*G \times T^*S$. It is easy to see that these actions commute and are free and proper.

Throughout this article, we will identify $TG$ with $G \times \mathfrak{g}$ and $T^*G$ with $G \times \mathfrak{g}^*$ by left trivialisation,

$$TG \xrightarrow{\cong} G \times \mathfrak{g} \quad \text{and} \quad T^*G \xrightarrow{\cong} G \times \mathfrak{g}^*,$$
$$TL_g(\xi) \longmapsto (g, \xi), \quad T^*L_{g^{-1}}(\nu) \longmapsto (g, \nu)$$

where $L_g$ is left multiplication by $g$. The following basic properties of the left and right multiplication actions are well known.

**Lemma 11.** *Let $G$ be a Lie group. With respect to the left trivialisations of $TG$ and $T^*G$, the left and right multiplication actions of $G$ on itself have the following lifted actions and infinitesimal lifted actions*:

$$\text{tangent:} \quad h^L \cdot (g, \xi) = (hg, \xi), \quad h^R \cdot (g, \xi) = (gh^{-1}, \mathrm{Ad}_h \xi),$$
$$\text{cotangent:} \quad h^L \cdot (g, \nu) = (hg, \nu), \quad h^R \cdot (g, \nu) = (gh^{-1}, \mathrm{Ad}^*_{h^{-1}} \nu),$$
$$\text{infinitesimal tangent:} \quad \eta^L \cdot (g, \xi) = (\mathrm{Ad}_{g^{-1}} \eta, 0), \quad \eta^R \cdot (g, \xi) = (-\eta, \mathrm{ad}_\eta \xi),$$
$$\text{infinitesimal cotangent:} \quad \eta^L \cdot (g, \nu) = (\mathrm{Ad}_{g^{-1}} \eta, 0), \quad \eta^R \cdot (g, \nu) = (-\eta, -\mathrm{ad}^*_\eta \nu).$$

*The cotangent-lifted actions have the following momentum maps, with respect to the canonical symplectic form on $T^*G$*:

$$J_L(g, \nu) = \mathrm{Ad}^*_{g^{-1}} \nu, \qquad J_R(g, \nu) = -\nu.$$

*The momentum map $J_L$ is invariant under the right multiplication action, and $J_R$ is invariant under the left multiplication action.*

There are obvious corresponding properties for the $G$ and $K$ actions on $G \times S$. In particular, we have the following:

**Remark 12.** Let $G$ and $K$ act on $G \times S$ as in Eq. (8). Then the momentum maps for the cotangent-lifted actions on $T^*(G \times S) \cong G \times \mathfrak{g}^* \times T^*S$, with respect to the canonical symplectic form on $T^*(G \times S)$, are

$$J_G(g, \nu, w) = \mathrm{Ad}^*_{g^{-1}} \nu, \qquad J_K(g, \nu, w) = -\nu|_\mathfrak{k} + J_S(w),$$

where $J_S$ is the momentum map for the cotangent-lifted action of $K$ on $T^*S$. The previous lemma implies that $J_G$ is invariant under the twist action of $K$ and $J_K$ is invariant under the left multiplication action of $G$. If $S$ is a vector space, we can identify $TS$ with $S \times S$ and $T^*S$ with $S \times S^*$, so

$$T(G \times S) \cong G \times \mathfrak{g} \times S \times S \quad \text{and} \quad T^*(G \times S) \cong G \times \mathfrak{g}^* \times S \times S^*,$$

where the first and third components are the base space, and the second and fourth are the (co-)tangent fibers. These identifications will be used throughout this article. In these coordinates, and using the diamond notation (see Eq. (2)), $J_K(g, \nu, a, \delta) = -\nu|_\mathfrak{k} + a \diamond_\mathfrak{k} \delta$.

We are now in a position to apply reduction theorems to the two actions on $T^*(G \times S)$. We begin by applying cotangent bundle reduction at zero momentum (Theorem 3) to the $K$ action. Note that $(G \times S)/K = G \times_K S$. The map $\varphi$ in Theorem 3 takes the following form, where $\pi_K : G \times S \to G \times_K S$ is projection:

$$\varphi : \left(J_K^{-1}(0) \subset T^*(G \times S)\right) \to T^*(G \times_K S), \qquad \langle \varphi(p), T\pi_K(v) \rangle = \langle p, v \rangle. \tag{9}$$

Recall that $G$ has a quotient action on $G \times_K S$, and so $G$ acts on $T^*(G \times_K S)$ by cotangent lifts. The projection $\pi_K$ is $G$-equivariant by definition of the $G$ action on $G \times_K A$, so $T\pi_K$ is $G$-equivariant with respect to the tangent lifted actions, from which it follows that $\varphi$ is $G$-equivariant. Since $J_K^{-1}(0)$ is $G$-invariant, the $G$ action descends to one on $J_K^{-1}(0)/K$. It is easily verified that this quotient action is symplectic; in fact this claim is part of Theorem 6 (regular commuting reduction). Applying Theorems 3 and 6 gives the following result.

**Proposition 13.** *Let $G$ and $K$ act on $T^*(G \times S)$ as above, with momentum maps $J_K$ and $J_G$ respectively. Let $\varphi$ be defined as in Eq. (9). Then $\varphi$ is a $G$-equivariant $K$-invariant surjective submersion that descends to a $G$-equivariant symplectomorphism*

$$\bar{\varphi} : J_K^{-1}(0)/K \to T^*(G \times_K S),$$

*with respect to the reduced symplectic form on $J_K^{-1}(0)/K$ and the canonical symplectic form on $T^*(G \times_K S)$. If $J'$ is the momentum map for the $G$ action on $T^*(G \times_K S)$, then the restriction of $J_G$ to $J_K^{-1}(0)$ equals $J' \circ \varphi$.*

We now return to the context of a proper $G$ action on $T^*Q$, where $G$ is any manifold. Let $J$ be the momentum map for this action, and let $z \in T_q^*Q$ and $\mu = J(z)$. Let $K = G_q$ and $H = G_z$, and let $\mathfrak{g}, \mathfrak{g}_\mu, \mathfrak{k}, \mathfrak{h}$ be the Lie algebras of $G, G_\mu, K$ and $H$. We begin with some basic facts concerning isotropy subgroups, which will be used in several contexts.

**Lemma 14.** (i) $H \subset K$. (ii) $H \subset G_\mu$. (iii) $\mathfrak{k} \subset \ker \mu$. (iv) *If $K$ is normal in $G$, then $K \subset G_\mu$.*

**Proof.** Claim (i) is clear from $z \in T_q^*Q$; (ii) follows from the equivariance of $J$. (iii) The definition of $J$ gives $\langle \mu, \xi \rangle = \langle z, \xi_Q(q) \rangle = 0$ for all $\xi \in \mathfrak{k}$. (iv) For every $g \in G$ and $k \in K$ we have $gkg^{-1}k^{-1} \in K$. Differentiating with respect to $g$ gives $\xi - \mathrm{Ad}_{k^{-1}}\xi \in \mathfrak{k}$. Thus, for every $k \in K$ and $\xi \in \mathfrak{g}$, we have $\langle \mathrm{Ad}^*_{k^{-1}}\mu - \mu, \xi \rangle = \langle \mu, \mathrm{Ad}_{k^{-1}}\xi - \xi \rangle = 0$, in other words $k \in G_\mu$.

We now begin the main task of studying reduced spaces and symplectic normal spaces. We apply Palais' slice theorem (Theorem 4) at $q \in Q$. Choose a $K$-invariant Riemannian metric on some neighbourhood of $q$ in $Q$, and let $A$ be the orthogonal complement to $\mathfrak{g} \cdot q$ in $T_qQ$, written $A = (\mathfrak{g} \cdot q)^\perp$. By the slice theorem, there exists a $K$-invariant neighbourhood $V$ of 0 in $A$ such that the map

$$s : G \times_K V \longrightarrow Q, \qquad [g, a]_K \longmapsto g \cdot \exp_q a \tag{10}$$

is a $G$-equivariant embedding. The cotangent lift $T^*s^{-1} : T^*(G \times_K V) \to T^*Q$ is a $G$-equivariant symplectic embedding onto a neighbourhood of $z$ (symplectic with respect to the standard cotangent bundle symplectic forms).

Applying Proposition 13, with $S = A$, and composing $\bar{\varphi}$ from that proposition with $T^*s^{-1}$, we have a $G$-equivariant symplectic embedding,

$$\big(\big(J_K^{-1}(0) \cap (G \times \mathfrak{g}^* \times V \times A^*)\big)\big)/K \xrightarrow{\bar{\varphi}} T^*(G \times_K V) \overset{T^*s^{-1}}{\hookrightarrow} T^*Q. \tag{11}$$

In particular, there exists an $x \in J_K^{-1}(0)$ such that $T^*s^{-1}(\varphi(x)) = z \in T_q^*Q$. Since $s([e,0]_K) = q$, we see that $\varphi(x)$ has base point $[e,0]_K$. Since $\varphi$ covers $\pi_K : G \times V \to G \times_K V$ and is $K$-invariant, we can choose $x$ to have base point $(e, 0)$; in fact, since $\varphi$ is injective, this uniquely determines $x$. So $x = (e, \nu, 0, \alpha)$, for some $\nu \in \mathfrak{g}^*$ and some $\alpha \in A^*$. Using Proposition 13, we have $\nu = J_G(x) = J'(\varphi(x)) = J(z) = \mu$. We can also show that $\alpha = z|_A$. Indeed, for every $v \in A$, we have

$$\begin{aligned} \langle \alpha, v \rangle &= \big\langle \varphi(e, \mu, 0, \alpha), T\pi_K(e, 0, 0, v) \big\rangle = \big\langle T^*s(z), T\pi_K(e, 0, 0, v) \big\rangle \\ &= \big\langle z, T(s \circ \pi_K)(e, 0, 0, v) \big\rangle = \langle z, v \rangle, \end{aligned}$$

where in the last line we have used $s \circ \pi_K(g, a) = g \cdot \exp_z a$, and the fact that the derivative at 0 of $\exp_z$ is the identity. In summary, we have shown:

**Lemma 15.** *Let $\alpha = z|_A$ and $x = (e, \mu, 0, \alpha)$. Then $T^*s^{-1}(\varphi(x)) = z$.*

**Remark 16.** Recall that $H = G_z$ and note that $G_z = G_{[x]_K}$, where $[x]_K = \pi_K(x)$. By definition of the $G$ and $K$ actions on $T^*(G \times A)$, we have $H = G_\mu \cap K_\alpha$. We note for later use that if $H = K$ then $K_\alpha = K$; while if $G_\mu = G$ then $H = K_\alpha$.

Applying singular commuting reduction (Theorem 7) gives the following picture:

$$\begin{tikzcd}
 & T^*(G \times A) \arrow[dl, "\text{reduction by} G"'] \arrow[dr, "\text{reduction by} K"] & \\
? \cong J_G^{-1}(\mu)/G_\mu \arrow[dr, "\text{reduction by} K"'] & & J_K^{-1}(0)/K \overset{local}{\cong} T^*Q \arrow[dl, "\text{reduction by} G"] \\
 & ? \overset{local}{\cong} J^{-1}(\mu)/G_\mu &
\end{tikzcd}$$

We now compute $J_G^{-1}(\mu)/G_\mu$. Note that the $G$ action leaves $A$ untouched, so

$$J_G^{-1}(\mu)/G_\mu \cong \left(J_L^{-1}(\mu)/G_\mu\right) \times T^*A,$$

where $J_L^{-1}(\mu)/G_\mu$ is the symplectic reduced space at $\mu$ for the lifted left multiplication action of $G$ on $T^*G$. It is well-known that $J_L^{-1}(\mu)/G_\mu$ is symplectomorphic to $\mathcal{O}_\mu$ with the KKS symplectic form $\omega^-_{\mathcal{O}_\mu}$ (defined in Eq. (1)); see for example, Appendix B.4 of [4]. The isomorphism is $[g, \nu]_{G_\mu} \mapsto \nu$ (using the left trivialisation of $T^*G$). We have almost proven the following proposition; the remaining claims in it are easily verified.

