Medial Axis Approximation
and
Unstable Flow Complex

Bardia Sadri

joint work with
Joachim Giesen and Edgar Ramos

Medial Axis
of
Shapes and Surfaces

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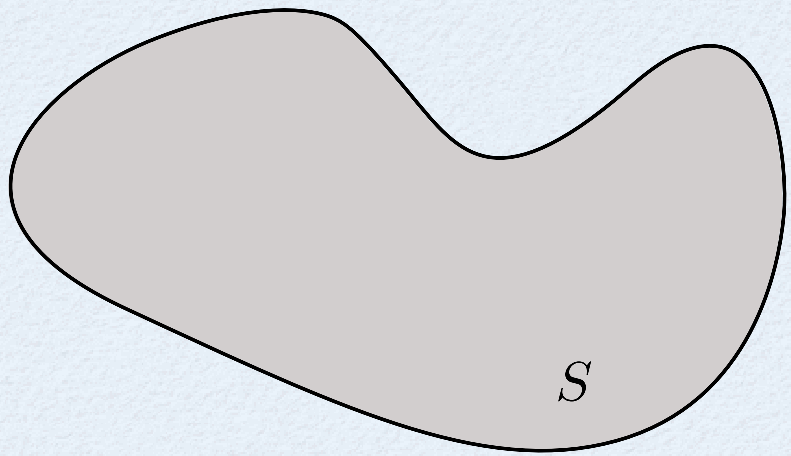

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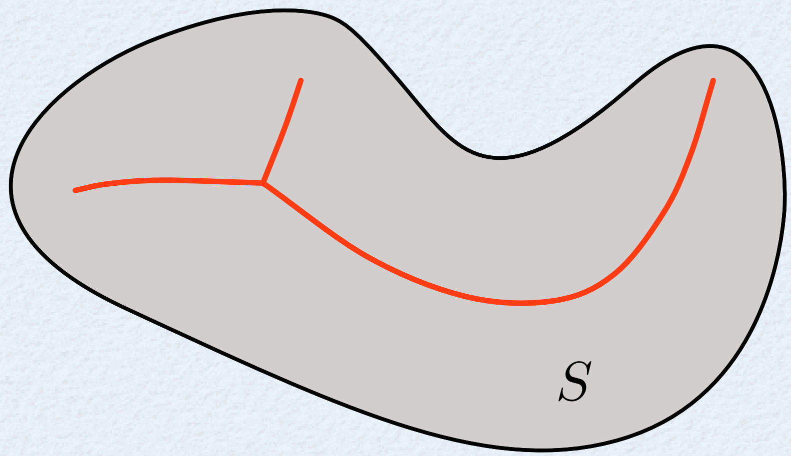

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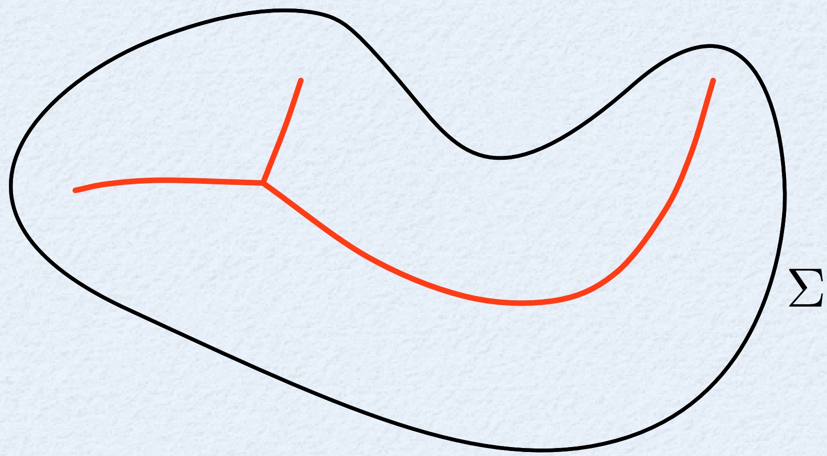

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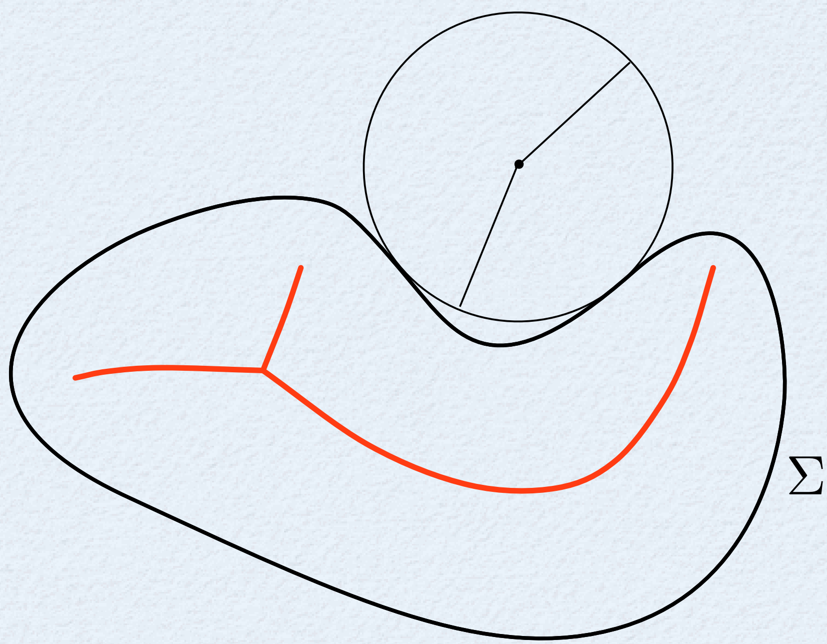

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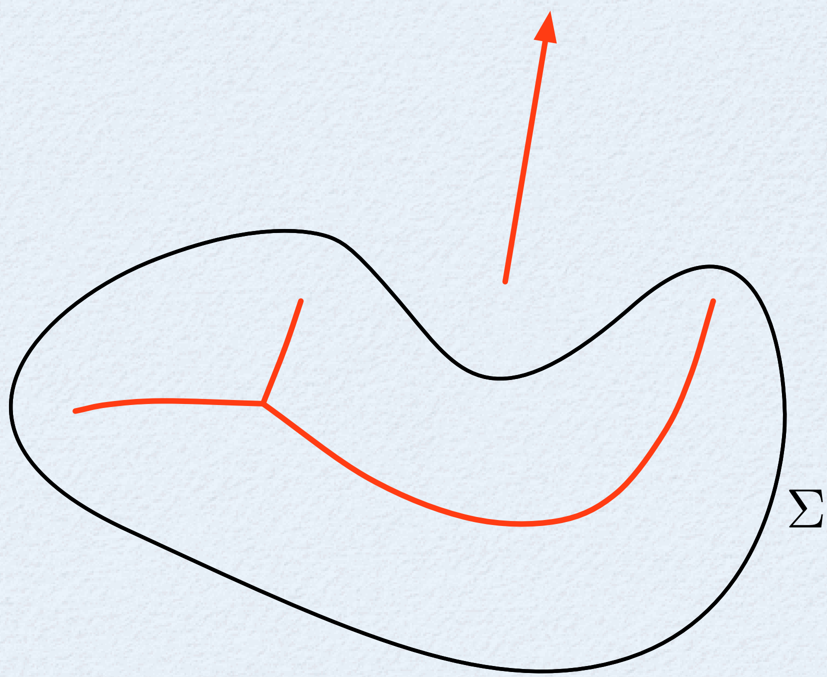
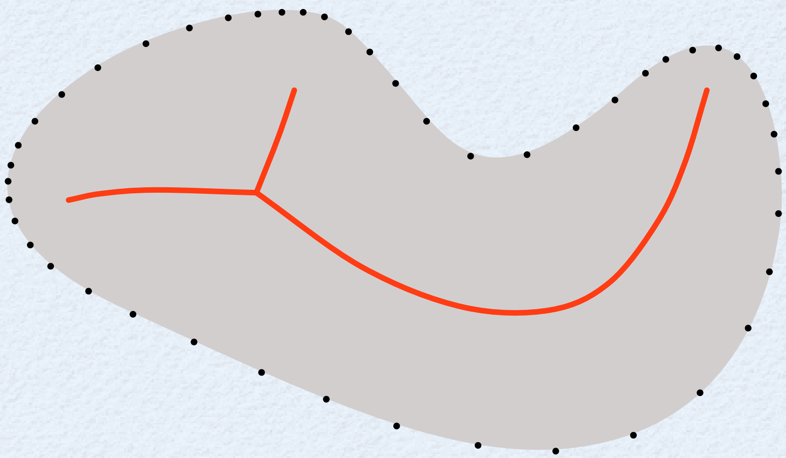


The medial axis of a surface Σ is the union of medial axes of all components of $\mathbb{R}^n \setminus \Sigma$.

Problem of Medial Axis Approximation

Given a sample of the surface enclosing a shape, we want to approxiamate the MA of shape geometrically and capture its topology.



Theorem. [Lieutier'03] Any bounded open subset of $\mathbb{R}^n$ has the same homotopy type as its medial axis.

◇ Applications: shape analysis, motion planning, mesh partitioning, ...

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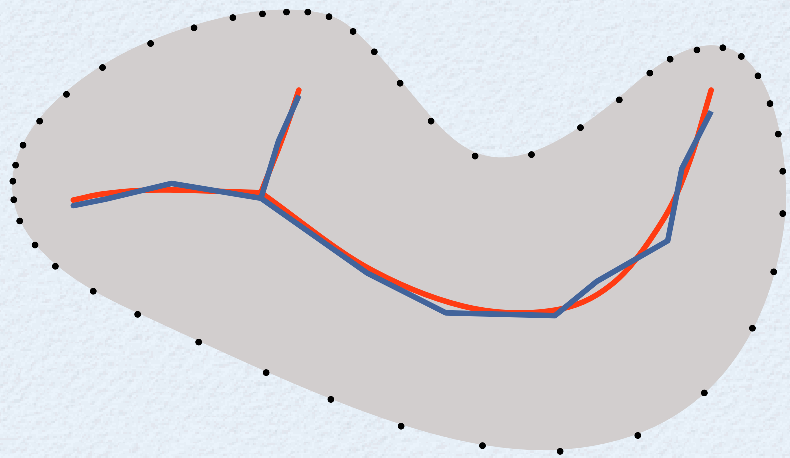

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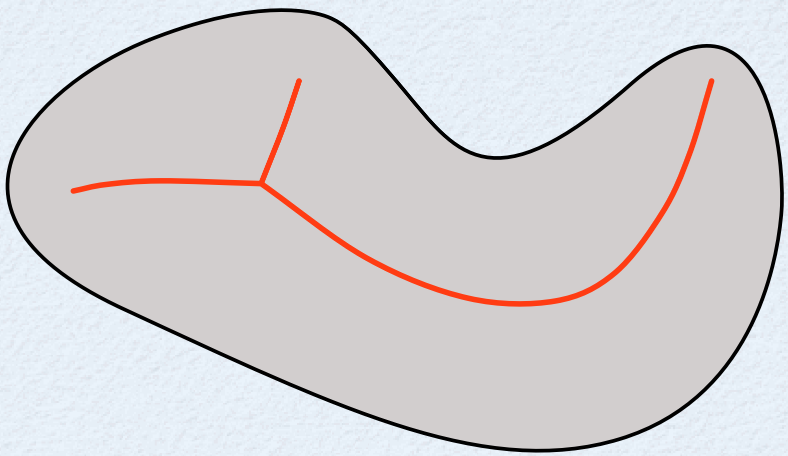
Challenge in Geometric Approximation

A small change in S can keep a sample valid but change $M(S)$ dramatically.



