Manifold Homotopy via the Flow Complex

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Abstract

It is known that the critical points of the distance function induced by a dense sample P of a submanifold Σ of $\mathbb{R}^n$ are distributed into two groups, one lying close to Σ itself, called the *shallow*, and other close to medial axis of Σ, called *deep* critical points. We prove that under (uniform) sampling assumption, the union of stable manifolds of the shallow critical points have the same homotopy type as Σ itself and the union of the stable manifolds of the deep critical points have the homotopy type of the complement of Σ. The separation of critical points under uniform sampling entails a separation in terms of distance of critical points to the sample. This means that if a given sample is dense enough with respect to two or more submanifold of $\mathbb{R}^n$, the homotopy type of all such submanifolds as well as that of their complements are captured as unions of stable manifolds of shallow critical points, in a filtration of the flow complex based on the distance of critical points to sample.

1 Introduction

The *flow complex* was introduced by Giesen and John [9] as a tool for geometry modeling. Much of the mathematical foundations behind the flow complex were well-explored; see [10] and references therein. Further important properties of the *flow map* induced by a generalized gradient of the distance function induced by compact sets have been subject of recent investigations, see e.g. [12]. In [8], it was noted empirically that the flow complex derived from a dense sample of a surface, though often much coarser than the Delaunay complex of the same point set, does contains a subcomplex that *approximates* the surface much in the same was as the Delaunay complex.

Surface reconstruction is the problem of producing from a discrete sample of a surface Σ a concisely represented surface $\tilde{\Sigma}$ that closely approximates Σ and shares its topology, provided that the sample is dense enough. This problem has a rich literature spanning several disciplines; see [2] for a survey of Delaunay-based algorithms which have particularly been the most successful in providing geometric and topological guarantees. Traditionally, "topological equivalence" is interpreted as *homeomorphism* or even *ambient isotopy*. This in particular requires the reconstructed object to also be a manifold and of the same dimension as the target surface. In this paper, we relax this interpretation to *homotopy equivalence* (See [11] for definitions). In other words, we seek to capture the homotopy type of a manifold by finding a topological space that that is not necessarily a manifold but approximates the original manifold in Hausdorff distance.

Prior to [8], flow methods were employed in surface reconstruction (e.g. [7]) but the first of such algorithms with geometric and topological guarantees was found by Dey et al. [5] who proved a sharp separation of critical points of the distance function induced by surface samples into two groups one lying close to the surface (shallow) and the other close to its medial axis (deep). They further showed that in 3D, the boundary of the union of stable manifolds of inner or outer deep critical points is homeomorphic to the original surface, provided that the sample is dense enough and *tight*. However, this does not generalize to higher dimensions. This paper aims to achieve this, albeit with certain modifications. On the down-side, we use *uniform* sampling (as opposed to adaptive sampling used in [5]) although we relax the tightness requirement. Moreover, homeomorphism is weakened to homotopy equivalence. On the upside, we prove that the union of stable manifolds of shallow critical points approximates the manifold and captures its topology while that of deep ones does the same for the complement of the manifold. Plus, we show that this works for any closed submanifold of a Euclidean space of any dimension not just for (codimension-1) surfaces. Capturing the homotopy type of the complement in addition to that of the manifold substantially improves the quality of topological guarantee. For example, all closed curves have the same homotopy type (in fact are homeomorphic) and its is the homotopy type of the complement of the curve that distinguishes knots from one another. Similarly, a knotted torus is homeomorphic to an unknotted one while their complements have different homotopy types.

For uniform samples, the separation of critical points which is determined in terms of their distance from the manifold translates into a separation in terms of distance from the sample itself. In other words, if one sorts the critical points in the order of their distance to the sample, shallow critical points make a prefix of this ordering. Thus if one filters the flow complex by putting together the stable man-

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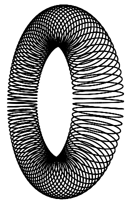
ifolds, i.e. cells associated to, critical points in all prefixes of this order, one is guaranteed to reach in this filtration a shape homotopy equivalent to the manifold in question.



As mentioned above, the union of stable manifolds of the remaining critical points then captures the homotopy type of the complement of the manifold. Since the filtration is regardless of the manifold, this statement is true for any manifold for which the given sample is a dense enough sample. For example, if the given sample is dense for a curve embedded on a torus and for the torus itself, the above filtration results homotopy equivalent reconstruction of both the curve and the torus, as well as their complements, in different stages.

