Forced orientation of graphs

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Abstract

The concept of forced orientation of graphs was introduced by G. Chartrand et al. in 1994. If, for a given assignment of directions to a subset S of the edges of a graph G, there exists an orientation of $E(G) \setminus S$, so that the resulting graph is strongly connected, then that given assignment is said to be extendible to a strong orientation of G. The forced strong orientation number $f_D(G)$, with respect to a strong orientation D of G, is the smallest cardinality among the subsets of $E(G)$ to which the assignment of orientations from D, can be uniquely extended to E. We use the term defining set instead of "forced orientation" to be consistent with similar concepts in other combinatorial objects. It is shown that any minimal strong orientation defining set is also smallest. We also study $\mathrm{Spec}(G)$, the spectrum of G, as the set of all possible values for $f_D(G)$, where D is taken over all strong orientations of G.

Key words: Forced orientation; defining set; strong orientation; algorithms; matroids.

1 Introduction and preliminaries

In this paper, we consider only connected graphs. The set of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively, or by V and E, when there is no ambiguity. For the notions not defined here, refer to [10]. An orientation of a graph G is a digraph D, with the same vertex set, whose underlying graph is G. A strong orientation is an orientation that is strongly connected, i.e., for every two vertices u and v there is a directed path from u to v and a directed path from v to u.

A partial orientation of an undirected graph G is a subset of the edges of an orientation of G. For a partial orientation F of G, we define G_F as the mixed graph whose underlying undirected graph is G and its set of directed edges is precisely F. A partial orientation F of G is called extendible if there is a strong orientation D of G that contains F. A partial orientation F is called a strong orientation forcing set or simply a forcing set for a strong orientation D of G, if D is the only strong orientation of G which contains F. A minimal forcing set is a forcing set containing no other forcing set as a proper subset.

Notions similar to forcing sets are studied under different names such as "defining sets" for combinatorial structures such as block designs [9] and graph colorings [5, 6, 7], and "critical sets" for latin squares [1, 4, 5]. In [3], Chartrand et al. introduced and studied this notion for orientations of graphs. Here we take on this last concept and investigate some of the remaining problems.

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A family of problems, analogous to those considered in this papers, can be introduced by replacing stronge orientaions with unilateral orientations in the defenitions given above. A unilateral orientation of a graph G, is an orientation of G in which for every pair of vertices $u, v \in V(G)$, there exist either a path from u to v, or one from v to u (or both). In [8] Pascovici considers the forced unilateral orientation number of graphs.

The smallest number of edges in any forcing set for a strong orientation D of G is called the forcing number of D, and is denoted by $f_D(G)$. We also define $\mathrm{Spec}(G)$, the spectrum of a graph G, as the set of all integers k for which G has a strong orientation of size k. The smallest and largest elements of $\mathrm{Spec}(G)$ are denoted by $f(G)$ (also known as the forcing number of G) and $F(G)$, respectively.

In [3] the following theorem is proved.

Theorem A [3]**.** *If G is a 2-edge-connected graph with n vertices and m edges, then $f(G) = m - n + 1$.*

In the present paper, we study $\mathrm{Spec}(G)$. In Section 2, some general results are established which will be used throughout the paper. In Section 3, the problem of minimal forcing sets is studied and our main result is proved which states that every minimal forcing set is also a smallest forcing set. Section 4 contains some results about $F(G)$. Finally, in section 5, the spectrum of some special graphs is studied.

2 General results

In this section we state some useful results about orientations of graphs and their extensions. The following theorem is a generalization of Robbins' theorem which states that every 2-edge-connected undirected graph has a strong orientation [2]. We need the following definition.