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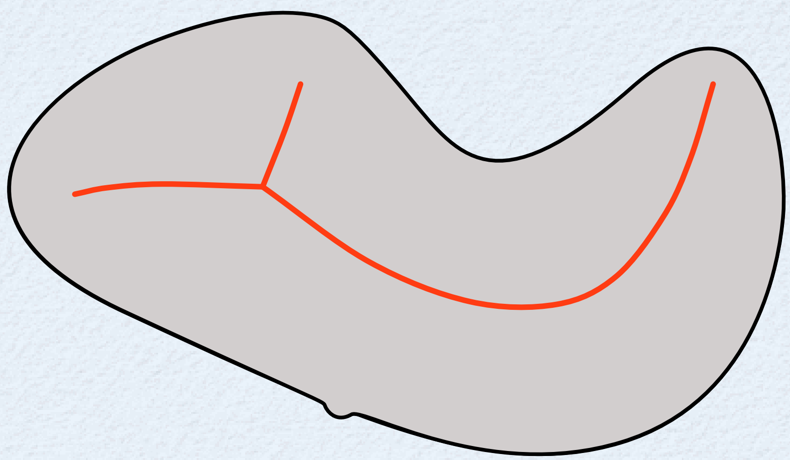

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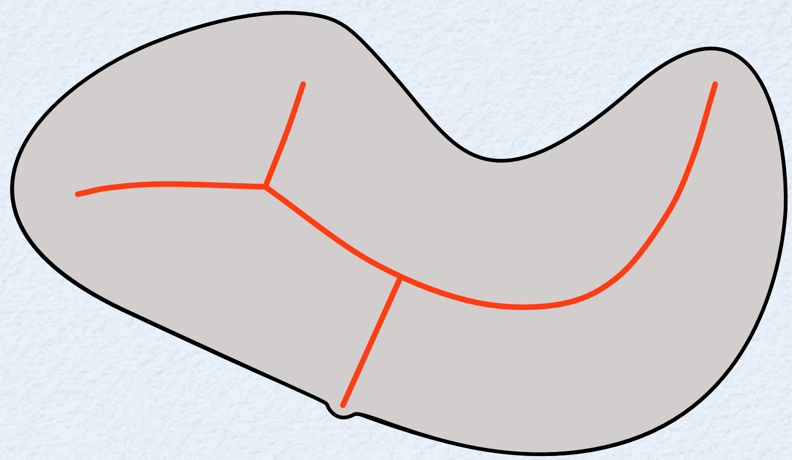

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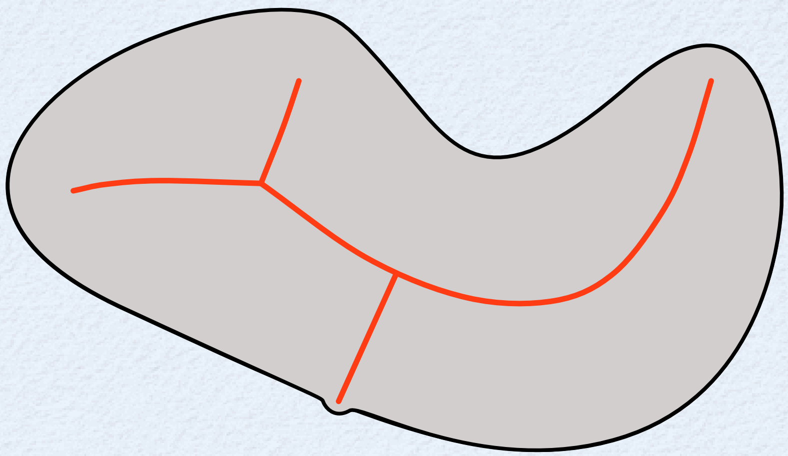
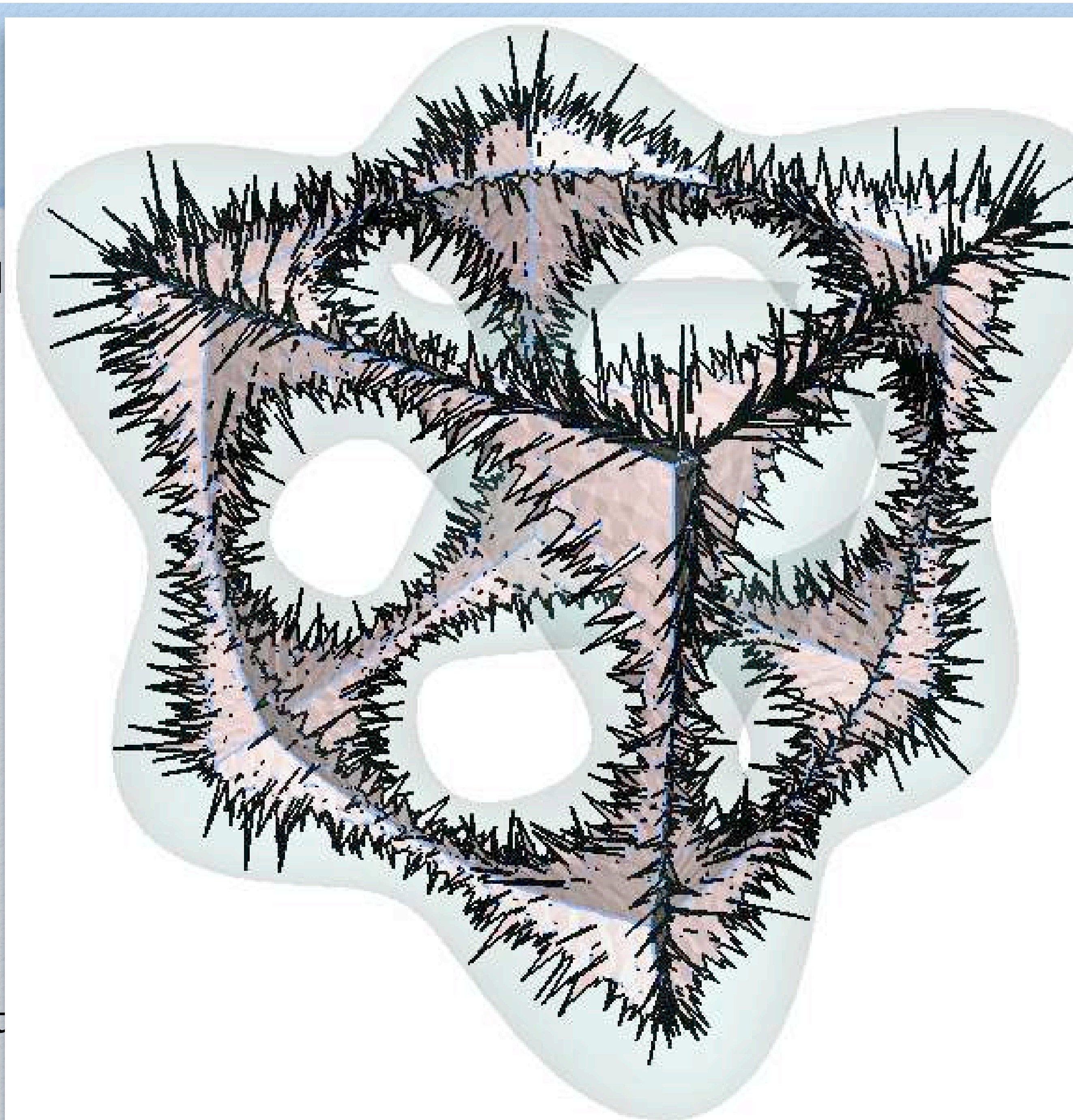


In practice, a filtered medial axis can be more interesting.

A small … matically.



Picture courtesy of Attali, Boissonnat, & Edelsbrunner

In pract…

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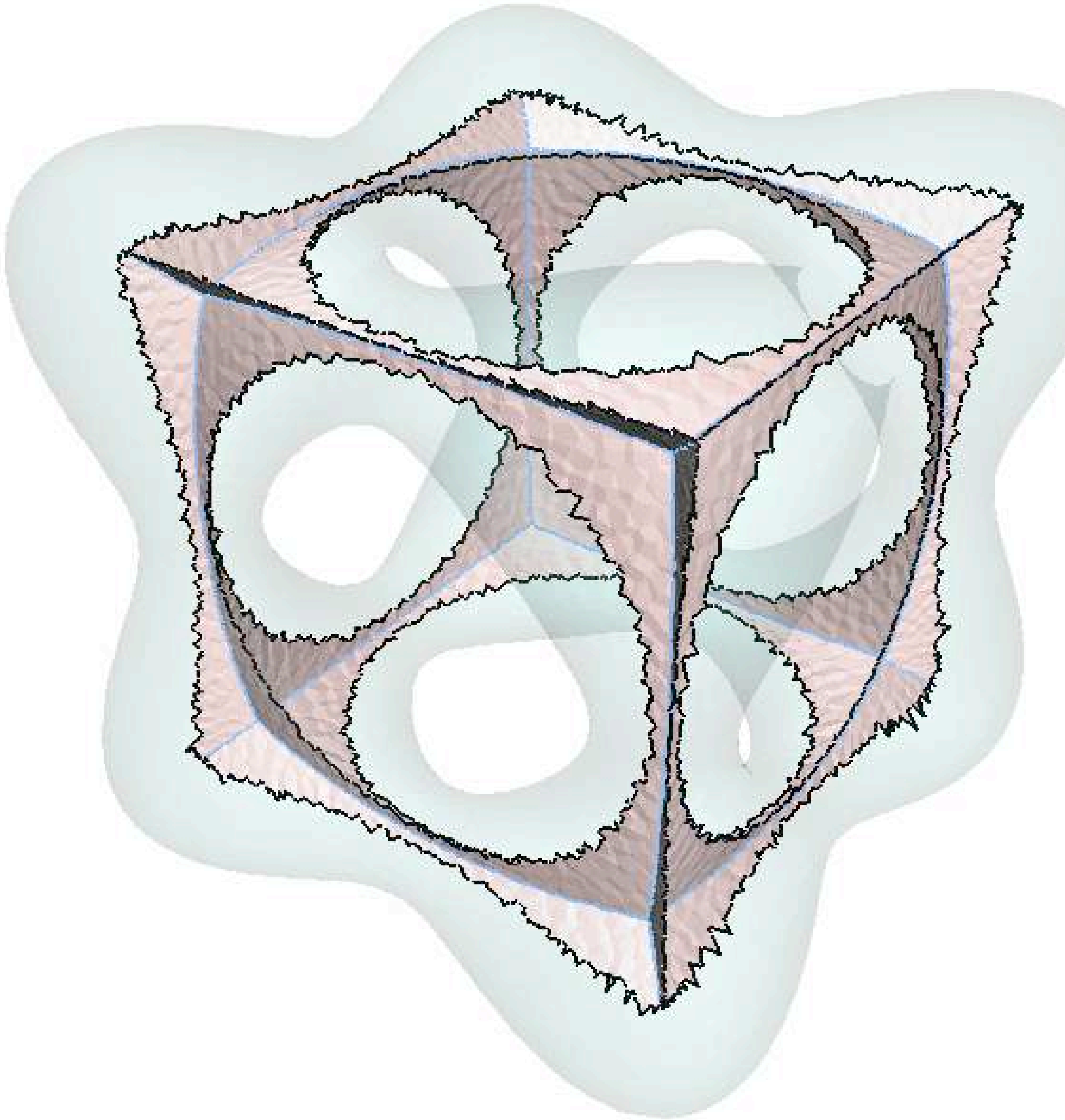