**Proposition 17.** *The map $\theta$ defined by*

$$\begin{aligned}
&\theta : J_G^{-1}(\mu) \longrightarrow \mathcal{O}_\mu \times T^*A,\\
&(g, \nu, a, \delta) \longmapsto (\nu, a, \delta)
\end{aligned} \tag{12}$$

*is a surjective submersion that descends to diffeomorphism*

$$\begin{aligned}
&\bar\theta : J_G^{-1}(\mu)/G_\mu \longrightarrow \mathcal{O}_\mu \times T^*A,\\
&[g, \nu, a, \delta]_{G_\mu} \longmapsto (\nu, a, \delta)
\end{aligned}$$

*that is symplectic with respect to the reduced symplectic form on the left and $\omega^-_{\mathcal{O}_\mu} + \omega_{T^*A}$ on the right. The pushed-forward $K$ action is*

$$k \cdot (\nu, a, \delta) = (\mathrm{Ad}^*_{k^{-1}} \nu, k \cdot a, k \cdot \delta).$$

*It has momentum map $J'_K(\nu, a, \alpha) = -\nu|_{\mathfrak{k}} + a \diamond_{\mathfrak{k}} \alpha$.*

Our results so far, combined with singular commution reduction (Theorem 7), give the following normal form for reduced spaces for cotangent-lifted actions.

**Theorem 18.** *Let $G$ act properly on a manifold $Q$ and by cotangent lifts on $T^*Q$ with momentum map $J$. Let $z \in T_q^*Q$ and $K = G_q$. Let $\mu = J(z)$ and suppose that $G_\mu$ is compact and $\mathcal{O}_\mu$ is locally closed. Let $A$ be a $K$-invariant complement to $\mathfrak{g} \cdot q$ with respect to some $K$-invariant metric. Then there is a local Poisson diffeomorphism between $J^{-1}(\mu)/G_\mu$ and the reduced space at $0$ for the product action of $K$ on the space $\mathcal{O}_\mu \times T^*A$ with symplectic form $\omega^-_{\mathcal{O}_\mu} + \omega_{T^*A}$, where $K$ has the coadjoint action on $\mathcal{O}_\mu$ and the cotangent lifted action on $T^*A$.*

Our main aim in this section is to characterise the symplectic normal space $N_s(z)$. To this end, we apply tangent-level commuting reduction (Theorem 10) to the actions of $G$ and $K$ on $T^*(G \times A)$. Recall from Lemma 15 and Remark 16 that $x = (e, \mu, 0, \alpha)$ and $H = G_z = G_{[x]_K} = G_\mu \cap K_\alpha$. It is easy to check that $(G \times K)_x = \{(h, h) \mid h \in H\}$, and that $H = K_{[x]_{G_\mu}}$ as well. A generalisation of this observation appears in Theorem 10. As in that theorem, we will identify $(G \times K)_x$ with $H$. The subgroup $H$ acts on all three of the symplectic normal spaces $N_s([x]_{G_\mu})$, $N_s(x)$ and $N_s([x]_K)$ in the usual way. Theorem 10 (tangent-level commuting reduction) implies that the following maps are $H$-equivariant vector space symplectomorphisms,

$$N_s\big([x]_{G_\mu}\big) \overset{\overline{T_x\pi_{G_\mu}}}{\overset{\longleftarrow}{\cong}} N_s(x) \overset{\overline{T_x\pi_K}}{\overset{\longrightarrow}{\cong}} N_s\big([x]_K\big), \tag{13}$$

where the overbars denote the quotient maps, as in Lemma 9.

Now recall from Eq. (11) that $T^*s^{-1} \circ \bar{\varphi}$ is a $G$-equivariant symplectomorphism from a neighbourhood of $[x]_K$ to a neighbourhood of $z$. It follows that $\overline{T_{[x]_K}(T^*s^{-1} \circ \bar{\varphi})}$ is an $H$-equivariant symplectomorphism from $N_s([x]_K)$ to $N_s(z)$. Since $\varphi = \bar{\varphi} \circ \pi_K$, we can compose this with $\overline{T_x\pi_K}$ from above to give $\overline{T_x(T^*s^{-1} \circ \varphi)} : N_s(x) \to N_s(z)$. Similarly, Proposition 17 implies that the map $\overline{T_{[x]_{G_\mu}}\bar{\theta}} : N_s([x]_{G_\mu}) \to N_s(\mu, 0, \alpha)$ is an $H$-equivariant symplectomorphism; and we can compose this with $\overline{T_x\pi_{G_\mu}}$ to give $\overline{T_x\theta} : N_s(x) \to N_s(z)$. Combining these results, we have the following,

**Theorem 19.** *In the above context (with $s, \varphi$ and $\theta$ defined by Eqs. (10), (9) and (12), respectively), the composition $\overline{T_x\theta} \circ \overline{T_x(T^*s^{-1} \circ \varphi)}^{-1} : N_s(z) \to N_s(\mu, 0, \alpha)$ is an $H$-equivariant symplectomorphism of symplectic normal spaces.*

The space $N_s(\mu, 0, \alpha)$ has simple forms in the special cases $K \subset G_\mu$ and $\alpha = 0$. When $K \subset G_\mu$, the $K$ action on $\mathcal{O}_\mu$ is trivial, so

$$N_s(\mu, 0, \alpha) \cong T_\mu\mathcal{O}_\mu \times N_s(0, \alpha), \tag{14}$$

the second summand being the symplectic normal space at $(0, \alpha)$ for the cotangent-lifted action of $K$ on $T^*A$. Recall that the momentum map for the latter action is $J_A(a, \gamma) = a \diamond \gamma$. It follows that $dJ_A(0, \alpha)(b, \beta) = 0 \diamond \beta + b \diamond \alpha = b \diamond \alpha$, so $\ker dJ_A(0, \alpha) = (\mathfrak{k} \cdot \alpha)^\circ \times A^*$. Hence

$$N_s(0, \alpha) = \ker dJ_A(0, \alpha)/\big(\mathfrak{k} \cdot (0, \alpha)\big) \cong (\mathfrak{k} \cdot \alpha)^\circ \times \big(A^*/(\mathfrak{k} \cdot \alpha)\big).$$

It is not hard to show that the dual $\iota^*$ of the inclusion $\iota : (\mathfrak{k} \cdot \alpha)^\circ \hookrightarrow A$ descends to an isomorphism $\overline{\iota^*} : A^*/(\mathfrak{k} \cdot \alpha) \cong ((\mathfrak{k} \cdot \alpha)^\circ)^*$, and the map

$$N_s(0, \alpha) \cong (\mathfrak{k} \cdot \alpha)^\circ \times \big(A^*/(\mathfrak{k} \cdot \alpha)\big) \xrightarrow{(id, \overline{\iota^*})} (\mathfrak{k} \cdot \alpha)^\circ \times \big((\mathfrak{k} \cdot \alpha)^\circ\big)^* \cong T^*(\mathfrak{k} \cdot \alpha)^\circ \tag{15}$$

is a symplectomorphism and is $H$-equivariant with respect to the cotangent lift of the restriction of the $K$ action on $A$ to a $H$ action on $(\mathfrak{k} \cdot \alpha)^\circ$. Thus we arrive at the following corollary to Theorem 19:

**Corollary 20.** *When $K \subset G_\mu$, there is an $H$-equivariant symplectomorphism*

$$N_s(z) \cong T_\mu\mathcal{O}_\mu \times T^*(\mathfrak{k} \cdot \alpha)^\circ.$$

**Remark 21.** In light of Lemma 14(iv), the above result applies whenever $K$ is normal in $G$.

**Remark 22.** This corollary generalises a splitting established for free actions by Montgomery et al. (see [18]).

We now consider the case $\alpha = 0$. Recall that $\alpha = z|_A$, where $A = (\mathfrak{g} \cdot q)^\perp$; so, with respect to our choice of metric, this is the case where the conjugate momentum $z$ is "purely in the group direction". Since $J'_K(\nu, a, \alpha) = -\nu|_\mathfrak{k} + a \diamond \alpha$, it follows that $dJ'_K(\mu, 0, 0)(\rho, b, \beta) = -\rho|_\mathfrak{k}$. Note that this equals $dJ_\mu(\mu)(\rho)$, where $J_\mu$ is the momentum map for the coadjoint action of $K$ on $(\mathcal{O}_\mu, \omega^-_{\mathcal{O}_\mu})$ namely $J_\mu(\nu) = -\nu|_\mathfrak{k}$. So $\ker dJ'_K(\mu, 0, 0) = \ker dJ_\mu(\mu) \times T^*A$. By Lemma 14 (iii), $J'_K(\mu) = J_\mu(\mu) = 0$, so $\mathfrak{k}_{J'_K(\mu)} = \mathfrak{k}_{J_\mu(\mu)} = \mathfrak{k}$. It follows that

$$N_s\big((\mu, 0, 0)\big) = \big(\ker dJ_\mu(\mu) \times T^*A\big)/\big(\mathfrak{k}_{J_\mu(\mu)\cdot\mu} \times \{(0, 0)\}\big) = N_s(\mu) \times T^*A.$$

By the Reduction lemma (or direct calculation), $N_s(\mu) = (\mathfrak{k} \cdot \mu)^{\omega^-}/(\mathfrak{k} \cdot \mu)$. Thus we have the following corollary to Theorem 19:

**Corollary 23.** *When $\alpha = 0$, the map in Theorem 19 is an $H$-equivariant symplectomorphism*

$$N_s(z) \cong N_s(\mu) \times T^*A = (\mathfrak{k} \cdot \mu)^{\omega^-}/(\mathfrak{k} \cdot \mu) \times T^*A,$$

*where $N_s(\mu)$ is the symplectic normal space at $\mu$ for the coadjoint action of $K$ on $(\mathcal{O}_\mu, \omega^-_{\mathcal{O}_\mu})$.*

**Remark 24.** The above corollary applies to all relative equilibria of simple mechanical systems. Indeed, if $z \in T^*_q Q$ is such a relative equilibrium then $z = \mathbb{F}L(\xi \cdot q)$ for some $\xi \in \mathfrak{g}$ (see [12]). For any $v \in A$ we have $\langle z, v\rangle = \langle\langle \xi \cdot q, v\rangle\rangle = 0$, since $A = (\mathfrak{g} \cdot q)^\perp$. Hence $\alpha = z|_A = 0$. More generally, the corollary applies to any point $z$ such that the kernel of $z$ includes some complement to $\mathfrak{g} \cdot q$, because we can choose our metric on $Q$ such that this complement is $(\mathfrak{g} \cdot q)^\perp$.