Definition. *A consistent path in a mixed graph G is a path in which the direction of every directed edge conforms with the direction of the path.*

Theorem 2.1 *Let G be a mixed graph. The following propositions are equivalent:*

(a) *The undirected edges of G can be oriented in such a way that the resulting digraph is strongly connected.*

(b) *The underlying undirected graph of G is 2-edge-connected and for every two vertices u and v, there is a consistent path from u to v and a consistent path from v to u.*

(c) *The underlying undirected graph of G is 2-edge-connected and there is no subset S of the vertices of G such that all of the edges in $[S, V(G) \setminus S]$ are directed from S to $V(G) \setminus S$.*

Proof. It is trivial that (a) implies (b) and (b) implies (c). We prove that (c) also implies (a). Suppose G has the property (c). To prove that G also has property (a), we use induction on k, the number of undirected edges of G.

If $k = 0$, the claim is trivial. Suppose that the theorem holds for every mixed graph with less than k undirected edges and let G be a mixed graph with k undirected edges which satisfies (c). Take any undirected edge uv of G. We prove that uv can be oriented in such a way that the resulting mixed graph still has the property (c), and then by induction hypothesis the statement is proved. If there exists a consistent path from u to v in $G - uv$, then we can orient the edge uv from v to u, and it is easy to verify that the resulting graph still has the property (c). Similarly, if there exists a consistent path from v to u, then we can orient uv from u to v. Therefore, if we prove that in $G - uv$, a consitent path exists either from u to v or from v to u, then the theorem will be proved.

To prove this claim let S_u be the set of all vertices x such that there is a consistent path from u to x in $G - uv$. Also, we define T_u as the set of all vertices x such that there is a consistent path from x to u in $G - uv$. S_v and T_v are defined similarly. It is evident from the definition of S_u that all of the edges in $[S_u, V(G) \setminus S_u]$ except uv are directed towards S_u, and the same property holds for S_v. Thus, the same property holds for $S_u \cup S_v$. But we know that G has the property (c). So, we should have $S_u \cup S_v = V(G)$. A similar argument shows that $T_u \cup T_v = V(G)$. If we do not have any consistent path neither from u to v, nor from v to u, then $S_u \cap T_v = S_v \cap T_u = \emptyset$. These equalities show that $V(G)$ is partitioned into two subsets $S_u = T_u$ and $S_v = T_v$, and the only edge between these sets is uv. But this contradicts the hypothesis that the underlying undirected graph of G is 2-edge-connected. ■

The proof of Theorem 2.1 leads us to the following definition.

Definition. *Let F be a partial orientation of G, and G_F denote the corresponding mixed graph. We say that an edge e of G is* forced *by F, if there is a cut $[S, V \setminus S]$ in G_F such that $e \in [S, V \setminus S]$ and all of the edges in $[S, V \setminus S]$, except e, are directed in the same direction.*

The following proposition provides an equivalent definition for an edge being forced by a partial orientation.

Proposition 2.2 *Let F be an extendible partial orientation of G and $e = uv$ be an edge in $E(G) \setminus F$. Then e is forced by F if and only if either there is no consistent path from u to v or from v to u in $G_F - e$.*

Proof. If e is forced by F, then for some $S \subset V$, $u \in S$ and $v \in V \setminus S$ and all of edges in $[S, V \setminus S]$, except e, are oriented by F in the same direction, say, without loss of generality, from S to $V \setminus S$. Then apparently, there is no consistent path in $G_F - e$ from v to u since every edge incident to $V \setminus S$ is directed towards it.

Conversely, suppose there is no consistent path from u to v in $G_F - e$. Let S be the set of all vertices of G to which there is a consistent path from u in $G_F - e$. Apparently $v \in V \setminus S$. Consider any edge xy with $x \in S$ and $y \in V \setminus S$. If F does not assign a direction to xy or assigns the direction from x to y, then the consistent path from u to x can be extended to a consistent path from u to y by adding xy to it. But then y must belong to S and this contradicts our choice of S. Thus every edge xy with $x \in S$ and $y \in V \setminus S$ must be oriented from y to x by F. ■

A very nice property of the forcing sets is their simultaneous "forcing" of the direction of every undirected edge of the graph. This is in contrast to the way most of the corresponding notions to forcing sets in other combinatorial contexts behave. For example, defining sets of graph coloring [5, 6, 7], do not necessarily force the color of every uncolored vertex at the same time and may instead only work in certain orders. The following theorem establishes this fact and is used in numerous places thorughout this paper.