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In pract

Some History on Medial Axis Approximation

◇ Exact Methods: for limited classes of shapes

- [Culver, Keyser, Manocha '04] for Polytopes

◇ Voronoi Filtering:

- [Amenta, Bern '99] 2d
- [Amenta, Choi, Kolluri '01] Power-Crust
- [Boissonnat, Cazals '02]
- [Dey, Zhao '04]
- [Lieutier, Chazal '05] λ-medial axis

◇ Other:

- Thinning Methods
- Grid Methods
- PDE Methods
- ...

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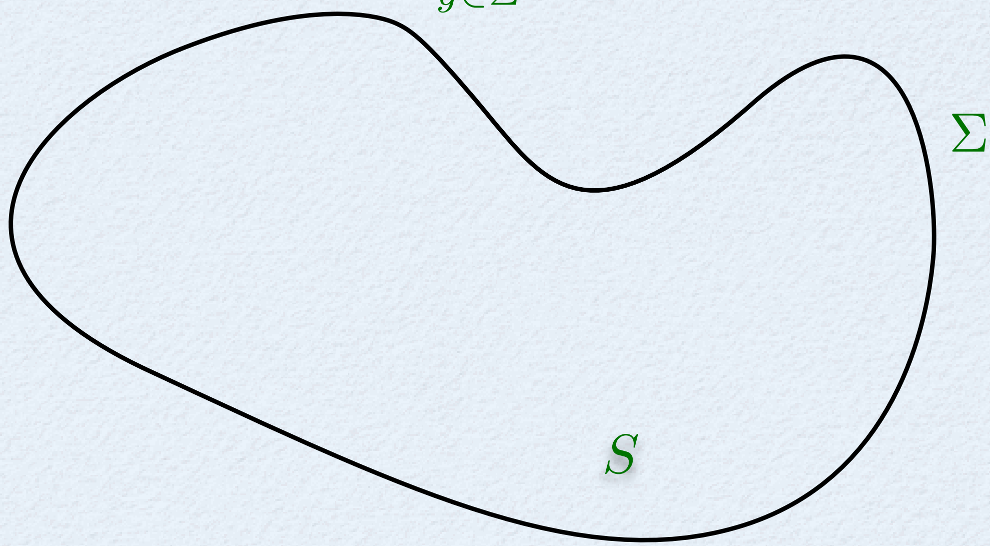
Our Contributions

- Separating geometry and topology:
  We introduce the MA core that captures the topology and can be used to topologically repair other geometric approximation methods.
- We use the (unstable) flow complex to do this for the first time.

Our Approach: Distance Functions

The key is to look at the distance function h induced by Σ:

$$h(x) := \inf_{y \in \Sigma} \|x - y\|$$



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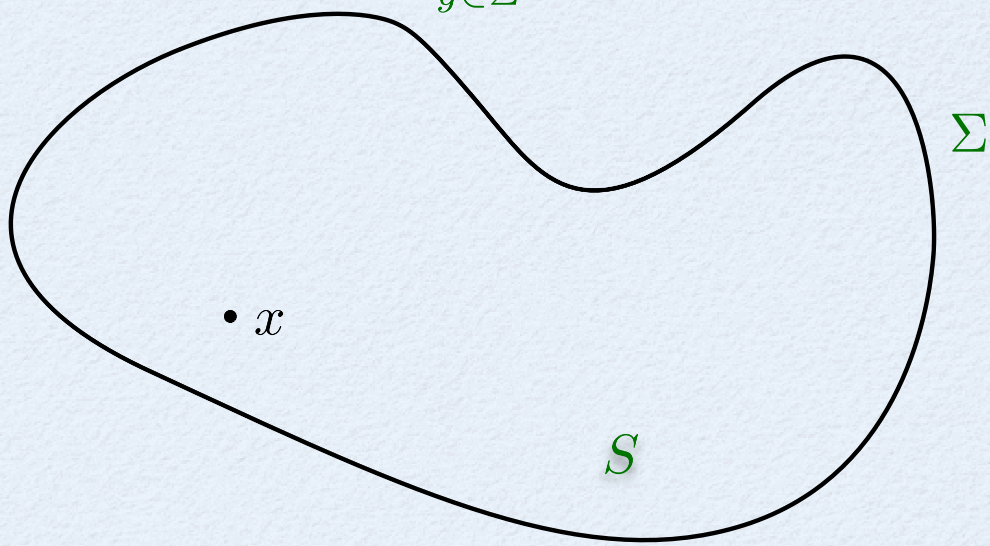

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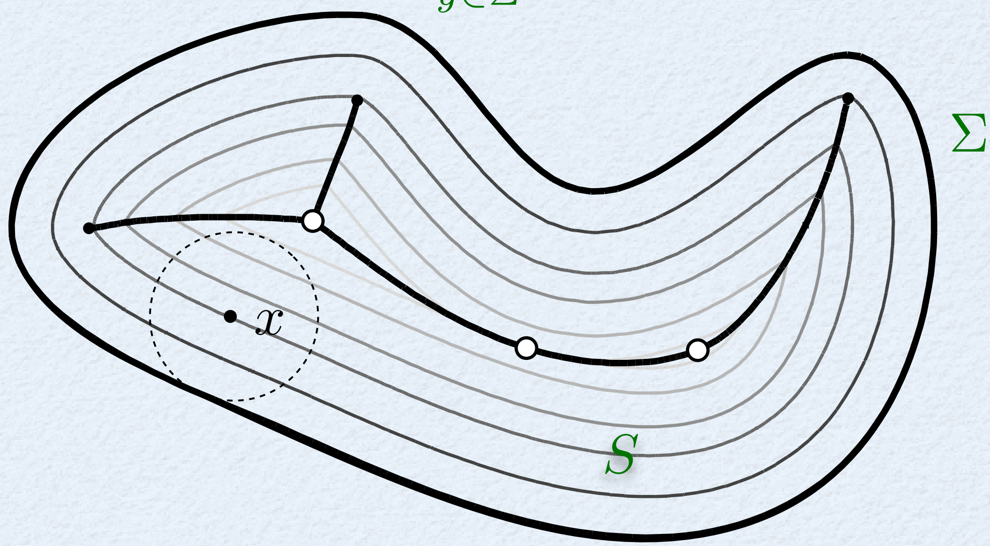

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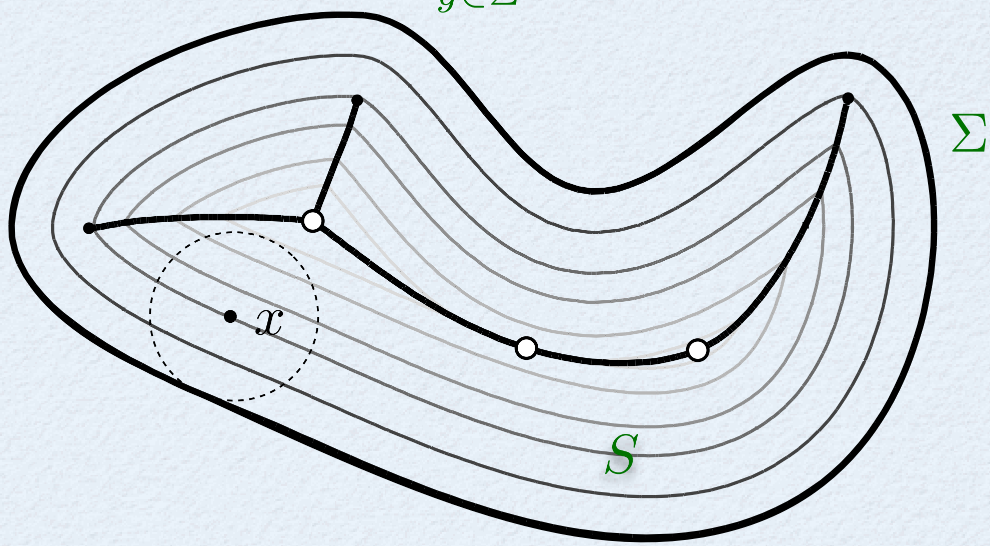

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- $M(\Sigma) \cup \Sigma = \{$points where h is not differentiable$\}$

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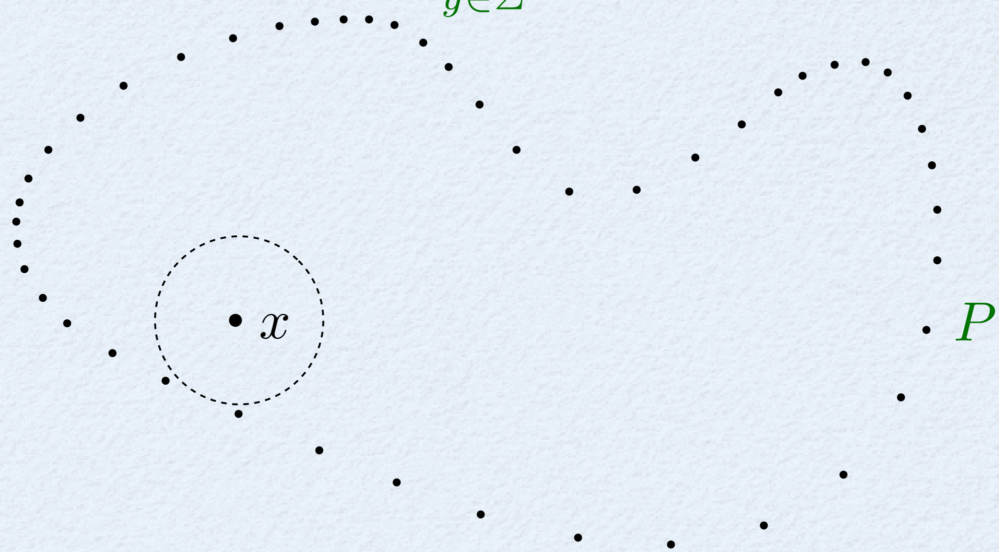