**Remark 25.** Theorem 19 and its corollaries, when combined with Theorem 8, give local models of the symplectic reduced spaces of cotangent bundles. Indeed, if $\mathcal{O}_\mu$ is locally closed, there is a local Poisson diffeomorphism between the reduced space $J^{-1}(\mu)/G_\mu$ and the reduced space at 0 for the $H$ action on $N_s(\mu, 0, \alpha)$. Note the similarity to Theorem 18, which shows that $J^{-1}(\mu)/G_\mu$ is isomorphic to the reduced space at 0 for the $K$ action on $\mathcal{O}_\mu \times T^*A$. Thus, symplectic reduced spaces for cotangent bundles have two local models, corresponding to the two isotropy subgroups $H$ and $K$. The model involving $H$ and $N_s(\mu, 0, \alpha)$ is more "economical" in that $H$ and $N_s(\mu, 0, \alpha)$ may be smaller than $K$ and $\mathcal{O}_\mu \times T^*A$, but on the other hand the latter space is "simpler" and might be easier to work with in some situations.

We end this section with the observation that Corollaries 20 and 23 lead to refinements of the *reconstruction equations* or *bundle equations* [19,22,28], which are a normal form for Hamilton's equations in the coordinates given by the Hamiltonian Slice Theorem (Theorem 5). Consider the local symplectomorphism $G \times_H (\mathfrak{m}^* \times N_s) \to P$ given by the Hamiltonian Slice Theorem, for any proper globally Hamiltonian action of $G$ on $P$, with $H = G_z$ as before; recall that $\mathfrak{m}$ is an $H$-invariant complement to $\mathfrak{h}$ in $\mathfrak{g}_\mu$, where $\mu = J(z)$. A Hamiltonian on $P$ pulls back to a Hamiltonian $h$ on a neighbourhood of $[e, 0, 0]_H$ in $G \times_H (\mathfrak{m}^* \times N_s)$, with corresponding Hamiltonian vector field $X_h$. Using a local bundle chart around $[e, 0, 0]_H$ for the principal bundle $\pi : G \times \mathfrak{m}^* \times N_s \to G \times_H (\mathfrak{m}^* \times N_s)$, we can lift $X_h$ to a smooth vector field on a neighbourhood of $(e, 0, 0)$ in $G \times \mathfrak{m}^* \times N_s$. This lift is not unique; however we can specify a unique lift by choosing an $H$-invariant complement $\mathfrak{q}$ to $\mathfrak{g}_\mu$, so that we now have $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{m} \oplus \mathfrak{h}$, and requiring that the component of the lifted vector field in the $\mathfrak{h}$ direction be zero. The lifted vector field can now be written as $X = (TL_g(X_\mathfrak{m} + X_\mathfrak{q}), X_{\mathfrak{m}^*}, X_{N_s})$.

If it is possible to choose $\mathfrak{q}$ to be $G_\mu$-invariant then we say that $\mu$ is *split*. We assume this now for simplicity; the general case is considered in [28]. Assuming $\mu$ is split, it can be shown that, at every point $(g, \rho, v) \in G \times \mathfrak{m}^* \times N_s$,

$$\begin{aligned}
&X_\mathfrak{q} = 0,\\
&X_\mathfrak{m} = D_{\mathfrak{m}^*}(h \circ \pi),\\
&X_{\mathfrak{m}^*} = \mathbb{P}_{\mathfrak{m}^*}(\mathrm{ad}^*_{D_{\mathfrak{m}^*}(h\circ\pi)}\, \rho) + \mathrm{ad}^*_{D_{\mathfrak{m}^*}(h\circ\pi)}\, J_{N_s}(v),\\
&\mathrm{i}_{X_{N_s}} \omega_{N_s} = D_{N_s}(h \circ \pi).
\end{aligned}$$

Now suppose $P = T^*Q$ and $z \in T^*_q Q$. If $G_q \subset G_\mu$, we know from Corollary 20 that $N_s$ is linearly symplectomorphic to $T_\mu \mathcal{O}_\mu \times T^*(\mathfrak{k} \cdot \alpha)^\circ$. Let $B = (\mathfrak{k} \cdot \alpha)^\circ$, so $N_s \cong T_\mu \mathcal{O}_\mu \times B \times B^*$. The vector field $X_{N_s}$ splits into three components $X_\mu$, $X_B$ and $X_{B^*}$ and the last displayed equation above splits into three corresponding equations,

$$\mathrm{i}_{X_\mu} \omega_{\mathcal{O}^-_\mu} = D_\mu(h \circ \pi); \qquad X_B = D_{B^*}(h \circ \pi); \qquad X_{B^*} = -D_B(h \circ \pi). \tag{16}$$

The case $\alpha = 0$ in Corollary 23 is similar: the splitting $N_s(z) \cong N_s(\mu) \times T^*A$ induces a three way split of the $X_{N_s}$ equation,

$$\mathrm{i}_{X_\mu} \omega_{\mathcal{O}^-_\mu, red} = D_\mu(h \circ \pi); \qquad X_A = D_{A^*}(h \circ \pi); \qquad X_{A^*} = -D_A(h \circ \pi). \tag{17}$$

## 5. A cotangent bundle slice theorem

In this section, we extend the Hamiltonian slice theorem (Theorem 5) in the case of a lifted action on a cotangent bundle. The main result is Theorem 31. We will consider only the case of fully isotropic momenta, $G_\mu = G$, for reasons that will be summarised in Remark 33. Our model for $T^*Q$ will be $G \times_H (\mathfrak{m}^* \times N_s)$, as in the general Hamiltonian slice theorem (Theorem 5) with the same symplectic form as in that theorem (definitions will be reiterated below). However, in contrast to the general Hamiltonian slice theorem, our isomorphism from the model space to $T^*Q$ will be constructed explicitly, apart from the use of a Riemannian exponential in the base space $Q$. The construction will use the decomposition of $N_s$ in Corollary 20.

As before, let $G$ be a Lie group acting smoothly and properly by cotangent lifts on $T^*Q$, with momentum map $J$. Let $z \in T_q^*Q$ and $\mu = J(z)$, and let $K = G_q$ and $H = G_z$. We assume $G_\mu = G$. Let $\mathfrak{g}, \mathfrak{g}_\mu, \mathfrak{h}$ and $\mathfrak{k}$ be the Lie algebras of $G, G_\mu, H$ and $K$ respectively. Fix a $K$-invariant inner product on $\mathfrak{g}$ and let $\mathfrak{m} = \mathfrak{k}^\perp$. Let $N_s$ be the symplectic normal space at $z$. Our goal is to find a symplectic tube from $G \times_H (\mathfrak{m}^* \times N_s)$ to $T^*Q$, possibly defined only on a neighbourhood of $[e, 0, 0]$, that maps $[e, 0, 0]$ to $z$.

We first apply Palais' slice theorem (Theorem 4) in the configuration space $Q$. Fix a $K$-invariant Riemannian metric on $Q$ and let $A = (\mathfrak{g} \cdot q)^\perp$. By the slice theorem, there exists a $G$-equivariant diffeomorphism $s : G \times_K V \to Q$ taking $[e, 0]_K$ to $q$, for some neighbourhood $V$ of 0 in $A$. The cotangent lift of $s$ is a $G$-equivariant symplectomorphism $T^*s^{-1} : T^*(G \times_K V) \to T^*Q$. Let $\varphi : J_K^{-1}(0) \to T^*(G \times_K A)$ be the cotangent bundle reduction map defined in Eq. (9), and let $\alpha = z|_A$. Recall from Lemma 15 that $T^*s^{-1} \circ \varphi(e, \mu, 0, \alpha) = z$. Hence it will suffice to find a symplectic tube

$$\begin{aligned} \tau : G \times_H (\mathfrak{m}^* \times N_s) &\longrightarrow T^*(G \times_K A), \\ [e, 0, 0]_H &\longmapsto \varphi(e, \mu, 0, \alpha). \end{aligned} \tag{18}$$

Since $G_\mu = G$, Corollary 20 says that the symplectic normal space $N_s$ is $H$-equivariantly symplectomorphic to $T^*B \cong B \times B^*$, where $B = (\mathfrak{k} \cdot \alpha)^\circ \subset A$; the symplectic form on $T^*B$ is the canonical one, and the $H$ action on $T^*B$ is the cotangent-lift of the restriction to $H$ and $B$ of the $K$ action on $A$. We will now show that the model space in the Hamiltonian slice theorem, $G \times_H (\mathfrak{m}^* \times T^*B)$, is a $H$-reduced space of $T^*(G \times B)$, and that our problem reduces to one of finding a certain symplectic local diffeomorphism from $T^*(G \times_H B)$ to $T^*(G \times_K A)$.

Recall that the presymplectic form on $G \times_H (\mathfrak{m}^* \times N_s)$ in the Hamiltonian slice theorem is defined using a symplectic form $\Omega_Z = \Omega_c + \Omega_\mu + \Omega_{N_s}$ on $Z = G \times \mathfrak{g}_\mu^* \times N_s$ (see Eq. (7)). In our case we have $Z = G \times \mathfrak{g}^* \times T^*B$, which we identify with $T^*(G \times B)$ by left trivialisation of $T^*G$. The twist action of $H$ on $Z$ becomes the cotangent lift of the twist action of $H$ on $G \times B$. The form $\Omega_\mu$ is a pull-back of a symplectic form on $\mathcal{O}_\mu$, which is trivial in this case, so $\Omega_Z = \Omega_c + \Omega_{N_s}$. Since $\Omega_c$ is the pull-back by left-trivialisation of the canonical symplectic form on $T^*G$, and $\Omega_{N_s}$ is the canonical symplectic form on $T^*B$, the identification of $Z$ with $T^*(G \times B)$ makes $\Omega_Z$ the canonical symplectic form on $T^*(G \times B)$. Note that, unlike in the general case, this $\Omega_Z$ is nondegenerate everywhere.

The symplectic form on $G \times_H (\mathfrak{m}^* \times N_s)$ is defined via an isomorphism with $J_H^{-1}(0)/H$, where $J_H$ is the momentum map of the $H$ action on $Z = T^*(G \times B)$. The isomorphism, defined earlier in Eqs. (6) and (7), is

$$\begin{aligned} L : G \times_H (\mathfrak{m}^* \times B \times B^*) &\longrightarrow J_H^{-1}(0)/H, \\ [g, \nu, a, \delta]_H &\longmapsto [g, \nu + a \diamond_{\mathfrak{h}} \delta, a, \delta]_H. \end{aligned}$$

The symplectic form on $G \times_H (\mathfrak{m}^* \times B \times B^*)$ is defined as the pull-back by $L$ of the reduced symplectic form on $J_H^{-1}(0)/H$. Since $L$ is clearly $G$-equivariant, it is a symplectic tube.

In the present case, cotangent bundle reduction (Theorem 3) shows that $J_H^{-1}(0)/H$ is isomorphic to $T^*(G \times_H B)$. Let $\psi$ and $\bar{\psi}$ be the maps in the cotangent bundle reduction theorem,

$$\begin{tikzcd} J_H^{-1}(0) \arrow[r, hook] \arrow[d, "\pi_{Z,H}"'] \arrow[dr, "\psi"] & T^*(G \times B) \\ J_H^{-1}(0)/H \arrow[r, "\bar{\psi}"] & T^*(G \times_H B). \end{tikzcd}$$

It is easily checked that $\bar\psi$ is $G$-equivariant, by the same reasoning as used in Proposition 13. Note that

$$(\bar\psi \circ L)\big([g,\nu,a,\delta]_H\big) = \psi(g,\nu + a \diamond_{\mathfrak h} \delta, a, \delta). \tag{19}$$

In particular, $(\bar\psi \circ L)([e,0,0,0]_H) = \psi(e,0,0,0)$. Thus, to find a tube $\tau$ as in Eq. (18), it suffices to find a $G$-equivariant symplectomorphism

$$\begin{aligned} \bar\sigma : T^*(G\times_H B) &\longrightarrow T^*(G\times_K A),\\ \psi(e,0,0,0) &\longmapsto \varphi(e,\mu,0,\alpha) \end{aligned} \tag{20}$$

(we will have to restrict the domain of $\bar\sigma$ in the general case).