Theorem 2.3 *An extendible partial orientation F of G is a strong orientation forcing set if and only if every edge $e \in E(G) \setminus F$ is forced by F.*

Proof. The "if" part is trivial. For the "only if" part, assume to the contrary that some edge uv in $E(G) \setminus F$ is not forced by F. By Proposition 2.2, there are consistent paths in $G_F - uv$ both from u to v and from v to u. Thus, if we orient uv in either direction, by Theorem 2.1 the resulting partial orientation can be extended into a strong orientation of G. But then, there is more than one way to extend F into a strong orientation. ■

3 Minimal forcing sets

In this section we study the properties of minimal forcing sets for any particular strong orientation of a graph. This leads into an efficient algorithm for finding a smallest forcing set for any given stronge orientation. First, we define a binary relation "$\preceq$" between the edges of a graph as follows.

Definition. *For any two edges e_1 and e_2 of a strongly connected digraph D, $e_1 \preceq e_2$ if every directed cycle C of D containing e_1 also contains e_2. Moreover, we write $e_1 \approx e_2$ if $e_1 \preceq e_2$ and $e_2 \preceq e_1$.*

The following proposition is trivial.

Proposition 3.1 *The relation $\preceq$ is a preorder, i.e. it is reflexive and transitive.*

This proposition implies that the relation $\approx$ is an equivalence relation and thus partitions the set of edges of D into some equivalence classes. These equivalence classes form a partial ordering under the relation $\preceq$. The following two lemmas give a characterization of these equivalence classes.

Lemma 3.2 *In a strongly connected digraph D we have $e_1 \preceq e_2$ if and only if there is a cut $[S, V \setminus S]$ such that e_1 is from S to $V \setminus S$, e_2 from $V \setminus S$ to S, and every other edge in the cut is from S to $V \setminus S$.*

Proof. The "if" part is trivial. For the "only if" part, let $e_1 = uv$ and suppose $e_1 \preceq e_2$. If there exists a path from v to u in $D - e_2$, this path together with e_1, would make a cycle containing e_1 but not e_2, contradicting to the assumption that $e_1 \preceq e_2$. Now, let S be the set of vertices that are *not* reachable from v in $D - e_2$. Then, $u \in S$ and $V \in V \setminus S$, and every edge in $[S, V \setminus S]$ except e_2, is directed from S to $V \setminus S$. On the other hand, since D is strongly connected, e_2 must be directed from $V \setminus S$ to S. ■

Lemma 3.3 *Let D be a strongly connected digraph. For any two edges e_1 and e_2 in D, $e_1 \approx e_2$ if and only if $\{e_1, e_2\}$ is a cut set.*

Proof. By Lemma 3.2 we know that there exits a cut $[S, V \setminus S]$ containing both e_1 and e_2 such that all of its edges except e_2 are directed from S to $V \setminus S$. We claim that $[S, V \setminus S]$ does not contain any edges other than e_1 and e_2. Assume to the contrary that there exist an edge uv in $[S, V \setminus S]$ other than e_1 and e_2. Strong connectivity of D implies that there is a path P_1 from the head of e_2 to u. This path cannot pass through $V \setminus S$ since the only edge from $V \setminus S$ to S is e_2. Similarly, there is a path P_2 in $V \setminus S$ from v to the tail of e_2. The two paths P_1 and P_2 along with e_2 and uv form a cycle which contains e_2, but not e_1 and this is a contradiction. ■

Corollary 3.4 *Every pair of edges from the same equivalence class of the $\approx$ relation form a cut set.*

Lemma 3.5 *Let e_1 and e_2 be two edges in a strongly connected digraph D such that $e_2 \not\preceq e_1$. If F is a forcing set for D containing e_2 but not e_1, then $F - e_2$ still forces the direction of e_1.*