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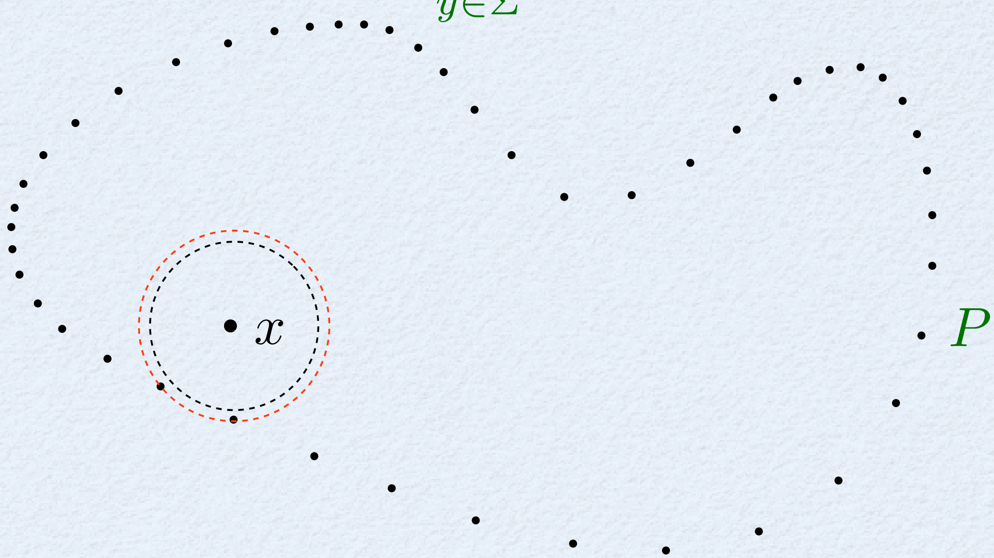

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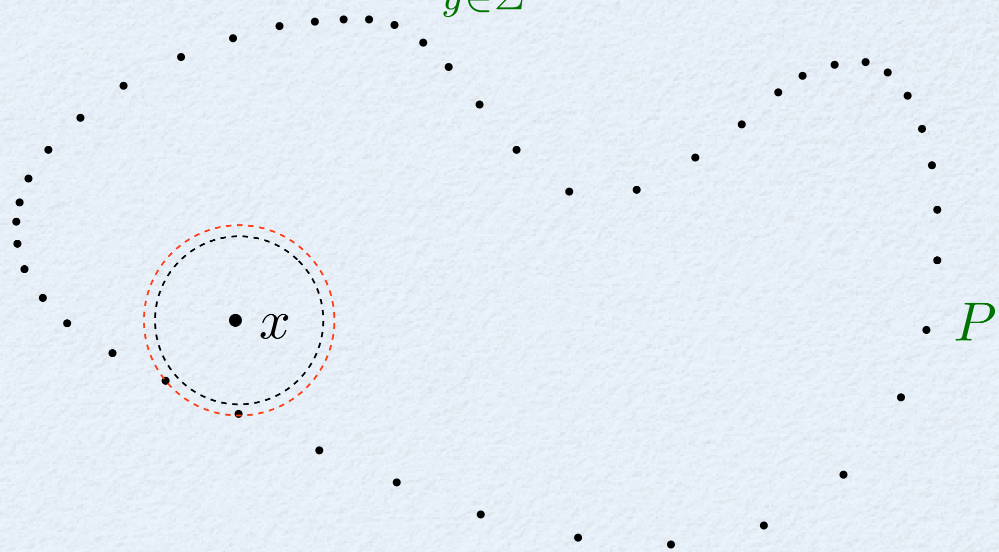
$$h(x) := \inf_{y \in \Sigma} \|x - y\|$$ continuous



- $\Sigma = h^{-1}(0)$
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$$\tilde{h}(x) = \min_{p \in P} \|x - p\|$$ discrete

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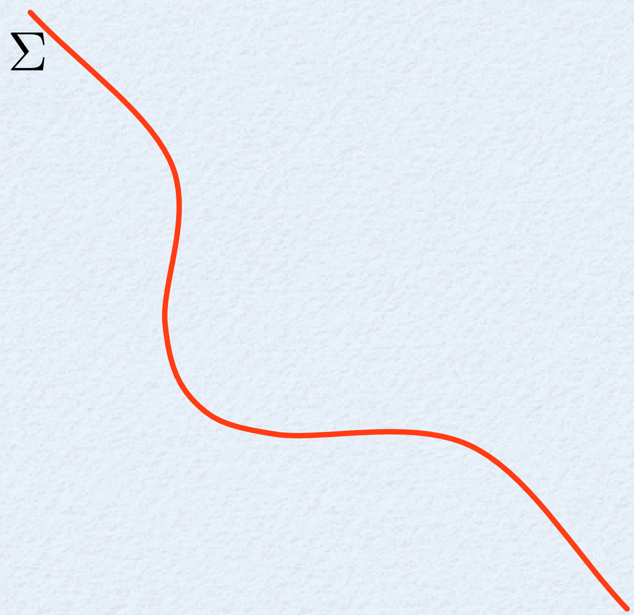
or
Generalizing the Gradient

A steepest ascent vector field v (or $\tilde{v}$) is defined everywhere that extends the gradient.



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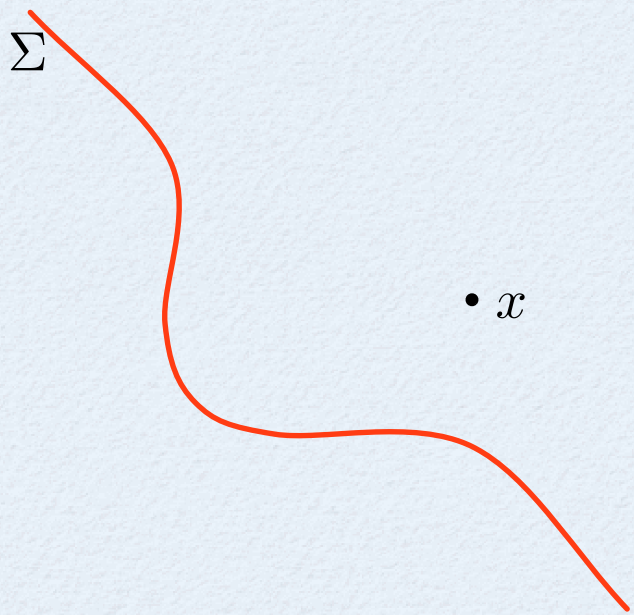

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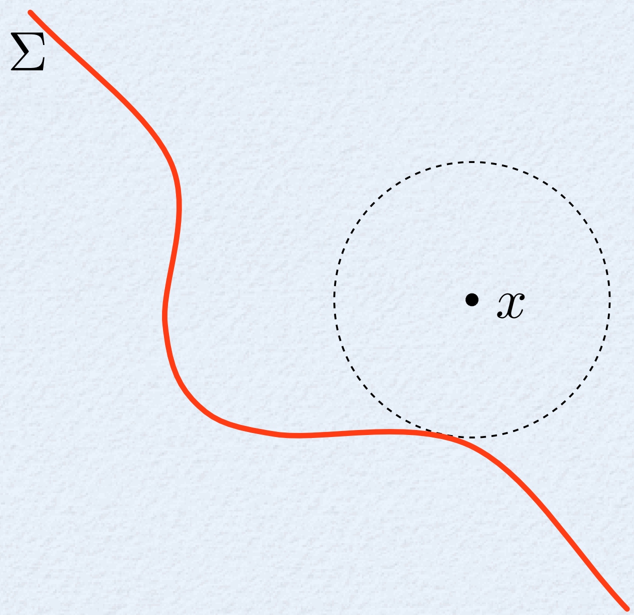

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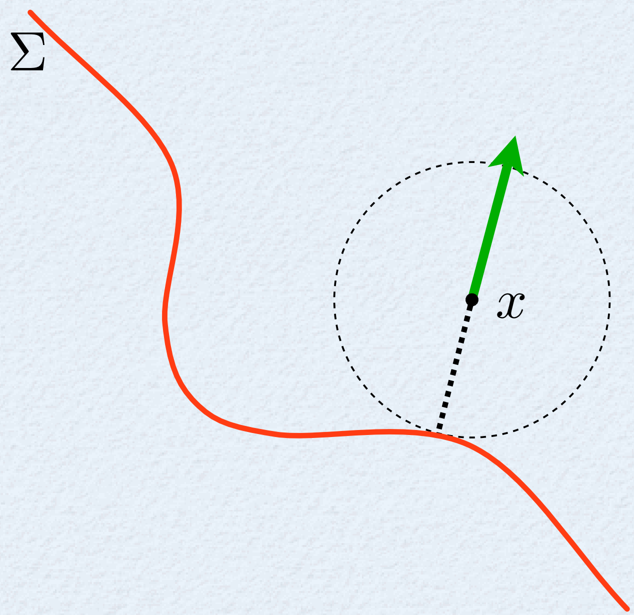

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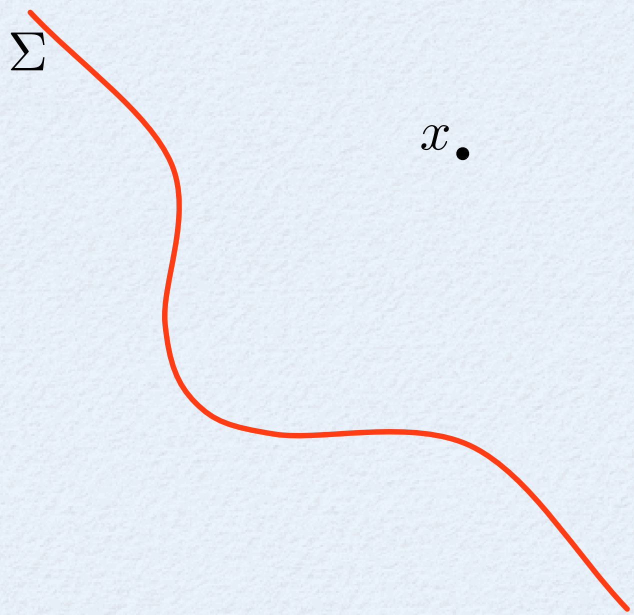