**The $z = 0$ case.** In the simplest case, $z = 0 \in T_q^*Q$, we have $\mu = 0, \alpha = 0,$ $H = K, B = A$ and $\psi = \varphi$, so $\bar\sigma$ may be chosen to be the identity map on $T^*(G\times_K A)$. Composing this with the maps $\bar\psi \circ L$ and $T^*s^{-1}$ gives the symplectic tube

$$\begin{aligned} G\times_K(\mathfrak m^*\times V\times A^*) &\longrightarrow T^*Q,\\ [g,\nu,a,\delta]_K &\longmapsto T^*s^{-1}\circ\varphi(g,\nu + a\diamond_{\mathfrak h}\delta, a,\delta), \end{aligned}$$

where $V$ is the neighbourhood of 0 in $A$ given by Palais' slice theorem applied at $q \in Q$.

**The case $H = K$.** Subcases include: $z = 0$; $\alpha = 0$; and all relative equilibria of simple mechanical systems (see Remark 24) (recall that we are assuming $G_\mu = G$ throughout this section). Note that, since $H = K_\alpha$ and $B = (\mathfrak k\cdot\alpha)^\circ$, the condition $H = K$ is equivalent to $B = A$. In this case, $\psi = \varphi$, and we may take $\bar\sigma$ to be a simple shift map, as in the following lemma.

**Lemma 26.** *If $G_\mu = G$ then the shift map $\Sigma_{(\mu,\alpha)} : (g,\nu,a,\delta) \mapsto (g,\mu+\nu,a,\alpha+\delta)$, from $T^*(G\times A)$ to itself, is symplectic and $G$-equivariant. If $H = K$ then $B = A$ and $\Sigma_{(\mu,\alpha)}$ leaves $J_H$ invariant and is $H$-equivariant. The "quotient" of $\Sigma_{(\mu,\alpha)}$ by $\psi$, the map*

$$\begin{aligned} \overline{\Sigma}_{(\mu,\alpha)} : T^*(G\times_H B) &\longrightarrow T^*(G\times_H B),\\ \psi(g,\nu,a,\delta) &\longmapsto \psi(g,\mu+\nu,a,\alpha+\delta) \end{aligned}$$

*is a $G$-equivariant symplectomorphism.*

**Proof.** It is clear from the local coordinate formula $dq^i \wedge dp_i$ that a canonical cotangent bundle symplectic form is invariant under shifts in the $p$ variable. The $G$-equivariance is also clear. Now suppose $H = K$, which implies $B = A$, as explained above. Recall that $J_H(g,\nu,a,\delta) = -\nu|_{\mathfrak h} + a\diamond_{\mathfrak h}\delta$. Since $\mathfrak k \subset \ker\mu$ (see Lemma 14(iii)), it follows that $-\mu|_{\mathfrak h} = 0$. Since $G_\mu = G$, it follows that $H = K_\alpha$ (see Remark 16), and hence that $a\diamond_{\mathfrak h}\alpha = 0$. Hence $J_H$ is invariant under $\Sigma_{(\mu,\alpha)}$. The $H$-equivariance follows from the linearity of the $H$ actions on $\mathfrak g^*$ and $B^*$ together with $H \subset G_\mu \cap K_\alpha$.

Since $\Sigma_{(\mu,\alpha)}$ is a $G$- and $H$-equivariant symplectomorphism leaving $J_H^{-1}(0)$ invariant, it descends to a $G$-equivariant symplectomorphism from $J_H^{-1}(0)/H$ to itself. This map induces $\overline{\Sigma}_{(\mu,\alpha)}$ via the $G$-equivariant symplectomorphism $\bar\varphi : J_H^{-1}(0)/H \to T^*(G\times_H B)$. $\square$

Composing $\overline{\Sigma}_{(\mu,\alpha)}$ from the above lemma with the maps $\bar\psi\circ L$ and $T^*s^{-1}$ gives the symplectic tube

$$\begin{aligned} G\times_K(\mathfrak m^*\times V\times A^*) &\longrightarrow T^*Q,\\ [g,\nu,a,\delta]_K &\longmapsto T^*s^{-1}\circ\varphi(g,\mu+\nu+b\diamond_{\mathfrak h}\delta, b,\alpha+\delta), \end{aligned}$$

where $V$ is the neighbourhood of 0 in $A$ given by Palais' slice theorem applied at $q\in Q$.

The general case is more difficult. We identify $B^*$ with $(\mathfrak k\cdot\alpha)^\perp \subset A^*$. It is easily checked that the shift formula $(g,\nu,a,\delta)\mapsto(g,\mu+\nu,a,\alpha+\delta)$, as a map from $T^*(G\times B)$ to $T^*(G\times A)$, need *not* map $J_H^{-1}(0)$ into $J_K^{-1}(0)$, so cannot be used directly to define a map $\bar\sigma$ as in Eq. (20). We will look for a map as close as possible to this shift map but with image contained in $J_K^{-1}(0)$. We will conclude in Lemma 28 that there is a unique map of the form $(g,\nu,a,\delta)\mapsto(g,\mu+\nu,a+c,\alpha+\delta)$, for $c\in B^\perp$, that accomplishes this.

We proceed by characterising the space $(G\times\mathfrak{g}^*\times A\times(\alpha+B^*))\cap J_K^{-1}(0)$. We decompose $A$ as $((\mathfrak{k}\cdot\alpha)^\perp)^\circ\oplus(\mathfrak{k}\cdot\alpha)^\circ=B^\perp\oplus B$; this splitting is $H$-invariant, since $H$ fixes $\alpha$ (see Remark 16). By definition of $\mathfrak{m}$, we have an $H$-equivariant splitting $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{h}$ (recall that we are assuming $G_\mu=G$). It is easily checked that $\mathfrak{k}$ splits $H$-equivariantly as $\mathfrak{k}=(\mathfrak{m}\cap\mathfrak{k})\oplus\mathfrak{h}$.

**Lemma 27.** $(G\times\mathfrak{g}^*\times A\times(\alpha+B^*))\cap J_K^{-1}(0)$ *is a submanifold of* $T^*(G\times A)$*, and it equals the set of all* $(g,\nu,b+c,\alpha+\delta)$ *such that* $b\in B$, $c\in B^\perp$, $J_H(g,\nu,b,\delta)=0$ *and* $(-\nu+b\diamond\delta+c\diamond(\alpha+\delta))|_{\mathfrak{m}\cap\mathfrak{k}}=0$.

**Proof.** Recall that $J_K(g,\nu,a,\beta)=-\nu|_{\mathfrak{k}}+a\diamond_{\mathfrak{k}}\beta$. The restriction of $J_K$ to $G\times\mathfrak{g}^*\times A\times(\alpha+B^*)$ is a submersion, since $\nu\mapsto\nu|_{\mathfrak{k}}$ is one. It follows that $(G\times\mathfrak{g}^*\times A\times(\alpha+B^*))\cap J_K^{-1}(0)$ is a submanifold of $G\times\mathfrak{g}^*\times A\times(\alpha+B^*)$, and hence of $G\times\mathfrak{g}^*\times A\times A^*$.

Now let $(g,\nu,b+c,\alpha+\delta)\in(G\times\mathfrak{g}^*\times A\times(\alpha+B^*))$, with $b\in B$ and $c\in B^\perp$. Then $J_K(g,\nu,b+c,\alpha+\delta)=-\nu|_{\mathfrak{k}}+(b+c)\diamond_{\mathfrak{k}}(\alpha+\delta)$. Since $b\in B=(\mathfrak{k}\cdot\alpha)^\circ$, it follows that $b\diamond_{\mathfrak{k}}\alpha=0$. For any $\xi\in\mathfrak{h}$ we have $\xi\cdot\alpha=0$ and $\xi\cdot\delta\in(\mathfrak{k}\cdot\alpha)^\perp$, and so $\langle c,\xi\cdot(\alpha+\delta)\rangle=0$; it follows that $c\diamond_{\mathfrak{h}}(\alpha+\delta)=0$. Thus

$$\begin{aligned}(g,\nu,b+c,\alpha+\delta)\in\ & J_K^{-1}(0)\\ &\Leftrightarrow \nu|_{\mathfrak{k}}=b\diamond_{\mathfrak{k}}\delta+c\diamond_{\mathfrak{m}\cap\mathfrak{k}}(\alpha+\delta)\\ &\Leftrightarrow \nu|_{\mathfrak{h}}-b\diamond_{\mathfrak{h}}\delta=-\nu|_{\mathfrak{m}\cap\mathfrak{k}}+b\diamond_{\mathfrak{m}\cap\mathfrak{k}}\delta+c\diamond_{\mathfrak{m}\cap\mathfrak{k}}(\alpha+\delta)\\ &\Leftrightarrow \nu|_{\mathfrak{h}}-b\diamond_{\mathfrak{h}}\delta=0\text{ and }-\nu|_{\mathfrak{m}\cap\mathfrak{k}}+b\diamond_{\mathfrak{m}\cap\mathfrak{k}}\delta+c\diamond_{\mathfrak{m}\cap\mathfrak{k}}(\alpha+\delta)=0\\ &\Leftrightarrow J_H(g,\nu,b,\delta)=0\text{ and }\big(-\nu+b\diamond\delta+c\diamond(\alpha+\delta)\big)\big|_{\mathfrak{m}\cap\mathfrak{k}}=0.\qquad\square\end{aligned}$$

**Lemma 28.** *Let* $U$ *be an* $H$*-invariant neighbourhood of* $0$ *in* $B^*$ *such that the map*

$$t:K\times_H U\longrightarrow A^*,\qquad [k,\delta]_H\longmapsto k\cdot(\alpha+\delta)$$

*is injective*; *such a* $U$ *always exists. Then*

(1) *For every* $\delta\in U$*, the map* $\Gamma_\delta^*:(\mathfrak{m}\cap\mathfrak{k})^*\longmapsto B^\perp$ *defined by*

$$\big\langle\Gamma_\delta^*(\nu),\xi\cdot(\alpha+\delta)+\varepsilon\big\rangle=\langle\nu,\xi\rangle,\tag{21}$$

*for every* $\xi\in\mathfrak{m}\cap\mathfrak{k}$ *and* $\varepsilon\in B^*$*, is* $H$*-equivariant and has an inverse given by* $c\longmapsto -c\diamond_{\mathfrak{m}\cap\mathfrak{k}}(\alpha+\delta)$.

(2) *The map* $\sigma$ *defined by*

$$\begin{aligned}&\sigma:(G\times\mathfrak{g}^*\times B\times U)\cap J_H^{-1}(0)\to\big(G\times\mathfrak{g}^*\times A\times(\alpha+U)\big)\cap J_K^{-1}(0),\\ &(g,\nu,a,\delta)\mapsto\big(g,\mu+\nu,a+\Gamma_\delta^*(-\nu|_{\mathfrak{m}\cap\mathfrak{k}}+a\diamond_{\mathfrak{m}\cap\mathfrak{k}}\delta),\alpha+\delta\big)\end{aligned}$$

*is the unique function of the form* $(g,\nu,a,\delta)\mapsto(g,\mu+\nu,a+c,\alpha+\delta)$*, for some* $c\in B^\perp$*, with this domain and range. It is a presymplectic diffeomorphism, with respect to the canonical symplectic forms on* $T^*(G\times B)$ *and* $T^*(G\times A)$*. It is equivariant with respect to both the left multiplication action by G and the twist action by* $H$*. It descends to a* $G$*-equivariant symplectic embedding* $\bar\sigma$ *defined by the following commutative diagram, where* $W=(G\times\mathfrak{g}^*\times B\times U)\cap J_H^{-1}(0)$*,*

$$\begin{tikzcd}(W\subset J_H^{-1}(0)) \arrow[r, hook, "\sigma"] \arrow[d, "\psi"] & J_K^{-1}(0) \arrow[d, "\varphi"]\\ (\psi(W)\subset T^*(G\times_H B)) \arrow[r, hook, "\bar\sigma"] & T^*(G\times_K A).\end{tikzcd}$$

*The image of* $\bar\sigma$ *is an open subset of* $T^*(G\times_K A)$.