Proof. By Theorem 2.3, there is a cut $[S, V \setminus S]$ containing e_1, such that every edge of this cut, except e_1 belongs to F and is directed from S to $V \setminus S$ while e_1 is directed from $V \setminus S$ to S. If $e_2 \notin [S, V \setminus S]$ we are done. Otherwise, if $e_2 \in [S, V \setminus S]$, then by Lemma 3.2 we obtain $e_2 \preceq e_1$, a contradiction. ■

Corollary 3.6 *If we remove an edge e from a forcing set F for a strongly connected digraph D, then the set of edges that are not forced by $F - e$ is a subset of the set $\{x \in E \mid e \preceq x\}$.*

Lemma 3.7 *Let D be a strongly connected digraph and e_1 and e_2 be two edges of D such that $e_1 \preceq e_2$. If F is a forcing set for D, then $F \cup \{e_1\} - e_2$ is also a forcing set for D.*

Proof. It is sufficient to prove that $F \cup \{e_1\} - e_2$ forces the direction of e_2. Assume to the contrary that this does not happen. By Lemma 3.2, we know that there is a cut $[S, V \setminus S]$, containing e_1 and e_2, so that all of its edges except e_2 are directed toward S. Let $e \in [S, V \setminus S]$ be an edge other than e_2. If $F - e_2$ does not force the direction of e, then by Corollary 3.6, we have $e_2 \preceq e$. On the other hand, by Lemma 3.2, we have $e \preceq e_2$. This means that $e \approx e_2$. An argument like the one in the proof of Lemma 3.3 shows that $[S, V \setminus S] = \{e, e_2\}$. But we knew that $e_1 \in [S, V \setminus S]$. So e cannot be any edge other than e_1. Thus every edge in $[S, V \setminus S] - e_2$ is forced by $F \cup \{e_1\} - e_2$. This, together with the fact that e_2 is the only edge in $[S, V \setminus S]$ directed toward $V \setminus S$, show that the direction of e_2 is forced by the set $F \cup \{e_1\} - e_2$. Thus, $F \cup \{e_1\} - e_2$ is a forcing set. ■

Lemma 3.8 *Let D be a strongly connected digraph and F be an arbitrary forcing set for D. Then, any minimal equivalence class under the relation $\preceq$, must have at least one edge in F.*

Proof. Assume to the contrary that C is a minimal equivalence class of the relation $\preceq$ and none of its elements is in the forcing set F. Let e be an edge in C. Then e must be forced by F. So, by Theorem 2.3, there exists a cut $[S, V \setminus S]$ which contains e and all of its edges except e are directed toward S and are in F. Let e' be an edge in $[S, V \setminus S] - e$. By Lemma 3.2, $e' \preceq e$. Also, we know that $e' \notin C$, because $e' \in F$ and $F \cap C = \emptyset$. This contradicts with minimality of the class C under the relation $\preceq$. ■

The following theorem characterizes the set of all minimal forcing sets of a given strong orientation.

Theorem 3.9 *A subset F of the edges of a strongly connected digraph D is a minimal forcing set for D if and only if F contains exactly one edge from each equivalence class of the relation $\approx$ which is minimal under the relation $\preceq$.*

Proof. Proof is trivial using Lemma 3.7 and Lemma 3.8. ■

Now we are in the position to state our main result of this section which follows immediately from Theorem 3.9.

Theorem 3.10 *Let D be a strong orientation of a graph G. Then all minimal forcing sets for D have the same size.*

Note that Theorem 3.10 yields an obvious efficient algorithm for constructing smallest forcing sets for a given strong orientation of a graph. Also, using the above results, it is easy to prove the following theorem.

Theorem 3.11 *For every strongly connected digraph D, the family of subsets of the edges of D whose complement is a forcing set for D, is a matroid.*

Note that by the theorem above a matroid can be assigned to every strongly connected digraph. A natural question arises that whether this matroid has any relation with the other known matroids assigned to graphs.