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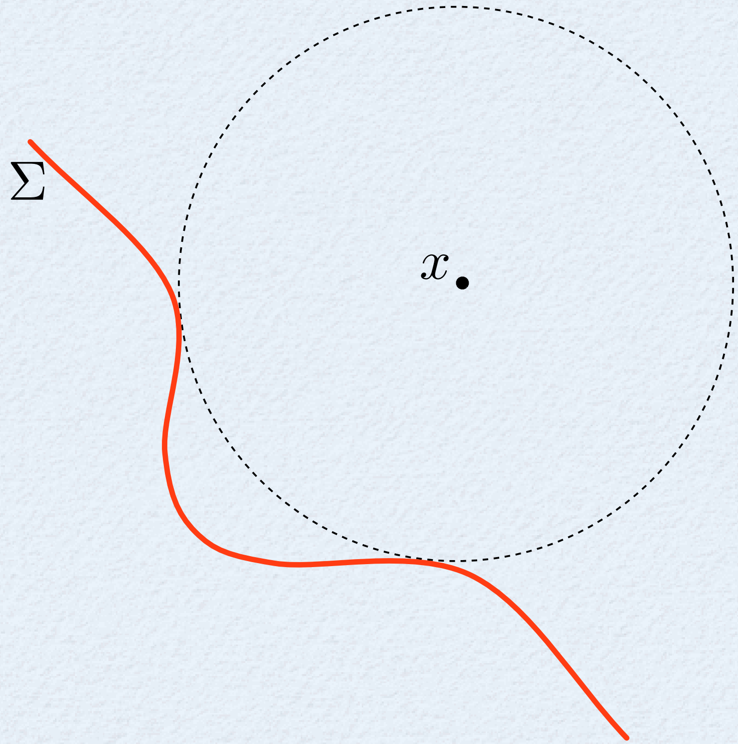

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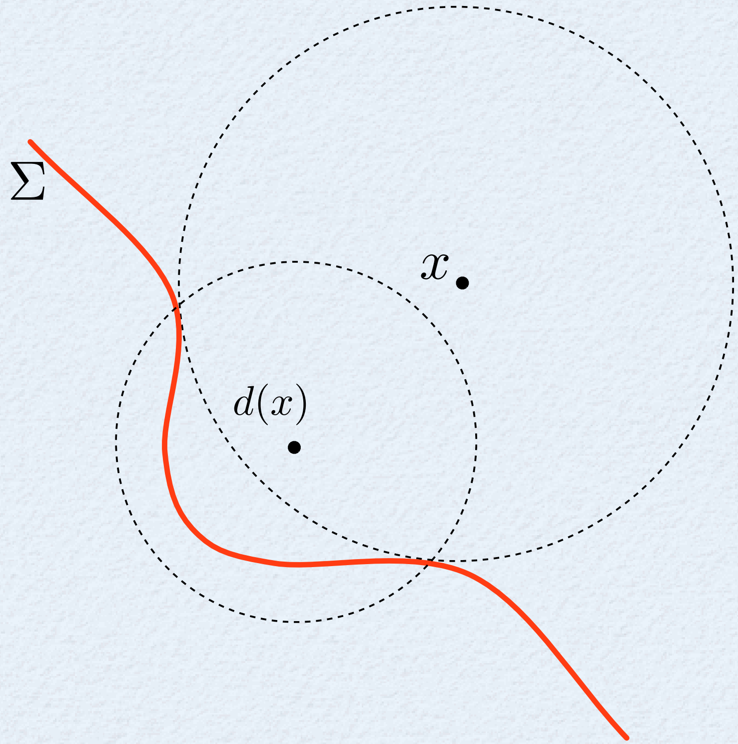

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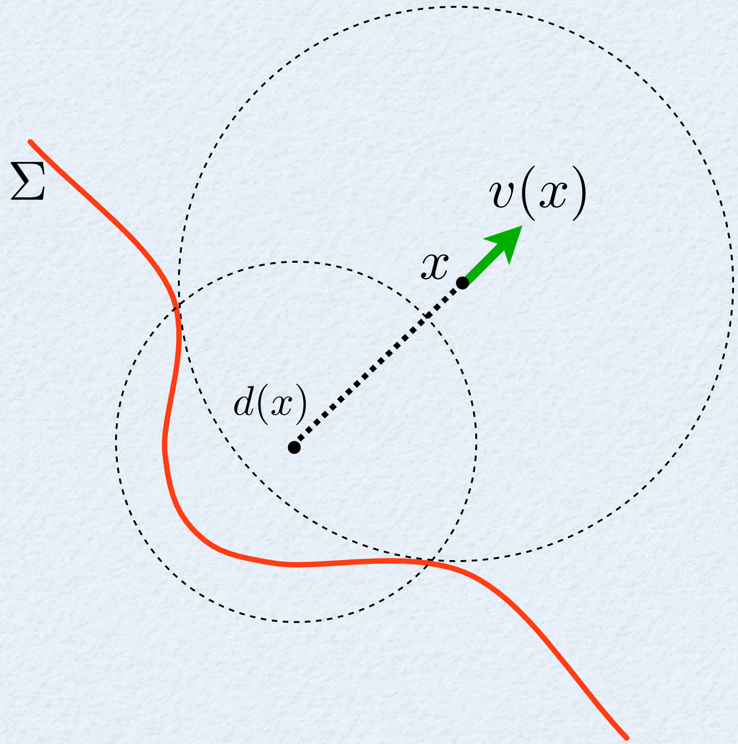

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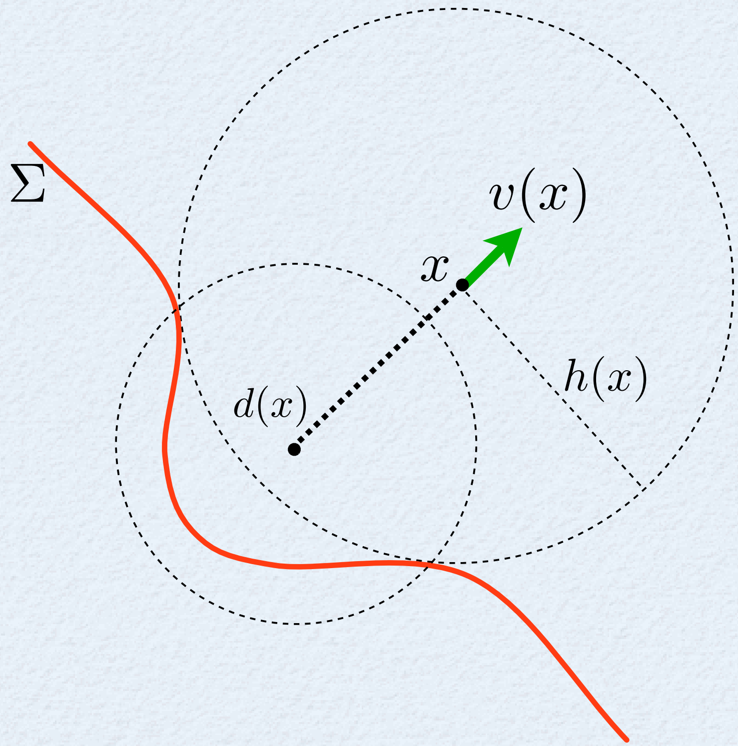

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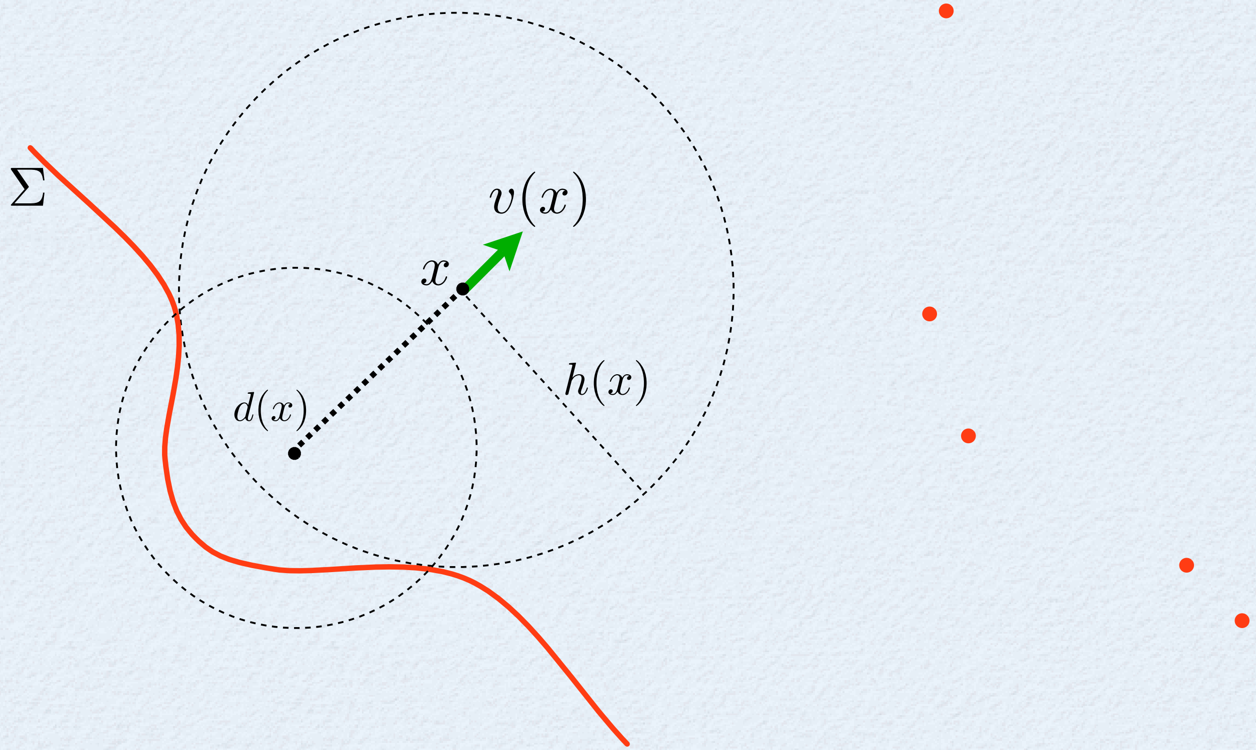

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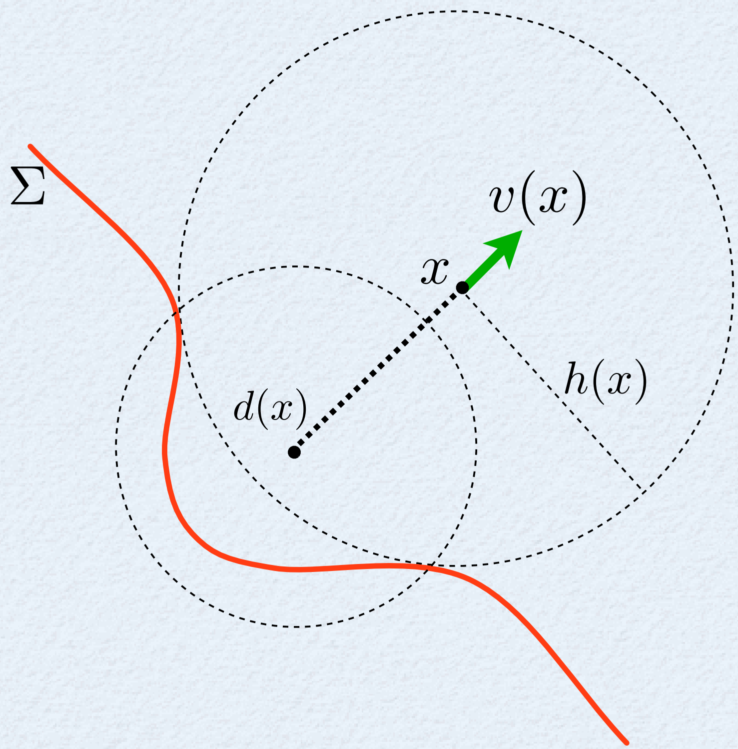