**Proof.** (1) Note that $H = K_\alpha$ and we are identifying $B^*$ with $(\mathfrak{k} \cdot \alpha)^\perp$, which is a linear slice for the $K$ action on $A^*$. The Palais' slice theorem (Theorem 4) shows the existence of an $H$-invariant neighbourhood $U$ of $0$ in $B^*$ such that the map $t$ above is injective and that, given any such $U$ the map $t$ is a $K$-equivariant diffeomorphism onto a $K$-invariant neighbourhood of $\alpha$. Let $\pi_H : K \times U \to K \times_H U$ be projection. The composition $t \circ \pi_H$ is a submersion. For any $k \in K$ and $\delta \in U$, the kernel of $T_{(k,\delta)}\pi_H$ is $\{(-\zeta, \zeta \cdot \delta) \in \mathfrak{k} \times B^*\colon\ \zeta \in \mathfrak{h}\}$, which is a complement to the space $(\mathfrak{m} \cap \mathfrak{k}) \times B^*$ in $\mathfrak{k} \times B^*$, so the following map is an isomorphism,

$$
\begin{aligned}
&T_{(e,\delta)}(t \circ \pi_H)|_{(\mathfrak{m}\cap\mathfrak{k})\times B^*} : (\mathfrak{m} \cap \mathfrak{k}) \times B^* \longrightarrow T_{(\alpha+\delta)}A^* \cong A^*,\\
&(\xi, \varepsilon) \longmapsto \xi \cdot (\alpha + \delta) + \varepsilon.
\end{aligned}
$$

Hence Eq. (21) defines a map $\Gamma_\delta^*$ from $(\mathfrak{m}\cap\mathfrak{k})^*$ to $A$. Its image is clearly contained in $(B^*)^\circ \cong B^\perp$. It is easily checked that $\Gamma_\delta^*$ is $H$-equivariant and has the stated inverse.

(2) We first check that $\sigma$ is well-defined; the only part that needs checking is that its image is contained in the target space. It follows from Claim 1 that the condition $(-\nu + b \diamond \delta + c \diamond (\alpha + \delta))|_{\mathfrak{m}\cap\mathfrak{k}} = 0$ in Lemma 27 is equivalent to $c = \Gamma_\delta^*((-\nu + b \diamond \delta)|_{\mathfrak{m}\cap\mathfrak{k}})$. The other condition in Lemma 27 that needs checking is $J_H(g, \mu + \nu, a, \delta) = 0$, for every $(g, \nu, a, \delta) \in (G \times \mathfrak{g}^* \times B \times U) \cap J_H^{-1}(0)$; this follows easily from the fact that $\mu|_\mathfrak{h} = 0$. It is easily checked that $\sigma$ has an inverse given by $(g, \mu + \nu, a + c, \alpha + \delta) \mapsto (g, \nu, a, \delta)$, where $a \in B$ and $c \in B^\perp$.

Part of that same argument, namely the fact that $\Gamma_\delta^*(\nu|_{\mathfrak{m}\cap\mathfrak{k}} - a \diamond_{\mathfrak{m}\cap\mathfrak{k}} \delta) = -c$ for any $(g, \mu + \nu, a + c, \alpha + \delta)$ in the range of $\sigma$, also proves that $\sigma$ is the unique function, with the given domain and range, of the form $(g, \nu, a, \delta) \longmapsto (g, \mu + \nu, a + c, \alpha + \delta)$ for $c \in B^\perp$.

We now show that $\sigma$ is a diffeomorphism. Note that its domain is a submanifold of $T^*(G \times B)$, being an open subset of a level set of the momentum map of a free action; similarly the range of $\sigma$ is a submanifold of $T^*(G \times A)$. Since the image of $\Gamma_\delta^*$ is $B^\perp$, its derivative is always in $B^\perp$, so for any $(g, \nu, a, \delta)$ in the domain of $\sigma$, and any tangent vector $(\dot g, \dot\nu, \dot a, \dot\delta) \in \mathfrak{g} \times \mathfrak{g}^* \times B \times B^*$, we have $T_{(g,\nu,a,\delta)}\sigma(\dot g, \dot\nu, \dot a, \dot\delta) = (\dot g, \dot\nu, \dot a + \dot c, \dot\delta)$ for some $\dot c \in B^\perp$. It is clear from this formula that $\sigma$ is an immersion. But any bijective immersion is a diffeomorphism (see Remark 30 below).

We next show that $\sigma$ is presymplectic. The canonical symplectic forms on the domain and codomain have the same formula,

$$
\Omega(g, \nu, a, \delta)\big((\dot g_1, \dot\nu_1, \dot a_1, \dot\delta_1), (\dot g_2, \dot\nu_2, \dot a_2, \dot\delta_2)\big) = \langle \dot\nu_2, \dot g_1\rangle - \langle \dot\nu_1, \dot g_2\rangle + \big\langle \nu, [\dot g_1, \dot g_2]\big\rangle + \langle \dot\delta_2, \dot a_1\rangle - \langle \dot\delta_1, \dot a_2\rangle.
$$

In calculating $\Omega(\sigma(g, \nu, a, \delta))((\dot g_1, \dot\nu_1, \dot a_1 + \dot c_1, \dot\delta_1), (\dot g_2, \dot\nu_2, \dot a_2 + \dot c_2, \dot\delta_2))$, with $\dot c_1, \dot c_2 \in B^\perp$ and $\dot\delta_1, \dot\delta_2 \in B^*$, the only part containing the $\dot c_i$'s is $\langle \dot\delta_2, \dot c_1\rangle - \langle \dot\delta_1, \dot c_2\rangle$, which equals zero. Since $G_\mu = G$, we always have $\langle \mu + \nu, [\dot g_1, \dot g_2]\rangle = \langle \nu, [\dot g_1, \dot g_2]\rangle$. It follows that $\sigma$ is presymplectic.

Since $\Gamma_\delta^*$ is $H$-equivariant and $\mu$ and $\alpha$ are $H$-invariant, it follows that $\sigma$ is $H$-equivariant, and hence that $\bar\sigma$ is well-defined by the diagram in the statement of the lemma. It is clear that $\sigma$ and $\bar\sigma$ are $G$-equivariant. Now, $K \cdot (\mathrm{Im}\,\sigma) = (G \times \mathfrak{g}^* \times A \times (\alpha + U)) \cap J_K^{-1}(0)$, which is an open subset of $J_K^{-1}(0)$. Since $\mathrm{Im}\,\bar\sigma = \varphi(\mathrm{Im}\,\sigma) = \varphi(K \cdot \mathrm{Im}\,\sigma)$, this implies that $\mathrm{Im}\,\bar\sigma$ is open. Hence $\bar\sigma$ is a surjective submersion onto an open subset of $T^*(G \times_K A)$. For injectivity, suppose $\bar\sigma(\psi(w_1)) = \bar\sigma(\psi(w_2))$, which is equivalent to $\varphi(\sigma(w_1)) = \varphi(\sigma(w_2))$. By definition of $\varphi$, this implies that $\sigma(w_1) = k \cdot \sigma(w_2)$ for some $k \in K$. If the $A^*$ coordinates of $w_1$ and $w_2$ are $\delta_1$ and $\delta_2$, this implies that $\alpha + \delta_1 = k \cdot (\alpha + \delta_2)$. Recall that $U$ was chosen so that the map $t : K \times_H U \to A^*$, $[k, \delta]_H \mapsto k \cdot (\alpha + \delta)$, is injective. Thus $[e, \delta_1]_H = [k, \delta_2]_H$, which implies $k \in H$. The $H$-equivariance of $\sigma$ implies that $\sigma(w_1) = \sigma(k \cdot w_2)$, which implies $w_1 = k \cdot w_2$, since we have shown that $\sigma$ is injective. Thus $\psi(w_1) = \psi(k \cdot w_2) = \psi(w_2)$, which proves injectivity of $\bar\sigma$. Therefore $\bar\sigma$ is a bijective submersion, and hence an embedding, onto an open subset of $T^*(G \times_K A)$. It is symplectic since $\sigma$ is presymplectic; in fact, this is an application of Lemma 9 at each base point.

**Remark 29.** The reason for the notation $\Gamma_\delta^*$ is the following: if $H$ is normal in $K$ then there is a free action of $K/H$ on $K \cdot (\alpha + U) \subset A^*$. The Riemannian metric defines a connection 1-form $T(K \cdot (\alpha + U)) \longrightarrow (\mathfrak{k}/\mathfrak{h}) \cong (\mathfrak{m} \cap \mathfrak{k})$ on the principal bundle $K \cdot (\alpha + U) \to K \cdot (\alpha + U)/(K/H)$, defined by orthogonal projection onto the vertical fibre followed by the inverse of the infinitesimal generator map. We re-package this connection 1-form as a map $K \cdot (\alpha + U) \longrightarrow L(A^*, \mathfrak{m} \cap \mathfrak{k})$ and compose with the shift map $(k, \delta) \mapsto k \cdot (\alpha + \delta)$, giving the map

$$
\Gamma : K \times U \longrightarrow L(A^*, \mathfrak{m} \cap \mathfrak{k}), \qquad \Gamma(k, \delta)\big(k \cdot (\xi \cdot (\alpha + \delta) + \varepsilon)\big) = \xi
$$

for every $\xi \in \mathfrak{m} \cap \mathfrak{k}$ and $\varepsilon \in (\mathfrak{k} \cdot \alpha)^\perp$. Define $\Gamma^* : K \times U \longrightarrow L((\mathfrak{m} \cap \mathfrak{k})^*, A)$ by $\Gamma^*(k, \delta) = (\Gamma(k, \delta))^*$. Then for every $\delta$, the map $\Gamma^*(e, \delta)$ equals $\Gamma_\delta^*$ as defined in the above lemma. The proof that $\Gamma^*(e, \delta) \in L((\mathfrak{m} \cap \mathfrak{k})^*, B^\perp)$ and not just $L((\mathfrak{m} \cap \mathfrak{k})^*, A)$ is identical to the proof, in the above lemma, that $\Gamma_\delta^*$ is well-defined.

**Remark 30.** The fact that every bijective immersion is a diffeomorphism (used in the proof of the above lemma) is well known (see [2]); however the following short proof for finite-dimensional manifolds seems not to be. Let $f : M \to N$ be a bijective immersion, and let $m$ and $n$ be the dimensions of $M$ and $N$ respectively. Since $f$ is an immersion, we have $m \leqslant n$. If $m$ were strictly less than $n$ then every point in $M$ would be a critical point, which would imply (since $f$ is surjective) that every point in $N$ was a critical value, contradicting Sard's theorem. Hence $m = n$, so $f$ is a local diffeomorphism at every point. Since $f$ is bijection, it is a diffeomorphism.