2 Background and Preliminaries

Let P be closed nonempty subset of $\mathbb{R}^n$. The complement of P is the open set $P^c = \mathbb{R}^n \setminus P$. For any point $x \in P^c$, let $h_P(x) = \inf_{y \in P} \|x - y\|$ be the *distance function* defined by P and let $A_P(x) = \{y \in P : \|x - y\| = h_P(x)\}$.

While the distance function h_P is not smooth, it induces a vector field v_P over P^c which behaves like the gradient of h_P in the sense that $v_P(x) \neq 0$ if and only if there is a unique direction of steepest ascent for h_P at x in which case the direction of this steepest ascent is given by $v_P(x)$ (See [10] for more general statement and details). The vector $v_P(x)$ at a point x is characterized by $v_P(x) = \frac{x - d_P(x)}{h_P(x)}$, where $d_P(x)$, called *driver of* x is the center of the smallest enclosing ball of $A_P(x)$, or equivalently, the closest point in $\operatorname{conv} A_P(x)$, the *convex hull* of $A_P(x)$, to x. The *critical points* of h_P are those points x for which $v_P(x) = 0$, or equivalently, $x = d_P(x) \in \operatorname{conv} A_P(x)$.

Lieutier [12] proved that if P^c is bounded, then Euler schemes defined by v_P on P^c uniformly converge and this results in a *flow map* $\phi_P : \mathbb{R}^+ \times P^c \to P^c$ (where $\mathbb{R}^+$ is the set of non-negative reals) which he also proved to be continuous (on both variables). Intuitively, $\phi_P(t, x)$ is the point y that is reached from following the vector field v_P for time interval of length t, starting at x, by infinitesimal movements proportional to the magnitude of v_P. The map ϕ_P has the classical properties of a flow map, namely $\phi_P(0, x) = x$, $\phi_P(s + t, x) = \phi_P(s, \phi_P(t, x))$, and for any point x and any $t \geq 0$, $v_P(\phi_P(t, x))$ is the *right-derivative* of $\phi_P(t, x)$. Lieutier also proved that h_P along any flow orbit, i.e. $t \mapsto h_P(\phi_P(t, x))$ is increasing and in addition satisfies

$$h_P(\phi_P(t, x)) = h_P(x) + \int_0^t \|v_P(\phi_P(\tau, x))\|^2 d\tau. \quad (1)$$

The special case where P is finite is of particular interest to us and the rest of this section goes over special properties of the flow maps in this case. Let $\operatorname{Vor} P$ and $\operatorname{Del} P$ respectively denote the Voronoi and Delaunay complexes induced by P. For any point $x \in \mathbb{R}^n$, we represent by $V_P(x)$ the *lowest dimensional face* of $\operatorname{Vor} P$ that contains x, and by $D_P(x)$ the face in $\operatorname{Del} P$ dual to $V_P(x)$. The set $A_P(x)$ is the vertex set of $D_P(x)$ and $d_P(x)$ becomes the closest point on $D_P(x)$ to x. It can be verified that all points in the relative interior of the same Voronoi face have the same driver. Since the affine hulls of a Voronoi face and its dual are orthogonal with total dimension n, they intersect in exactly one point. Thus if $V_P(x)$ and $D_P(x)$ intersect, then this intersection consists of a single critical point which is the driver of x. All critical points (except for the maximum at infinity) are characterized the same way (as intersection points of duals). Following [9], we make a *general position assumption* that all pairs of Voronoi and Delaunay objects that are dual to and intersect each other, do so in their relative interiors. The *index* of a critical point c is defined as the dimension of $D_P(c)$.

For a given flow map ϕ_P, the *flow orbit* of a regular point x, denoted $\phi_P(x)$ is defined as $\phi_P([0, +\infty), x)$. For a set T we use $\phi_P(T)$ for $\bigcup_{x \in T} \phi_P(x)$. Notice that by this definition $T \subseteq \phi_P(T)$.