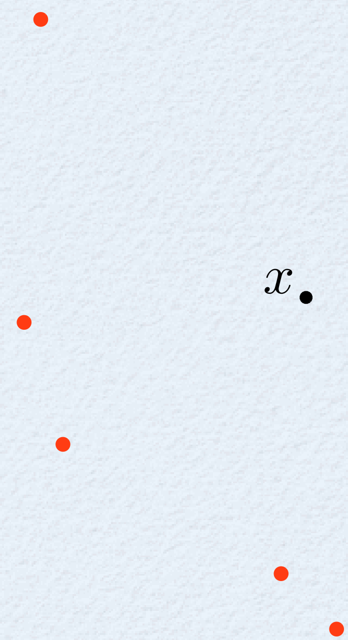

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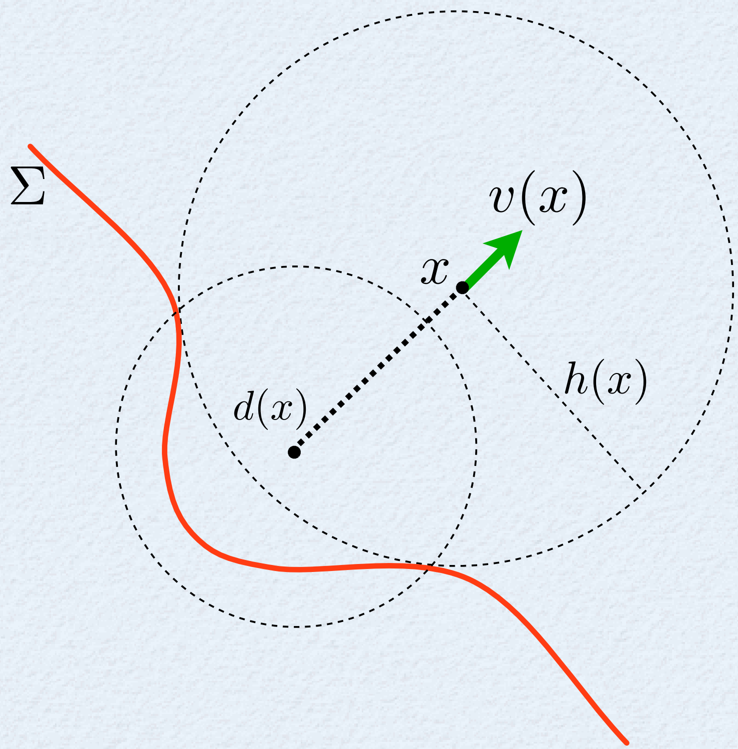

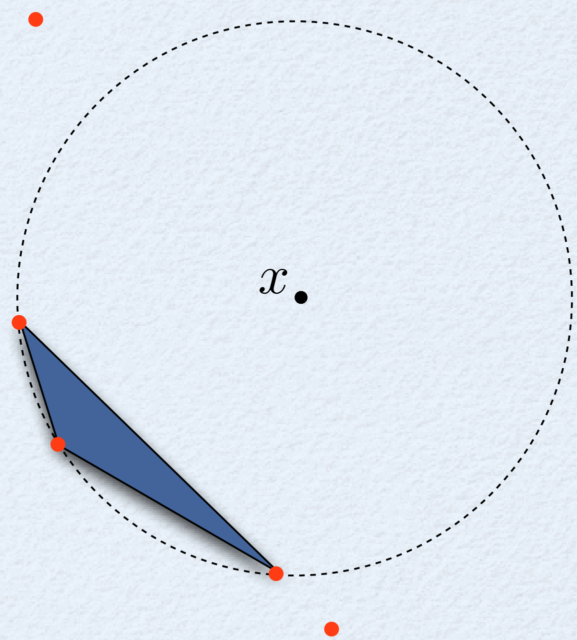

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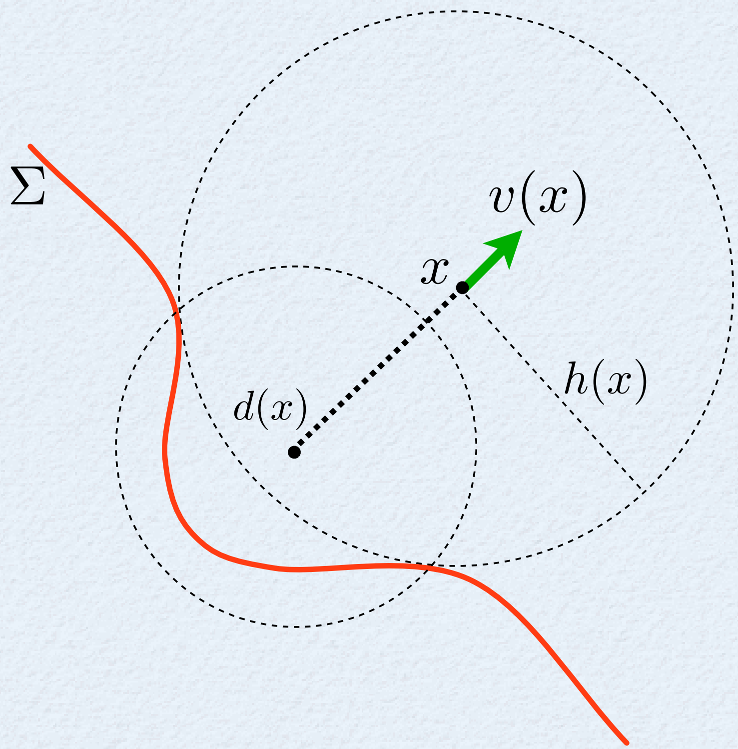

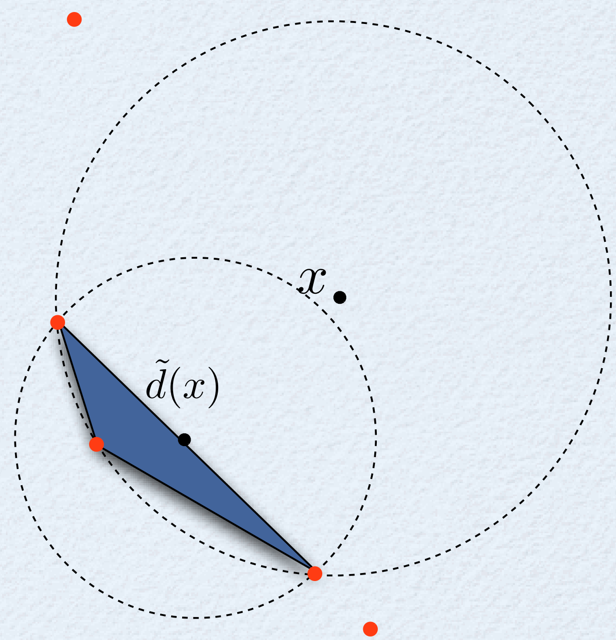

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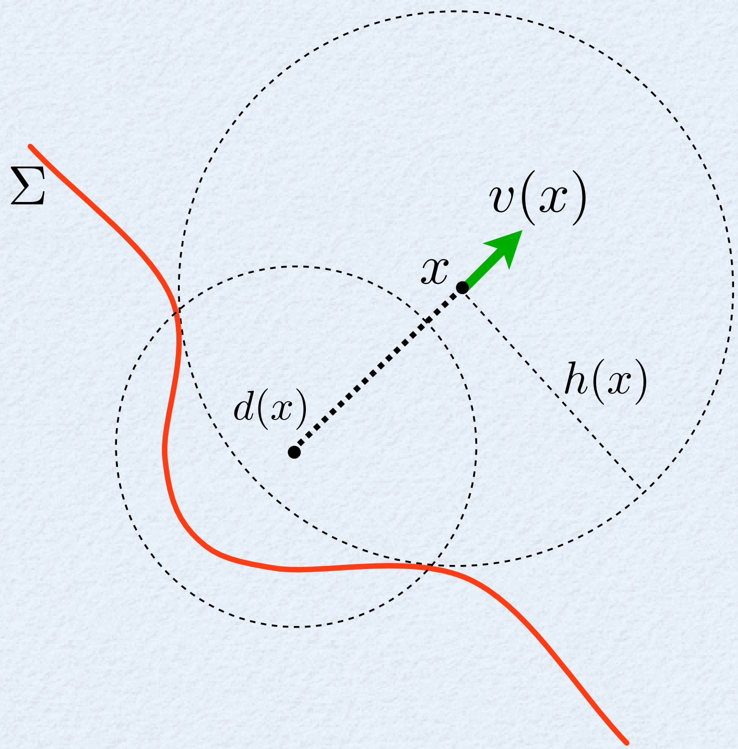