The composition $\tau = \bar{\sigma} \circ \bar{\psi} \circ L$ of the map $\bar{\sigma}$ from Lemma 28 with $\bar{\psi} \circ L$ from Eq. (19) is the $G$-equivariant embedding

$$\begin{aligned} &\tau : G \times_H (\mathfrak{m}^* \times B \times U) \longrightarrow T^*(G \times_K A), \\ &[g, \nu, a, \delta]_H \longmapsto \varphi(g, \mu + \nu + a \diamond_{\mathfrak{h}} \delta, a - \Gamma_\delta^*(\nu|_{\mathfrak{m}\cap\mathfrak{k}} - a \diamond_{\mathfrak{m}\cap\mathfrak{k}} \delta), \alpha + \delta), \end{aligned} \tag{22}$$

where $U$ and $\Gamma_\delta^*$ are as in Lemma 28. Since $\tau$ maps $[e, 0, 0, 0]_H$ to $\varphi(e, \mu, 0, \alpha)$, and its image is an open subset of $T^*(G \times_K A)$, it is a symplectic tube.

Recall that there is a $G$-equivariant symplectomorphism $T^*s^{-1} : T^*(G \times_K V) \to T^*Q$, for some neighbourhood $V$ of 0 in $A$. The composition of $\tau$ with $T^*s^{-1}$ will give our final result. Unfortunately, the preimage $\tau^{-1}(T^*(G \times_K V))$ doesn't have a simple description in general, so we can only say that $T^*s^{-1} \circ \tau$ is defined on some neighbourhood of $[e, 0, 0, 0]_H$. However, in the special case $H = K$, the $\Gamma_\delta^*$ term disappears, so the domain of $T^*s^{-1} \circ \tau$ is $G \times_H (\mathfrak{m}^* \times (B \cap V) \times U)$. A second special case occurs if the domain of $s$ is the entire space $(G \times_K A)$, which occurs, for example, if $K = G$ and $G$ acts linearly on $Q$. In this case the domain of $T^*s^{-1} \circ \tau$ is simply $G \times_H (\mathfrak{m}^* \times B \times U)$. We have proven the following:

**Theorem 31** *(Cotangent Bundle Slice Theorem). Let $G$ be a Lie group acting properly on a manifold $Q$ and by cotangent lifts on $T^*Q$, which we give the canonical cotangent symplectic form. Let $J$ be the momentum map for the $G$ action, and let $z \in T_q^*Q$ and $\mu = J(z)$. Assume that $G_\mu = G$. Let $H = G_q$ and $K = G_z$, and let $\mathfrak{h}$ and $\mathfrak{k}$ be their Lie algebras. Choose an $H$-invariant metric on $\mathfrak{g}$ and let $\mathfrak{m}$ be the orthogonal complement to $\mathfrak{h}$. Choose a $K$-invariant metric on $Q$, and let $A = (\mathfrak{g} \cdot q)^\perp$. By Palais' slice theorem, there exists a $K$-invariant neighbourhood $V$ of $0$ in $A$ such that the map $s : G \times_K V \to Q, [g, a]_K \mapsto g \cdot \exp_q a$, is a $G$-equivariant diffeomorphism onto a neighbourhood of $q$. Let $J_K$ be the momentum map for the cotangent lift of the twist action of $K$ on $G \times A$, and let $\varphi : (J_K^{-1}(0) \subset T^*(G \times A)) \to T^*(G \times_K A)$ be the cotangent bundle reduction map, defined in Theorem 3. Let $\alpha = z|_A$ and $B = (\mathfrak{k} \cdot \alpha)^\circ$. There exists an $H$-invariant neighbourhood $N$ of $(0, 0, 0)$ in $\mathfrak{m}^* \times B \times B^*$ such that the map $T^*s^{-1} \circ \bar{\sigma} \circ \bar{\psi} \circ L : G \times_H N \to T^*Q$ given by*

$$[g, \nu, a, \delta]_H \mapsto T^*s^{-1} \circ \varphi\big(g, \mu + \nu + a \diamond_{\mathfrak{h}} \delta, a - \Gamma_\delta^*(\nu|_{\mathfrak{m}\cap\mathfrak{k}} - a \diamond_{\mathfrak{m}\cap\mathfrak{k}} \delta), \alpha + \delta\big),$$

*with $\Gamma_\delta^*$ as in Lemma 28, is a symplectic tube around $z$. If $H = K$ or $V = A$ then $N$ may be taken to equal $\mathfrak{m}^* \times (B \cap V) \times U$, where $U \subset B^*$ is chosen as in Lemma 28.*

**Remark 32.** There are three new aspects of this result, when compared with the general Hamiltonian slice theorem (Theorem 5). First, the symplectic tube is explicitly constructed, up to the cotangent lift of a Riemannian exponential on the configuration space. Second, we have used the cotangent-bundle-specific splitting $N_s \cong T^*B$ in the model space. Third, the tube has the uniqueness property stated in Lemma 28.

**Remark 33.** This result depends crucially on the condition $G_\mu = G$, for the following reasons. The isomorphism $N_s \cong T^*B$ depends on $G_\mu = G$ (see Corollary 20) and the isomorphism $G \times \mathfrak{g}_\mu^* \times N_s \cong T^*(G \times B)$ depends on $N_s \cong T^*B$ and also requires $\mathfrak{g}_\mu^* = \mathfrak{g}^*$. The condition $G_\mu = G$ is used twice in the construction of $\sigma$: in the splitting $\mathfrak{k} = \mathfrak{m} \cap \mathfrak{k} \oplus \mathfrak{h}$; and in the application of Palais' slice theorem to the $K$ action on $A^*$, where it is required that $K_\alpha = H$. Finally, $G_\mu = G$ is needed to guarantee that the map $\sigma$ is symplectic, since this map involves a shift by $\mu$ (see the last paragraph of the proof of Lemma 28).

When computing the symplectic tube in the cotangent bundle slice theorem in an example, it is easiest to compute the composition $T^*s^{-1} \circ \varphi$ directly, using the formula

$$\big\langle T^*s^{-1} \circ \varphi(g, \nu, a, \beta), T(s \circ \pi_K)(g, \xi, a, \dot{a})\big\rangle = \langle \nu, \xi\rangle + \langle \beta, \dot{a}\rangle,$$

which follows directly from the definitions of the cotangent lift and the map $\varphi$. Since the kernel of $T(s \circ \pi_K)$ is $\mathfrak{k} \cdot (G \times A)$, all elements of $TQ$ can be written as $T(s \circ \pi_K)(g, \xi_\perp, a, \dot{a})$ for some $\xi_\perp \in \mathfrak{k}^\perp$. Note that, when $\xi_\perp \in \mathfrak{k}^\perp$, the $\mathfrak{k}^*$ component of $\nu$ is irrelevant in the above equation, and in particular, the term $a \diamond_{\mathfrak{h}} \delta$ in the formula in the cotangent bundle slice theorem is irrelevant.

A particularly simple case is when $G$ acts linearly in a vector space $Q$ and $K = G$. In this case, $A = T_q Q \cong Q$, and all elements of $TQ$ can be written as $T(s \circ \pi_K)(g, 0, a, \dot{a})$. Recalling that for linear actions, $s \circ \pi_K(g, a) = g \cdot (q + a)$, and identifying $A$ with $Q$, we have $T(s \circ \pi_K)(g, 0, a, \dot{a}) = (g \cdot (q + a), g \cdot \dot{a})$. So the above equation becomes

$$\big\langle T^*s^{-1} \circ \varphi(g, \nu, a, \beta), (g \cdot (q + a), g \cdot \dot{a})\big\rangle = \langle \beta, \dot{a}\rangle,$$

for all $\dot{a} \in Q$, which is equivalent to

$$T^*s^{-1} \circ \varphi(g, \nu, a, \beta) = \big(g \cdot (q + a), g \cdot \beta\big). \tag{23}$$

*An alternative construction*. We now give an alternative formulation and proof of Theorem 31. The new construction is more elegant but less concrete. We will produce another $G$-equivariant local symplectomorphism from $G \times_H (\mathfrak{m}^* \times N_s)$ to $T^*Q$ taking $[e, 0, 0]_H$ to $z$, and then show that it is the same as the one in Theorem 31.

We retain all of the definitions from earlier in this section, as well as the assumption $G_\mu = G$. We have seen that $G \times_H (\mathfrak{m}^* \times N_s)$ is isomorphic to $T^*(G \times_H B)$, so that it suffices to find a $G$-equivariant local symplectomorphism from $T^*(G \times_H B)$ to $T^*(G \times_K A)$ taking $\psi(e, 0, 0, 0)$ to $\varphi(e, \mu, 0, \alpha)$. It is natural to consider the cotangent lift of some $G$-equivariant diffeomorphism from $G \times_H B$ to $G \times_K A$, since cotangent lifts are automatically symplectic. However, the cotangent lift of any map from $G \times_H B$ to $G \times_K A$ must map $\psi(e, 0, 0, 0)$, which is in the zero section of $T^*(G \times_H B)$, to some element of the zero section of $T^*(G \times_K A)$, i.e., an element of the form $\varphi(g, 0, a, 0)$; but the target point $\varphi(e, \mu, 0, \alpha)$ is in general not of this form. We might try a momentum shift, but note that the shift $(g, \nu, a, \delta) \mapsto (g, \mu + \nu, a, \alpha + \delta)$ need not preserve $J_K^{-1}(0)$ (see Lemma 27), so the "map" $\varphi(g, \nu, a, \delta) \mapsto \varphi(g, \mu + \nu, a, \alpha + \delta)$ is ill-defined.

The idea of using cotangent lifts can be made to work, by "switching the roles of $A$ and $A^*$": modelling $G \times_H (\mathfrak{m}^* \times N_s)$ as $T^*(G \times_H B^*)$ instead of $T^*(G \times_H B)$, and $T^*Q$ as $T^*(G \times_K A^*)$ instead of $T^*(G \times_K A)$. The advantages of this approach will be: (i) $z \in T^*Q$ will correspond to a point in the zero section of $T^*(G \times_K A^*)$; and (ii) there is a simple local diffeomorphism from $G \times_H B^*$ to $G \times_K A^*$, namely $[g, \delta]_H \to [g, \alpha + \delta]_K$ (see Lemma 35).

Our starting point is the isomorphism in the following lemma, which is easily verified.