For a critical point c of h_P, the set of all points x whose flow orbit converges to c is called the *stable manifold* of c and denoted by $\operatorname{Sm}(c) = \{x : \phi_P(+\infty, x) = c\}$. Although there is no flow out of a critical point c, we study the orbits of points very close to c. Some of these points flow into c while other flow away from it. We define the *unstable manifold* $\operatorname{Um}(c)$ of a critical point c, as the set of all points into which points arbitrarily close to c flow. Formally, $\operatorname{Um}(c) = \bigcap_{\varepsilon > 0} \phi(B(c, \varepsilon))$, where $B(c, \varepsilon)$ denotes the open ball of radius ε centered at c. In other words, the unstable manifold of c consists of c and all the integral lines that start infinitesimally close to c.

Proposition 1 *Let P be finite. For a critical point c of h_P, $Um(c) = \phi_P(V_P(c))$.*

A set T is said to be *flow-tight* for ϕ_P if $\phi_P(T) = T$. Stable and unstable manifolds of critical points and their union and intersections are flow tight. Let $\mathcal{C}_P$ be the set of critical points of h_P induced by P (including the critical point at infinity). The *(stable) flow complex* of P, denoted $\operatorname{Sfc} P$ is the collection of stable manifolds of all critical points in $\mathcal{C}_P$. Generically, the cell associated to an index k critical point is a topological open k-ball. Moreover, if for critical points $c, c' \in \mathcal{C}_P$, $c \in \partial \operatorname{Sm}(c')$, then $\operatorname{Sm}(c) \subset \partial \operatorname{Sm}(c')$.

Lemma 1 *If for $c \in \mathcal{C}_P$, $\operatorname{ind} c = k$, then every critical point $c' \in \partial \operatorname{Sm}(c)$ has index less than k, provided that $Sm(c)$ does not intersect the $(n - k - 1)$-skeleton of*

$\mathrm{Vor}\, P$. *Under the same assumption, if* $c \in \partial\, \mathrm{Um}(c')$, *then* $\mathrm{ind}\, c' < \mathrm{ind}\, c$.

All but a measure-0 set of points P satisfy the requirement that $\mathrm{Sm}(c)$ must stay clear from faces of $\mathrm{Vor}\, P$ of dimension $n-k-1$ or smaller (See [14]).

By a *manifold* we refer to a C^2-smooth closed submanifold Σ of $\mathbb{R}^n$. The medial axis $M(\Sigma)$ of Σ consists of points in space with 2 or more closest points in Σ. The *reach* of Σ is the distance between Σ and $M(\Sigma)$. We assume the reach of Σ is strictly positive. Any point $x \notin M(\Sigma)$, has a unique closest point $\hat{x}$ in Σ. The half-line bounded at $\hat{x}$ through x hits $M(\Sigma)$ for the first time at a point $\check{x}$ (or at infinity).

A point set $P \subset \Sigma$ is a *uniform ξ-sample* of Σ if $\forall x \in \Sigma\ \exists p \in P : \|x - p\| \leq \xi$. For a given parameter $r \geq 0$, the union of balls $\bigcup_{p\in P} B(p,r)$ is denoted by $B^{(r)}(P)$. The *α-shape* of P of parameter r, denoted $K^{(r)}(P)$ is the underlying space of *restriction* of $\mathrm{Del}\, P$ to $B^{(r)}(P)$ (See [6]). The *flow shape* of P for parameter r, denoted $F^{(r)}(P)$ is the union of stable manifolds of critical points at distance $\leq r$ from P (See [4]).

3 Shallow versus deep critical points

For any point $x \in \mathbb{R}^n \setminus (\Sigma \cup M(\Sigma))$ let $\mu(x) = \|\check{x} - \hat{x}\|$. If $\check{x}$ is at infinity, then $\mu(x) = \infty$. Otherwise, the ratio $0 < \frac{\|x-\hat{x}\|}{\|\check{x}-\hat{x}\|} < 1$, is a relative measure of how close to Σ or $M(\Sigma)$ the point x is. It turns out [5, 3] that when a (possibly noisy) sample P of Σ satisfies some density requirements, then critical points of h_P are distributed, according to the above measure, into two distinguishable groups, one lying very close to Σ and the other to $M(\Sigma)$. We use a weaker version of the Lemma for uniform samples here.