4 Properties of $F(G)$

In this section, we study the properties of $F(G)$, the largest element of Spec(G). Theorem A establishes a simple formula for $f(G)$, but for $F(G)$ we do not have any such formula. For every graph G with n vertices and m edges, we know that $m - n + 1 \leq F(G) \leq m$. In this section, we characterize the graphs for which equality holds in the right hand side inequality.

Lemma 4.1 *Let D be a strong orientation of an undirected graph G with m edges. A necessary and sufficient condition for D such that $f_D(G) = m$ is that for every edge e, $D - e$ is a strongly connected digraph.*

Proof. To prove sufficiency, assume that in contrary there is a forcing set F of size less than m for D. This means that there is an edge e which does not belong to F. But, we know that $D - e$ is strongly connected and thus e is not forced by F. This is in contradiction with Theorem 2.3.

For the necessity, assume that there is an edge e in D, such that $D-e$ is not strongly connected. Therefore, there is no path in $D - e$ from one end of e to the other end. But this means that e is forced by $D - e$. So, $D - e$ is a forcing set of size less than m for D. ■

In [10], a digraph D is called i-strongly connected, if for any set S of $i - 1$ edges of D, the graph $D \setminus S$ is strongly connected. By this notation, Lemma 4.1 states that $f_D(G) = m$ if and only if D is a 2-strongly connected orientation of G. The next theorem gives a necessary and sufficient condition for a graph to have a 2-strong orientation.

Theorem 4.2 *For a graph G with m edges we have $F(G) = m$ if and only if G is a 4-edge-connected graph.*

Proof. Assume that G is 4-edge-connected. Then by a theorem of Nash-Williams (see [10] page 154) G has a 2-strong orientation D. Therefore by Lemma 4.1, $f_D(G) = m$. Thus, $F(G) = m$.

Now assume that $F(G) = m$, and in contrary G has a cut set $[S, V - S]$ of size 3 or less. Let D be a strong orientation of G for which $f_D(G) = m$. In D there are two possibilities for the direction of the edges of this cut (from S to $V \setminus S$, or from $V \setminus S$ to S). Thus, one of these cases consists of exactly one edge, say e. This means that $D-e$ is not strongly connected, and therefore, e is forced by $D - e$. So, by Lemma 4.1, $f_D(G) < m$, which is a contradiction. ■

5 The spectrum of some special graphs

In this section we find Spec(G) for some special graphs.

Theorem 5.1 Spec$(K_3) = \{1\}$, Spec$(K_4) = \{3\}$, *and for* $n \geq 5$, Spec$(K_n) = \{m - n + 1, \ldots, m\}$, *where m is the number of edges of K_n.*

Proof. When G is K_3 or K_4, it is easy to see that there is a unique strong orientation of G, up to isomorphism. If $n \geq 5$, let $V(K_n) = \{1, 2, \ldots, n\}$. For every i, $m - n + 1 \leq i \leq m$, we construct a strong orientation D_i of K_n with $f_{D_i}(K_n) = i$ as follows:

- If $m-n+1 \leq i \leq m-n+4$, let S_i be the following set of edges of K_n:

$$S_i = \begin{cases} \{\{j,j+1\} : 1 \leq j < n\} & \text{if } i = m-n+1 \\ \{\{j,j+1\} : 1 \leq j < n\} \cup \{\{n-2,n\}\} & \text{if } i = m-n+2 \\ \{\{j,j+1\} : 1 \leq j < n\} \cup \{\{n-2,n\},\{n-3,n-1\}\} & \text{if } i = m-n+3 \\ \{\{j,j+1\} : 1 \leq j < n\} \cup \{\{n-3,n\},\{n-3,n-1\}\} & \text{if } i = m-n+4 \end{cases}$$

  We direct the edges in S_i from the lower numbered vertex to the higher numbered vertex, and every other edge of K_n from the higher numbered vertex to the lower numbered vertex. Let D_i be the resulting orientation.