**Lemma 34.** *Let $G$ act linearly on a vector space $W$ and by cotangent lifts on $T^*W$. With respect to the inverse dual action of $G$ on $W^*$ and the corresponding cotangent lifted action on $T^*W^*$, the map*

$$\chi : T^*W \cong W \times W^* \longrightarrow W^* \times W \cong T^*W^*,$$
$$(a, \alpha) \longmapsto (\alpha, -a)$$

*is a $G$-equivariant symplectomorphism, with respect to the standard symplectic forms. If $J$ and $J_*$ are the standard momentum maps for the $G$ actions on $T^*W$ and $T^*W^*$ respectively, then $J_* \circ \chi = J$, and in particular, $J_*^{-1}(0) = \chi(J^{-1}(0))$.*

It follows that

$$\chi_0 : T^*(G \times A) \cong G \times \mathfrak{g}^* \times A \times A^* \longrightarrow G \times \mathfrak{g}^* \times A^* \times A \cong T^*(G \times A^*),$$
$$(g, \nu, a, \alpha) \longmapsto (g, \nu, \alpha, -a)$$

is symplectic with respect to the canonical symplectic forms, and that $\chi_0(J_K^{-1}(0)) = J_{K,*}^{-1}(0)$, where $J_{K,*}$ is the momentum map of the cotangent lift of the twist action of $K$ on $G \times A^*$. Also, $\chi_0$ is clearly $G$-equivariant. Applying point cotangent bundle reduction to both sides, $\chi_0$ induces a $G$-equivariant symplectomorphism

$$\bar{\chi}_0 : T^*(G \times_K A) \longrightarrow T^*(G \times_K A^*).$$

By similar reasoning, the symplectic isomorphism

$$\begin{aligned}&\chi_Z : T^*(G \times B) \cong G \times \mathfrak{g}^* \times B \times B^* \longrightarrow G \times \mathfrak{g}^* \times B^* \times B \cong T^*(G \times B^*),\\ &(g, \nu, b, \beta) \longmapsto (g, \nu, \beta, -b)\end{aligned}$$

maps $J_H^{-1}(0)$ to $J_{H,*}^{-1}(0)$, where $J_{H,*}$ is the momentum map for the cotangent-lift of the twist action of $H$ on $G \times B^*$, and induces a $G$-equivariant symplectomorphism

$$\bar{\chi}_0 : T^*(G \times_H B) \longrightarrow T^*(G \times_H B^*).$$

Thus, in order to find a $G$-equivariant local diffeomorphism from of $T^*(G \times_H B)$ to $T^*(G \times_K A)$ that maps $\psi(e, 0, 0, 0)$ to $\varphi(e, 0, 0, \alpha)$, it suffices to find one, call it $\tau_2$, from $T^*(G \times_H B^*)$ to $T^*(G \times_K A^*)$ that maps $\psi_*(e, 0, 0, 0)$ to $\varphi_*(e, 0, \alpha, 0)$, where $\psi_*$ and $\varphi_*$ are the maps that appear in cotangent bundle reduction (Theorem 3), with domain and range as in the following summary diagram,

$$\begin{tikzcd} J_H^{-1}(0) \arrow[r, "\chi_Z"] \arrow[d, "\psi"] & J_{H,*}^{-1}(0) \arrow[r] \arrow[d, "\psi_*"] & J_{K,*}^{-1}(0) \arrow[r, "\chi_0^{-1}"] \arrow[d, "\varphi_*"] & J_K^{-1}(0) \arrow[d, "\varphi"] \\ T^*(G \times_H B) \arrow[r, "\bar{\chi}_Z"] & T^*(G \times_H B^*) \arrow[r, "\tau_2"] & T^*(G \times_K A^*) \arrow[r, "\bar{\chi}_0^{-1}"] & T^*(G \times_K A). \end{tikzcd}$$

The map $\tau_2$ will be the cotangent lift of the diffeomorphism in the following lemma.

**Lemma 35.** *Let $U$ be an $H$-invariant neighbourhood of $0$ in $B^*$ such that the map*

$$t : K \times_H U \longrightarrow A^*, \qquad [k, \delta]_H \longmapsto k \cdot (\alpha + \delta)$$

*is injective; such a $U$ always exists. Then the map*

$$\begin{aligned}&F : G \times_H U \longrightarrow G \times_K \big(K \cdot (\alpha + U)\big) \subset G \times_K A^*,\\ &[g, \delta]_H \longmapsto [g, \alpha + \delta]_K\end{aligned}$$

*is a $G$-equivariant diffeomorphism of $G$-invariant neighbourhoods of $[e, 0]_H$ and $[e, \alpha]_K$.*

**Proof.** By Palais' slice theorem (Theorem 4), the map $t$ is a tube. It follows that the map

$$\begin{aligned}&G \times_K (K \times_H U) \longrightarrow G \times_K \big(K \cdot (\alpha + U)\big),\\ &\big[g, [k, \delta]_H\big]_K \longmapsto \big[g, k \cdot (\alpha + \delta)\big]_K\end{aligned}$$

is a $G$-equivariant diffeomorphism. It thus suffices to show that the following map is a $G$-equivariant diffeomorphism,

$$G \times_H U \longrightarrow G \times_K (K \times_H U), \qquad [g, \delta]_H \longmapsto \big[g, [e, \delta]_H\big]_K.$$

This is not hard to verify; a proof appears in [23]. $\square$

Let $F$ be as in the previous lemma. Its cotangent lift is the $G$-equivariant symplectomorphism

$$T^*F^{-1} : T^*(G \times_H U) \longrightarrow T^*\big(G \times_K \big(K \cdot (\alpha + U)\big)\big).$$

Since $F$ maps $[e, 0]_H$ to $[e, \alpha]_K$, it follows from the definitions of $\psi_*$ and $\varphi_*$ that $T^*F^{-1}$ maps $\psi_*(e, 0, 0, 0)$ to $\varphi_*(e, 0, \alpha, 0)$. The composition $\bar{\chi}_0^{-1} \circ T^*F^{-1} \circ \bar{\chi}_Z$ maps $\psi_*(e, 0, 0, 0)$ to $\varphi_*(e, 0, 0, \alpha)$. We compose this with the shift map

$$\begin{aligned}&\overline{\Sigma}_{(\mu,0)} : T^*(G \times_K A) \longrightarrow T^*(G \times_K A),\\ &\psi(g, \nu, a, \delta) \longmapsto \psi(g, \mu + \nu, a, \delta)\end{aligned}$$

which is easily shown to be a $G$-equivariant symplectomorphism, by an argument similar to that in Lemma 26. The composition $\overline{\Sigma}_{(\mu,0)} \circ \bar{\chi}_0^{-1} \circ T^*F^{-1} \circ \bar{\chi}_Z$ maps $\psi_*(e,0,0,0)$ to $\varphi_*(e,\mu,0,\alpha)$. Composing with $\bar{\psi} \circ L$, defined in Eq. (19), gives a map

$$\overline{\Sigma}_{(\mu,0)} \circ \bar{\chi}_0^{-1} \circ T^*F^{-1} \circ \bar{\chi}_Z \circ \bar{\psi} \circ L : G \times_H (\mathfrak{m}^* \times B \times B^*) \longrightarrow T^*(G \times_K V)$$

taking $[e,0,0,0]_H$ to $\varphi_*(e,0,\alpha,0)$. Finally, we compose with the map $T^*s^{-1} : T^*(G \times_K V) \to T^*Q$, which forces us to restrict the domain of the composition. The result is an alternative version of the cotangent bundle slice theorem (Theorem 31):

**Theorem 36.** *Under the conditions of the Theorem 31, there exists an $H$-invariant neighbourhood $N$ of $(0,0,0)$ in $\mathfrak{m}^* \times B \times B^*$ such that the map*

$$T^*s^{-1} \circ \overline{\Sigma}_{(\mu,0)} \circ \bar{\chi}_0^{-1} \circ T^*F^{-1} \circ \bar{\chi}_Z \circ \bar{\psi} \circ L : G \times_H N \longrightarrow T^*Q$$

(*defined above*) *is a symplectic tube around $z$.*

We will now show that the symplectic tubes in Theorems 31 and 36 are the same. Note that the definition of $U$ is the same in Lemma 35 as in Lemma 28. Hence it suffices to show that $\overline{\Sigma}_{(\mu,0)} \circ \bar{\chi}_0^{-1} \circ T^*F^{-1} \circ \bar{\chi}_Z = \bar{\sigma}$, or equivalently,

$$\bar{\chi}_0 \circ \overline{\Sigma}_{(\mu,0)}^{-1} \circ \bar{\sigma} \circ \bar{\chi}_Z^{-1} = T^*F^{-1}. \tag{24}$$

It is straight-forward to check that

$$\bar{\chi}_0 \circ \overline{\Sigma}_{(\mu,0)}^{-1} \circ \bar{\sigma} \circ \bar{\chi}_Z^{-1}\big(\psi_*(g,\nu,\delta,a)\big) = \varphi_*\big(g,\nu,\alpha+\delta, a + \Gamma_\delta^*(\rho|_{\mathfrak{m}\cap\mathfrak{k}} + a \diamond_{\mathfrak{m}\cap\mathfrak{k}} \delta)\big), \tag{25}$$

for every $\psi_*(g,\nu,\delta,a) \in T^*(G \times_H U)$.

To compute $T^*F^{-1}$, let $U$ and $F$ be as above, and define $f : G \times U \to G \times (\alpha + U)$ by $f(g,\delta) = (g, \alpha+\delta)$. It is clear that the following diagram commutes,

$$\begin{array}{ccc} G \times U & \xrightarrow{f} & G \times (\alpha + U) \\ \downarrow{\scriptstyle \pi_H} & & \downarrow{\scriptstyle \pi_K} \\ G \times_H U & \xrightarrow{F} & G \times_K (K \cdot (\alpha + U)) \end{array}$$

where $\pi_H$ and $\pi_K$ are restrictions of the canonical projections. Since $F$ is invertible, we have $\pi_H = F^{-1} \circ \pi_K \circ f$. The surjectivity of $F$ implies that every element of $T(G \times_K (K \cdot (\alpha + U)))$ can be expressed as $T(\pi_K \circ f)(g,\xi,\delta,\epsilon) = T\pi_K(g,\xi,\alpha+\delta,\epsilon)$ for some $(g,\xi,\delta,\epsilon) \in T(G \times U)$. Hence we can compute $T^*F^{-1}$ as follows: for any $(g,\nu,\delta,a) \in T^*(G \times U) \cap J_{K,*}^{-1}(0)$ and any $(g,\xi,\delta,\epsilon) \in T(G \times U)$,

$$\begin{aligned} &\big\langle T^*F^{-1} \circ \psi_*(g,\nu,\delta,a), T\pi_K(g,\xi,\alpha+\delta,\varepsilon)\big\rangle \\ &\quad = \big\langle \psi_*(g,\nu,\delta,a), T(F^{-1} \circ \pi_K \circ f)(g,\xi,\delta,\varepsilon)\big\rangle \\ &\quad = \big\langle \psi_*(g,\nu,\delta,a), T\pi_H(g,\xi,\delta,\varepsilon)\big\rangle \\ &\quad = \langle \nu,\xi\rangle + \langle a,\varepsilon\rangle. \end{aligned} \tag{26}$$

Now we make the corresponding computation with the right-hand side of Eq. (25), namely $\varphi_*(g,\nu,\alpha+\delta, a + \Gamma_\delta^*(\rho|_{\mathfrak{m}\cap\mathfrak{k}} + a \diamond_{\mathfrak{m}\cap\mathfrak{k}} \delta))$. Since $\Gamma_\delta^*(\rho|_{\mathfrak{m}\cap\mathfrak{k}} + a \diamond_{\mathfrak{m}\cap\mathfrak{k}} \delta) \in B^\perp$, which annihilates $\varepsilon \in U \subset B^*$, we have,

$$\begin{aligned} &\big\langle \varphi_*\big(g,\nu,\alpha+\delta, a + \Gamma_\delta^*(\rho|_{\mathfrak{m}\cap\mathfrak{k}} + a \diamond_{\mathfrak{m}\cap\mathfrak{k}} \delta)\big), T\pi_K(g,\xi,\alpha+\delta,\varepsilon)\big\rangle \\ &\quad = \langle \nu,\xi\rangle + \big\langle a + \Gamma_\delta^*(\rho|_{\mathfrak{m}\cap\mathfrak{k}} + a \diamond_{\mathfrak{m}\cap\mathfrak{k}} \delta), \varepsilon\big\rangle \\ &\quad = \langle \nu,\xi\rangle + \langle a,\varepsilon\rangle. \end{aligned} \tag{27}$$

The calculations in Eqs. (26) and (27) prove that

$$T^*F^{-1}\big(\psi_*(g,\nu,\delta,a)\big) = \varphi_*\big(g,\nu,\alpha+\delta, a + \Gamma_\delta^*(\rho|_{\mathfrak{m}\cap\mathfrak{k}} + a \diamond_{\mathfrak{m}\cap\mathfrak{k}} \delta)\big), \tag{28}$$

which, together with Eq. (25), proves Eq. (24). Thus we have shown:

**Theorem 37.** *The symplectic tubes in Theorems 31 and 36 are identical.*

**Remark 38.** Using Lemma 27, it can be easily checked that $\Gamma_\delta^*(\rho|_{\mathfrak{m}\cap\mathfrak{k}} + a \diamond_{\mathfrak{m}\cap\mathfrak{k}} \delta)$ is the unique element $c \in B^\perp$ such that $(g, \nu, \alpha+\delta, a+c) \in J_{K,*}^{-1}(0)$. Thus the calculations in Eqs. (26) and (27) show that the formula in Eq. (28) is the unique one expressing $T^*F^{-1}(\psi_*(g,\nu,\delta,a))$ as the $\varphi_*$-image of an element of $T^*_{(g,\alpha+\delta)}(G\times A)$.