Theorem 2 *Let P be an $\varepsilon\tau$-sample of a manifold Σ of reach τ with $\varepsilon \leq 1/\sqrt{3}$. Then for every critical point c of h_P, either $\|c - \hat{c}\| \leq \varepsilon^2\tau$, or $\|c - \hat{c}\| \geq (1 - 2\varepsilon^2)\tau$. In the former case we call c a* shallow *critical point and a in the latter case a* deep *one.*

Corollary 1 *Under the settings of Theorem 2, for every shallow critical point c of h_P, $h_P(c) \leq \sqrt{5/3}\cdot\varepsilon\tau$, and for every deep critical point c' of h_P, $h_P(c') \geq (1 - 2\varepsilon^2)\tau$.*

For any $0 \leq \delta < 1$, the *δ-tubular neighborhood* of a manifold Σ of reach τ is defined as the set $\Sigma_\delta = \{x \in \mathbb{R}^n : \|x - \hat{x}\| \leq \delta\tau\}$. Notice that $M(\Sigma) \subset \Sigma_\delta^c$.

Lemma 3 *For any $0 \leq \delta < 1$, $cl\,\Sigma_\delta^c$ is homotopy equivalent to Σ^c. In fact, the former is a strong deformation retract of the latter.*

Lemma 4 *Let P be an $\varepsilon\tau$-sample of a manifold Σ of reach τ with $\varepsilon \leq 1/(1+\sqrt{2})$. Then, $cl\,\Sigma_\delta^c$ is flow-tight under the flow ϕ_P, for any $\frac{\varepsilon^2}{1-\varepsilon} < \delta < 1 - \varepsilon - \frac{\varepsilon^2}{1-\varepsilon}$. In particular this is true for $\delta = 1/2$.*

The above lemma implies that union of stable manifolds of shallow critical points is contained in Σ_δ for $\delta = \varepsilon^2/(1-\varepsilon)$ thus providing the Hausdorff distance guarantee for our reconstructions.

4 Homotopy Type of the Manifold

In this section we show that in a dense enough sample of a submanifold of $\mathbb{R}^n$, the union of stable manifolds of the shallow critical points has the same homotopy type as the manifold itself. This statement follows from the following sequence of results.

Lemma 5 [13] *Let Σ be a manifold of reach τ and let P be an $\varepsilon\tau$-sample of Σ for any $\varepsilon \leq \frac{1}{2}\sqrt{3/5}$. Then $B^{(r)}(P)$ deformation retracts (and is in particular homotopy equivalent) to Σ, for any $2\varepsilon\tau < r < \sqrt{3/5}\cdot\tau$.*

Lemma 6 [6] *For any $r \geq 0$, $B^{(r)}(P)$ and the α-shape $K^{(r)}(P)$ are homotopy equivalent.*

Lemma 7 [4, 1] *For any r, the flow shape $F^{(r)}(P)$ and the α-shapes $K^{(r)}(P)$ are homotopy equivalent.*

Theorem 8 *Let Σ be a manifold of reach τ and let P be an $\varepsilon\tau$-sample of Σ for $\varepsilon \leq \frac{1}{2}\sqrt{3/5}$. Then Σ is homotopy equivalent to the union U of stable manifolds of shallow critical points of h_P.*

Proof. For a critical point c of h_P, by Corollary 1 $h_P(c) \leq \sqrt{5/3}\cdot\varepsilon\tau$ if c is shallow and $h_P(c) \geq (1-2\varepsilon^2)\tau$ if c is deep. For $\varepsilon < \frac{1}{2}\sqrt{3/5}$ the latter bound is strictly greater than the former and therefore there is a positive value r for which $h_P(c) < r$ for every shallow critical point c and $h_P(c') > r$ for every deep critical point c'. Thus the flow shape $F^{(r)}(P)$ is precisely the union of stable manifolds of shallow critical points of h_P with respect to Σ. Lemmas 5, 6, 7 now imply that this union is homotopy equivalent to Σ. $\square$

5 Homotopy Type of the Complement of the Manifold

In this section we prove that the union of stable manifolds of deep critical points has the homotopy type of Σ^c using the continuity of the flow map ϕ_P. The technique is inspired from the work of Lieutier [12]. A proof can found in [14].