- If $m-n+5 \leq i \leq m$, orient the edges between the vertices $m-i+1$, $m-i+2$, …, and n in such a way that the digraph induced by this set is 2-strongly connected. Also, for each $1 \leq j < m-i$ orient the edge between the vertices j and $j+1$ from j to $j+1$, and orient the remaining edges from the higher numbered vertex to the lower numbered vertex. Let D_i be the resulting orientation.

In each case, it is not difficult to see that the directions of the edges in the set $F_i = \{\{j,j+1\} : 1 \leq j \leq m-i\}$ are forced by the edges in $E(K_n) - F_i$, and $E(K_n) - F_i$ is in fact a minimal (and therefore, by Theorem 3.10, a smallest) forcing set of size i for D_i. ■

In the next two theorems we discuss the spectrum of some complete bipartite graphs.

Theorem 5.2 $\mathrm{Spec}(K_{2,2}) = \{1\}$*; and for* $n > 2$, $\mathrm{Spec}(K_{2,n}) = \{n-1, n\}$.

Proof. The graph $K_{2,n}$ consists of two vertices u and v, connected by n paths of length two. Therefore, any strong orientation of $K_{2,n}$ must be an orientation in which i paths are oriented from u to v, and $n-i$ paths from v to u, where $0 < i < n$. It is easy to verify (using Theorem 3.10) that the smallest size of any forcing set for this orientation is n, when $1 < i < n-1$, and $n-1$, otherwise. ■

Theorem 5.3 $\mathrm{Spec}(K_{3,3}) = \{4, 6\}$*; and for* $n \geq 4$, $\mathrm{Spec}(K_{3,n}) = \{2n-2, 2n-1, 2n\}$.

Proof. For $\mathrm{Spec}(K_{3,3})$ the proof is easy. So, we assume that $n \geq 4$. First observation is that if D is a strong orientation of $K_{3,n}$ and F is a smallest forcing set for D, then $|F| \leq 2n$. To see this, note that as in the proof of Theorem 4.2 if v is a vertex in the second part of $K_{3,n}$ (which has n vertices), then at least one of the edges incident to v is not in F. Thus, the size of F is at most $3n - n = 2n$.

Also, it is clear from Theorem A that $|F| \geq 2n-2$ and there exists a strong orientation D for which $|F| = 2n-2$. Now we show that if $n \geq 4$, for every $i \in \{2n-1, 2n\}$, there is an orientation D_i of $K_{3,n}$ such that $f_{D_i}(K_{3,n}) = i$. Let $\{a, b, c\}$ and $\{1, 2, \ldots, n\}$ be the set of vertices in the first and the second part of $K_{3,n}$, respectively, and S_i be the following set of edges.

$$S_i = \begin{cases} \{\{a,i\} : 2 \leq i \leq n\} \cup \{\{b,1\},\{c,1\},\{c,2\}\} & \text{if } i = 2n-1 \\ \{\{a,i\} : 3 \leq i \leq n\} \cup \{\{b,1\},\{b,2\},\{c,1\},\{c,2\}\} & \text{if } i = 2n \end{cases}$$

Let D_i be the orientation that is obtained by directing the edges of S_i toward the second part, and the remaining edges toward the first part. It is not difficult to see (using Theorem 3.10) that, for each i, the set of edges $E(K_{3,n}) - F_i$, where F_i is defined by

$$F_i = \begin{cases} \{\{a,i\} : 3 \leq i \leq n\} \cup \{\{1,a\},\{b,1\},\{2,b\}\} & \text{if } i = 2n-1 \\ \{\{a,i\} : 3 \leq i \leq n\} \cup \{\{1,a\},\{2,a\}\} & \text{if } i = 2n \end{cases}$$

is a smallest forcing set of size i for D_i. ■

The graph $K_{3,3}$ shows that there are some graphs whose spectrum is not a set of successive integers. It will be interesting to characterize such graphs.

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