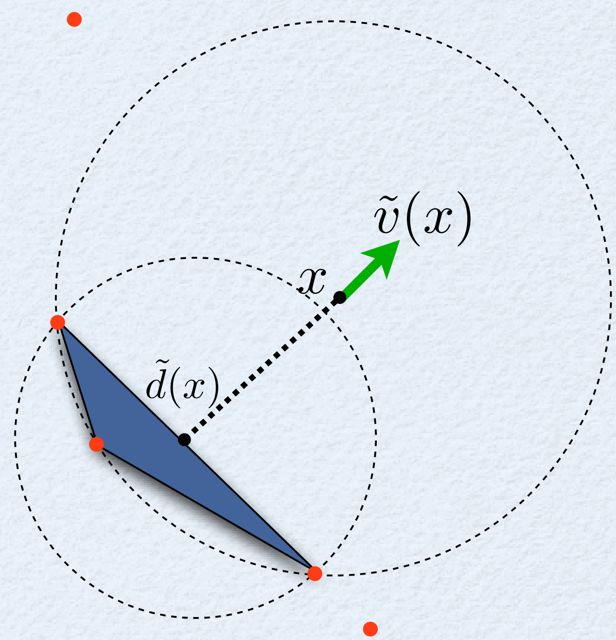

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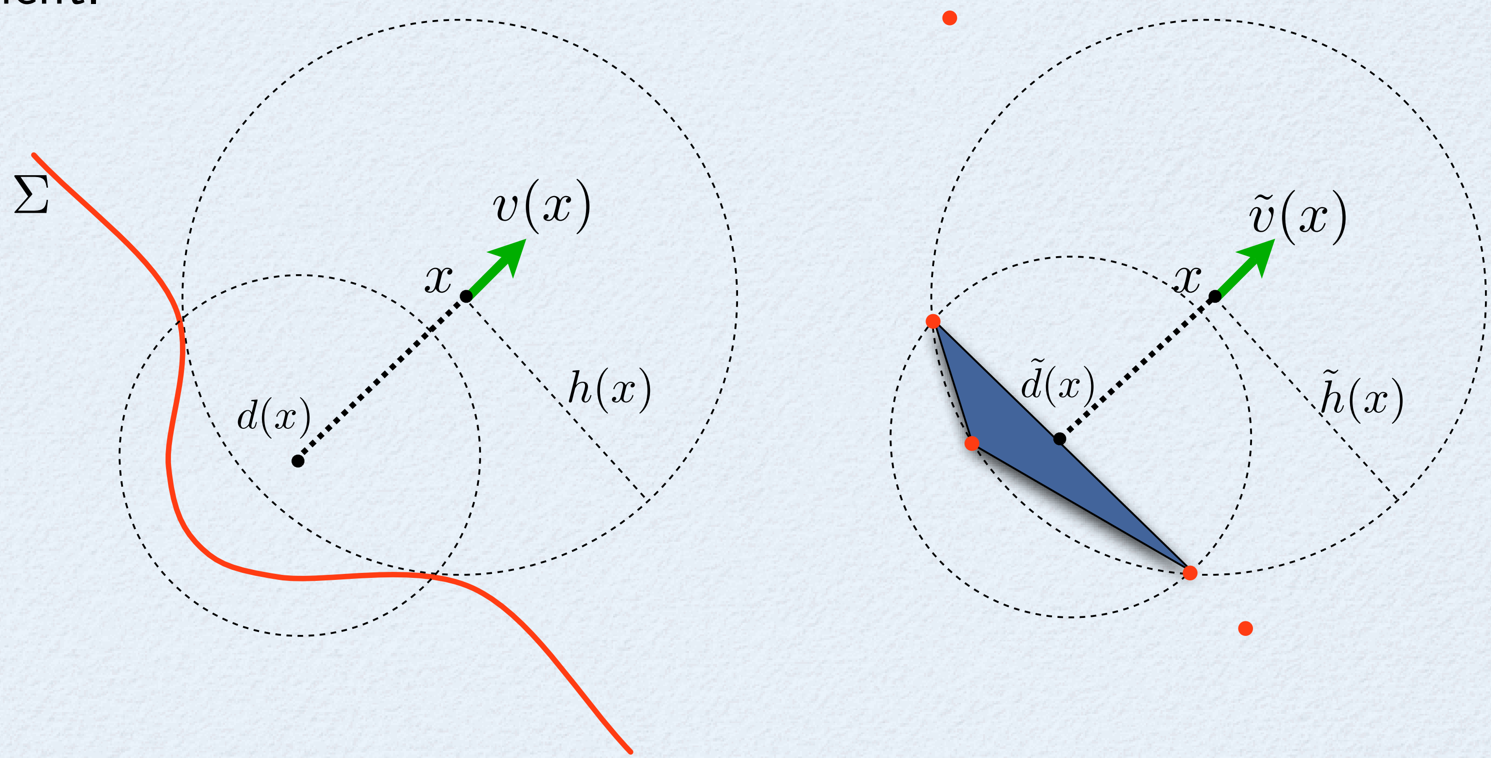
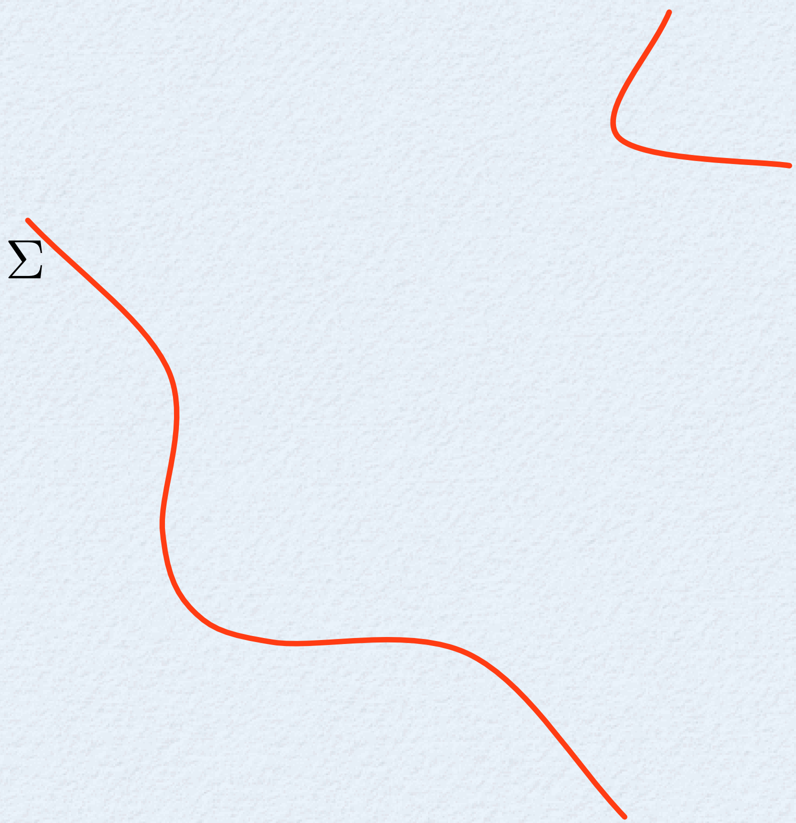


$$v(x) = \frac{x - d(x)}{h(x)} \qquad \tilde{v}(x) = \frac{x - \tilde{d}(x)}{\tilde{h}(x)}$$

Critical Points of Distance Functions

Point x is a critical point of h iff $v(x) = 0$ or equivalently if $x = d(x)$.



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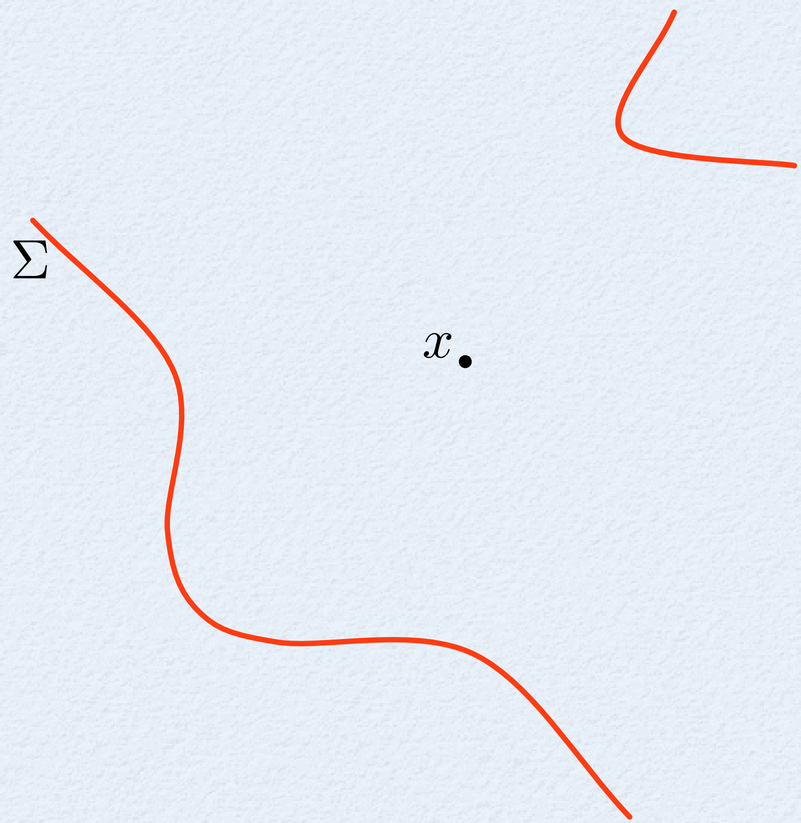

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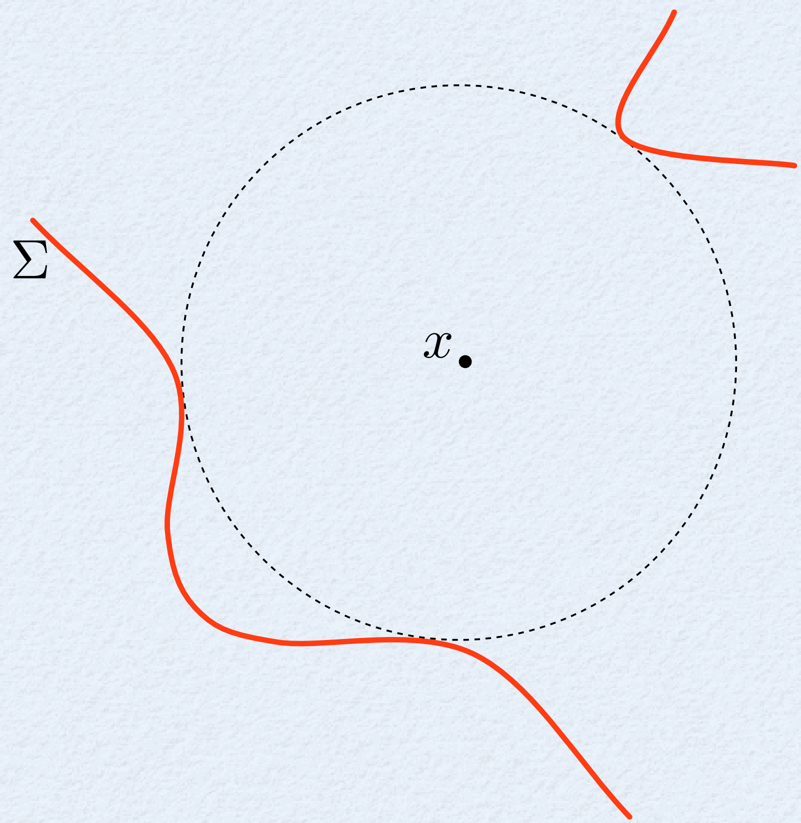