*Example*. We conclude this section with a simple example of the Cotangent bundle slice theorem (Theorem 31). Consider $G = SO(3)$ acting on in the standard way on $Q = \mathbf{R}^3$, and by cotangent lifts on $T^*\mathbf{R}^3$. The momentum map is $\mu = J(q,p) = q \times p$. In order to apply Theorem 31 we require $G_\mu = G$; the coadjoint action of $SO(3)$ is such that this condition is satisfied only at $\mu = 0$. Thus $q$ and $p$ must be parallel, or at least one of them must be zero. We will present the case $q = 0$ and $p \neq 0$, and then state without details the results of similar calculations for the other cases. We will implicitly use the Euclidean inner product in several places, to define orthogonal complements and to identify spaces with their duals.

Assume $q = 0$ and $p \neq 0$. Without loss of generality, $(q,p) = ((0,0,0),(\lambda,0,0))$ for some $\lambda \neq 0$. We have $K = G = SO(3)$, and $H$ is the circle group of rotations around the $x$-axis. Since $G$ fixes $q$, we have $A = (\mathfrak{g}\cdot q)^\perp = \mathbf{R}^3$. Also, $\alpha = z|_A = (\lambda,0,0) \in A^*$. Since $K = SO(3)$, the group orbit $K\cdot\alpha$ is the sphere of radius $\lambda$, so $B := (\mathfrak{k}\cdot\alpha)^\circ$ is the $x$-axis (identifying $(\mathbf{R}^3)^*$ with $\mathbf{R}^3$); the space $B^*$ is also the $x$-axis. We make the standard identification

$$\mathfrak{g} = \mathrm{so}(3) \to \mathbf{R}^3, \qquad \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix} \mapsto (\xi_1,\xi_2,\xi_3).$$

Since $H$ is the group of rotations around the $x$-axis, its Lie algebra, $\mathfrak{h}$, is the $x$-axis. Now $\mathfrak{m} = \mathfrak{h}^\perp$ so, we can identify $\mathfrak{m}$ and $\mathfrak{m}^*$ with the $yz$-plane. We now calculate the map $\tau$ from Eq. (22), beginning with the subset $U \subset B^*$ and the map $\Gamma_\delta^*$ in Lemma 28. Let $U = (-\lambda,\infty)\times\{(0,0)\} \subset B^*$; this is the largest neighbourhood in $B^*$ such that the tube $t : K\times_H U \longrightarrow A^*, [g,\delta]_H \longmapsto g\cdot(\alpha+\delta)$ is injective. Since $K = G$, we have $\mathfrak{m}\cap\mathfrak{k} = \mathfrak{m}$. The map $\Gamma_\delta^* : (\mathfrak{m}\cap\mathfrak{k})^* \to B^\perp$ is defined by $\langle \Gamma_\delta^*(\nu), \xi\cdot(\alpha+\delta)+\varepsilon\rangle = \langle \nu,\xi\rangle$, for every $\xi \in \mathfrak{m}\cap\mathfrak{k}$, $\nu \in (\mathfrak{m}\cap\mathfrak{k})^*$, $\delta \in U$ and $\varepsilon \in B^*$. Let $\nu = (0,\nu_2,\nu_3)$, $\xi = (0,\xi_2,\xi_3)$, $\delta = (\delta_1,0,0)$ and $\varepsilon = (\varepsilon_1,0,0)$, and recall that $\alpha = (\lambda,0,0)$. Then

$$\xi\cdot(\alpha+\delta)+\varepsilon = \xi\cdot(\lambda+\delta_1,0,0) + (\varepsilon_1,0,0) = \bigl(\varepsilon_1, \xi_3(\lambda+\delta_1), -\xi_2(\lambda+\delta_1)\bigr),$$

so $\langle \Gamma_\delta^*(\nu), (\varepsilon_1, \xi_3(\lambda+\delta_1), -\xi_2(\lambda+\delta_1))\rangle = \langle\nu,\xi\rangle$. It follows that $\Gamma_\delta^*(\nu) = (0, \frac{\nu_3}{\lambda+\delta_1}, -\frac{\nu_2}{\lambda+\delta_1})$. Now $a\diamond_{\mathfrak{g}}\delta = J_K(a,\delta) = a\times\delta$. Since $B$ is the $x$-axis and $U\subset B^*$, we have $a\times\delta = 0$ for all $a\in B$ and $\delta\in U$. Putting these calculations together,

$$\tau\bigl[g,\nu,\bigl((a_1,0,0)\bigr),(\delta_1,0,0)\bigr]_H = \varphi\left(g,\nu,\left(a_1, \frac{\nu_3}{\lambda+\delta_1}, -\frac{\nu_2}{\lambda+\delta_1}\right),(\lambda+\delta_1,0,0)\right).$$

The symplectic tube in Theorem 31 is $T^*s^{-1}\circ\tau$. Since $K = G = SO(2)$, we know from Eq. (23) that $T^*s^{-1}\circ\varphi(g,\nu,a,\beta) = (g\cdot a, g\cdot\beta)$. So we obtain the following symplectic tube for the $G$ action around $z$ (dropping the subscript-1's):

$$T^*s^{-1}\circ\tau : G\times_H(\mathfrak{m}^*\times B\times U) \longrightarrow T^*\mathbf{R}^3,$$
$$\bigl[g,(\nu_1,\nu_2,\nu_3),(a,0,0),(\delta,0,0)\bigr]_H \longmapsto \left(g\cdot\left(a, \frac{\nu_3}{\lambda+\delta}, -\frac{\nu_2}{\lambda+\delta}\right), g\cdot(\lambda+\delta,0,0)\right).$$

The other nontrivial subcase of the $Q = \mathbf{R}^3$, $G = SO(3)$ example occurs when $q \neq 0$ and $p \parallel q$; it turns out that it makes no difference whether or not $p = 0$. In this case, $H = K \cong SO(2)$. Since $H = K$, the map $\Gamma_\delta^*$ is trivial. However, since $K$ is neither $G$ nor $\{e\}$, the calculation of $T^*s^{-1}\circ\varphi$ is nontrivial, though not difficult. For brevity, we state only the final result for this case: if $z = ((\kappa,0,0),(\lambda,0,0))$, then $\mathfrak{m}^*$ may be identified with the $yz$-plane, and $A$ and $B$ with the $x$-axis. The map $T^*s^{-1}\circ\tau$ given by

$$G\times_H\bigl(\mathfrak{m}^*\times(-\kappa,\infty)\times B^*\bigr) \longrightarrow T^*\mathbf{R}^3,$$
$$\bigl[g,(\nu_1,\nu_2,\nu_3),(a,0,0),(\delta,0,0)\bigr]_H \longmapsto \left(g\cdot(\kappa+a,0,0), g\cdot\left(\lambda+\delta, \frac{\nu_3}{\kappa+a}, -\frac{\nu_2}{\kappa+a}\right)\right)$$

is a symplectic tube for the $G$ action around $z$.

The only remaining subcase is $q = p = 0$. In this case, $H = K = SO(3)$ and $A = B = \mathbf{R}^3$ and $U = B^*$. The map $T^*s^{-1} \circ \tau : G \times_G (A \times A^*) \to T^*A$ is the trivial one $[g, a, \delta]_G \mapsto (a, \delta)$.

## 6. Conclusion

We have investigated the local structure of cotangent-lifted Lie group actions. We proved a "tangent-level" commuting reduction result, Theorem 10, and then used it to characterise the symplectic normal space, in Theorem 19. In two special cases, we arrived at splittings of the symplectic normal space. One of these, Corollary 20, applies whenever the configuration isotropy group $G_q$ is contained in the momentum isotropy group $G_\mu$, and generalises the splitting given for free actions by Montgomery et al. [18]. The other splitting, in Corollary 23, applies at all points $z \in T^*Q$ that are "purely in the group direction", meaning $z|_A = 0$, where $A = (\mathfrak{g} \cdot g)^\perp$; this condition is satisfied by all relative equilibria of simple mechanical systems. In both of these special cases, the new splitting leads to a refinement of the reconstruction equations (bundle equations), as explained at the end of Section 4. We also noted in Section 4 two cotangent-bundle-specific local normal forms for the symplectic reduced space, in Theorem 18 and Remark 25.

Our main result is the Cotangent bundle slice theorem, Theorem 31, which applies at all points with fully isotropic momentum values, $G_\mu = G$. This theorem extends the Hamiltonian slice theorem of Marle, Guillemin and Sternberg (Theorem 5) in three ways. First, it is constructive, apart from the use of the cotangent lift of a Riemannian exponential. Second, it includes a cotangent-bundle-specific splitting of the symplectic normal space (a special case of one of the first of the splittings described in the previous paragraph). Third, our construction has a uniqueness property, contained in Lemma 28. In Theorems 36 and 37, we gave an alternative construction of the symplectic tube in the main theorem, showing that it is essentially a cotangent lift of a simple map between certain twisted products. The example presented at the end of the Section 5 shows that our construction is feasible; we believe that this is the first time that symplectic tubes have been computed in an example.

A number of open questions remain, the most salient of which is: what happens when $\mu$ is not fully isotropic? We have so far only been able to formulate our cotangent bundle slice theorem for the case of a fully-isotropic momentum value, for reasons summarised in Remark 33. Our characterisation of the symplectic normal space $N_s$ is also incomplete in the general case. We have found a splitting of $N_s$ that applies to all relative equilibria of simple mechanical systems, but what about relative equilibria of other systems? Even for the simple mechanical case, what form do the reconstruction equations take if $\mu$ is not split?

One possible application of this work is to the problem of singular cotangent bundle reduction (this was in fact our initial motivation for this research). Local normal forms given in Section 4 are a start, but do not address the global structure.

Dynamical applications seem promising. The constructive nature of the cotangent bundle slice theorem should allow us to apply theoretical results on stability, bifurcations and persistence, such as those referred to in the Introduction, to specific examples. Also, the refinement of the reconstruction equations in the cotangent bundle case may lead to extensions of the theory. In particular, the relationship between our splitting of the symplectic normal space at a relative equilibrium of a simple mechanical system, and the splitting used in the Lagrangian Block Diagonalisation [10] method for testing stability, deserves investigation.

### Acknowledgements

I would like to thank Tudor Ratiu and Juan Pablo Ortega for introducing me to this subject, and for many helpful suggestions. This article is based on part of my Ph.D. thesis, written at the Ecole Polytechnique Fédérale de Lausanne under the supervision of Tudor Ratiu, supported by the Swiss National Fund grant 21-54'138. Thanks also to Mark Roberts and Claudia Wulff for clarifying the "tangent-level commuting reduction" viewpoint, and to M. Esmeralda Sousa-Dias, Richard Cushman, and Miguel Rodriquez-Olmos for helpful comments on the manuscript.

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