Theorem 9 *Let P be a finite set of points in $\mathbb{R}^n$. If for sets $Y \subset X \subset \mathbb{R}^n$, X and Y are both flow-tight for ϕ_P, i.e. $\phi_P(X) = X$ and $\phi_P(Y) = Y$, and if $X \setminus Y$ is bounded, and, finally, if there is a constant $c > 0$ for which $\|v_P(x)\| \geq c$ for all $x \in X \setminus Y$, then X and Y are homotopy equivalent.*

A difficulty in using the above theorem is that ϕ_P is proven in [12] to be continuous on P^c as long as it is a *bounded* set. This can be overcome by clipping the space with a very large ball, thus letting $P_0 = P \cup B^c$ where B is a very large ball satisfying $P \subset \frac{1}{5}B$. It can then be verified that within $\frac{1}{2}B$, ϕ_P and ϕ_{P_0} agree which is enough for what we want to prove. In the sequel $\mathcal{C}_\Sigma$ denotes the set of shallow critical points of P where P is an $\varepsilon\tau$-sample of a manifold σ of reach τ. The value of ε is determined later. For shorthand, we write S for Σ^c as well S_δ for Σ^c_δ.

Lemma 10 *Let c be a critical point of h_P and let $U \subseteq \mathbb{R}^n$ be a flow-tight set for ϕ_P with $c \notin U$. Let $V = \operatorname{rel\,int} V_P(c)$. For $r \geq 0$, let $V_r = V \cap B(c, r)$. Then for every $r \geq 0$, U and $U \setminus V_r$ have the same homotopy type and if $U \cap B(c, r) \subset V$ then $U \setminus V_r$ is flow-tight for ϕ_P.*

Theorem 11 *Let $\varepsilon \leq \frac{1}{2}\sqrt{3/5}$. Let $\tilde{S} = \bigcup_{c \in \mathcal{C} \setminus \mathcal{C}_\Sigma} Sm(c)$ be the union of stable manifolds of all deep critical points of h_P with respect to Σ. Let $U_\Sigma = \bigcup_{c \in \mathcal{C}_\Sigma} Um(c)$ be the union of unstable manifolds of all shallow critical points. Then $\tilde{S}$ is homotopy equivalent to S.*

Proof. (*sketch*) We use Theorem 9 to show that $X = S$ is homotopy equivalent to $Y = \tilde{S}_{1/2}$ which is itself homotopy equivalent to S by Lemma 3. The main difficulty in the proof is that although both X and Y are flow-tight for ϕ_P, $\|v_P\|$ can be arbitrarily small in $X \setminus Y$ because the boundary of X can contain critical points that drive points in the interior of X arbitrarily close to them. To handle this difficulty, a first idea is to use Lemma 10 to remove from X a neighborhood of these critical points, thus creating a strictly positive distance between these critical points and points in the trimmed X. However, in order to do this in a manner that ensures the trimmed X is still flow tight, this has to be done in several steps where the i-th step gets rid of critical points of index i. We thus first delete from X a neighborhood of every critical point of index-0 to get a flow-tight set X_0 that by Theorem 9 will be homotopy equivalent to the set $\tilde{X}_0$ consisting of the union of Y and the *unstable* manifolds of critical points of index 1 and higher on boundary of X, restricted to X. One can then remove from x a neighborhood of all index-1 critical points resulting a set X_1 that using Lemmas 1 and 10 is flow-tight for ϕ_P. Applying Theorem 9 then results a set $\tilde{X}_1$ consisting of the union of Y and unstable manifolds of critical points of index 2 or higher clipped by X. Continuing this way, all critical points on the boundary of X can be eliminated resulting a sequence of homotopy equivalent shapes $X_0, \tilde{X}_0, X_1, \tilde{X}_1, \ldots, X_n, \tilde{X}_n$ the first of which is homotopy equivalent to X and the last one to Y. $\square$

Corollary 2 *Let $\Sigma_1, \ldots, \Sigma_s$ be manifolds of various dimensions for all of which the same sample P is an $\varepsilon\tau_i$-sample where τ_i is the reach of Σ_i, $i = 1, \ldots, s$. If $c_1, \ldots, c_m$ are the set of critical points of h_P sorted such that $h_P(c_1) < \cdots < h_P(c_m)$, then for each i, there is a j_i such that $\bigcup_{j \leq j_i} Sm(c_j) \simeq \Sigma_i$ and $\bigcup_{j > j_i} Sm(c_j) \simeq \Sigma_i^c$.*

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