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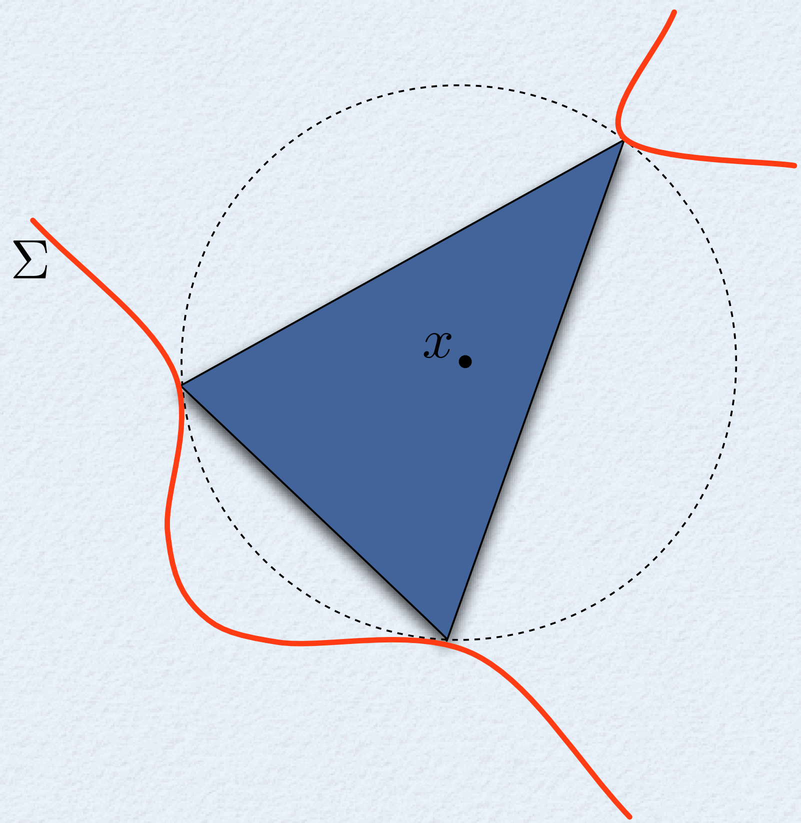

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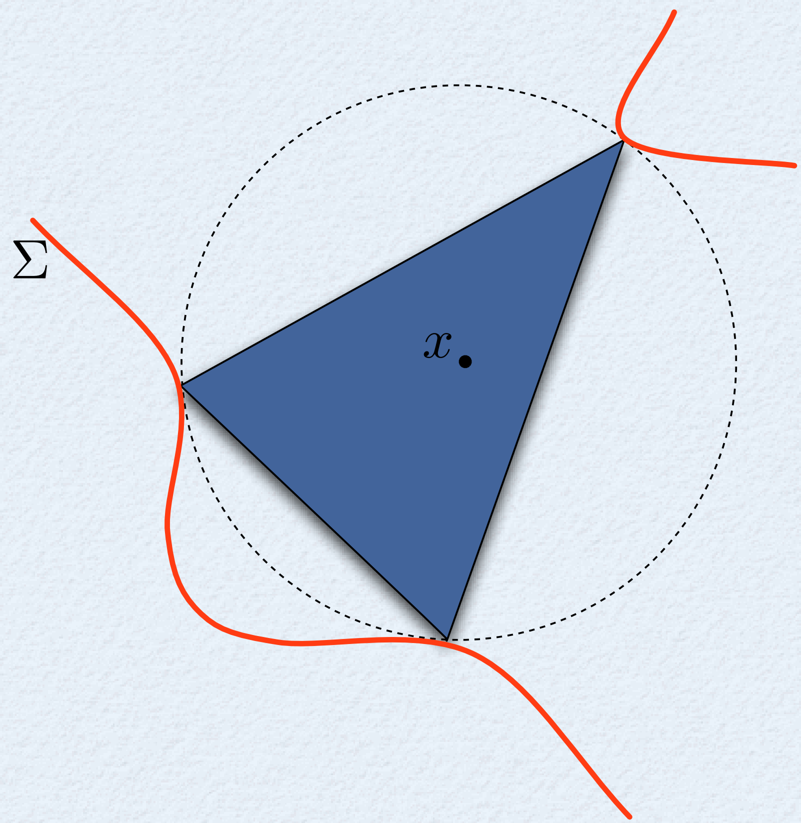

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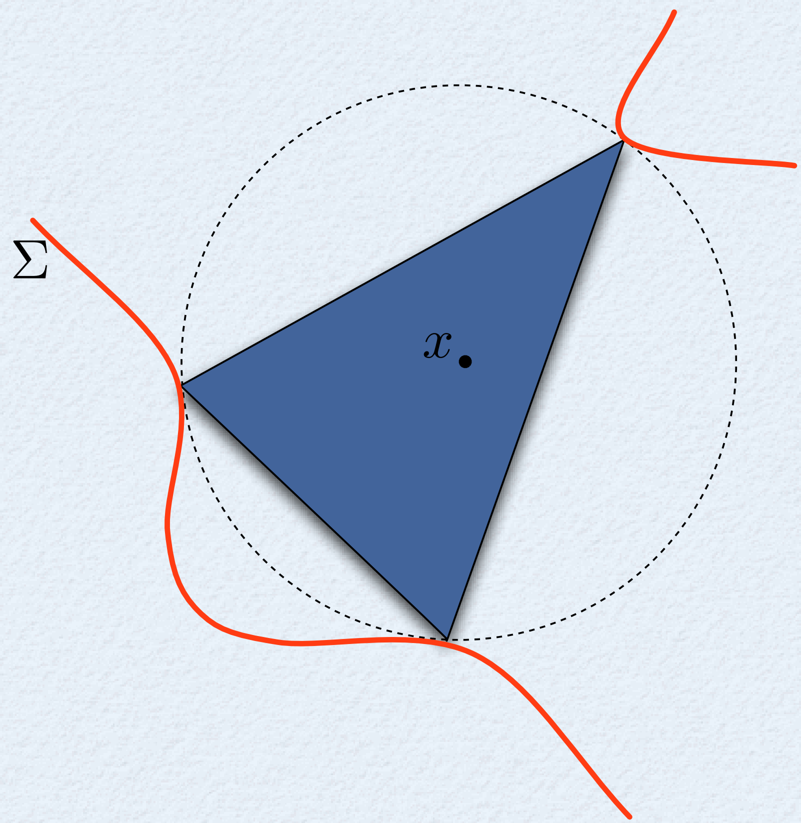

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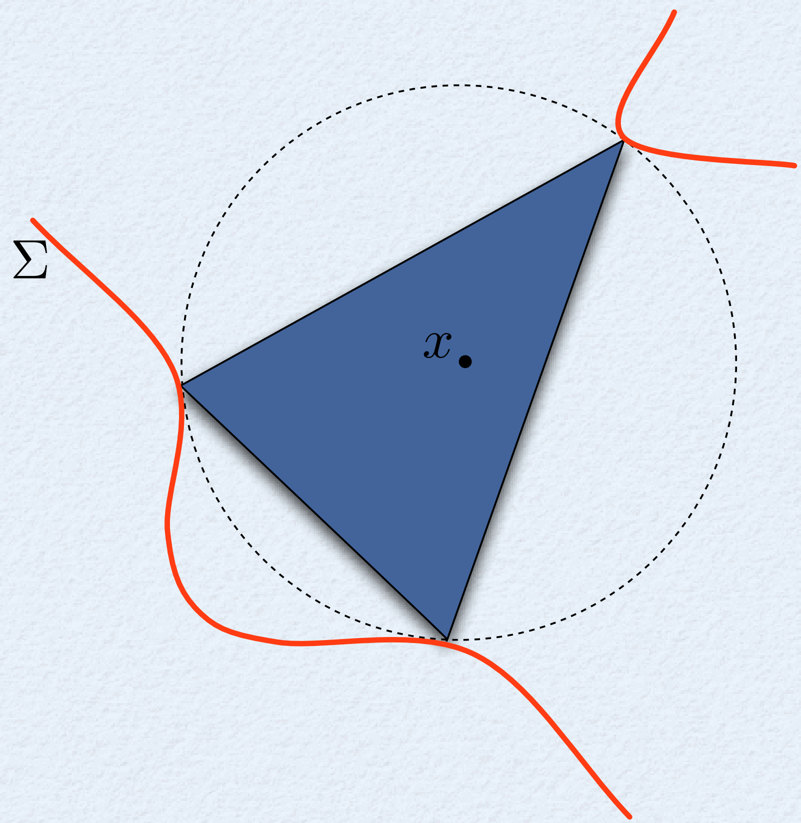

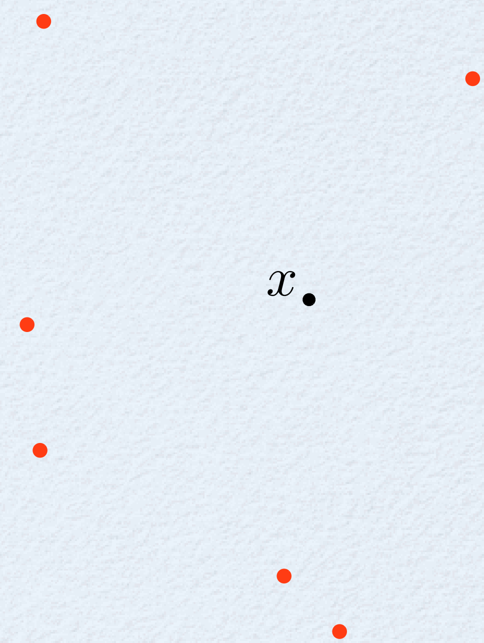

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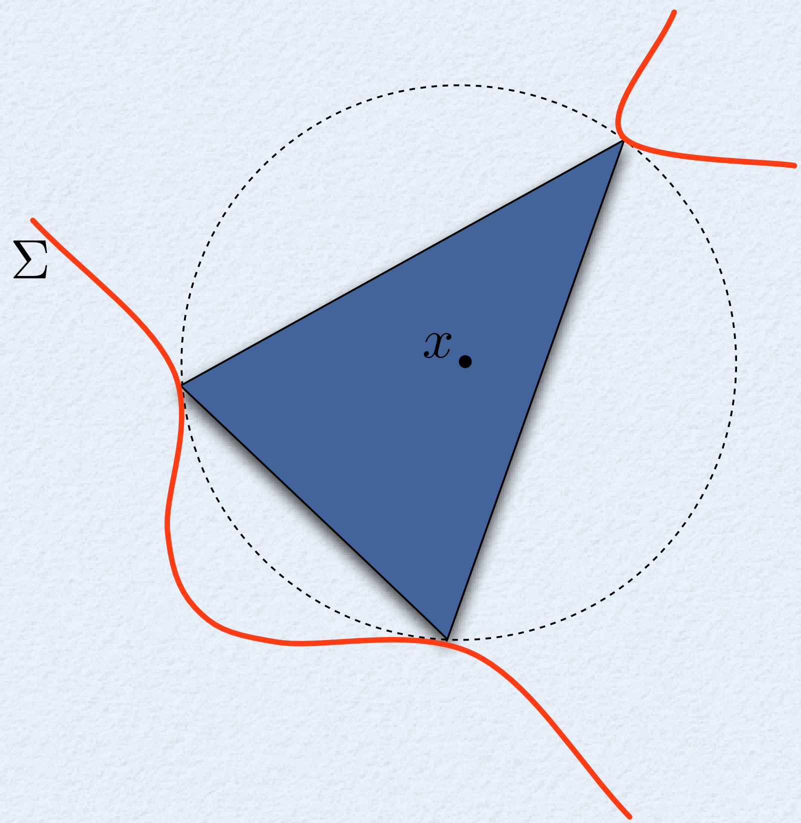

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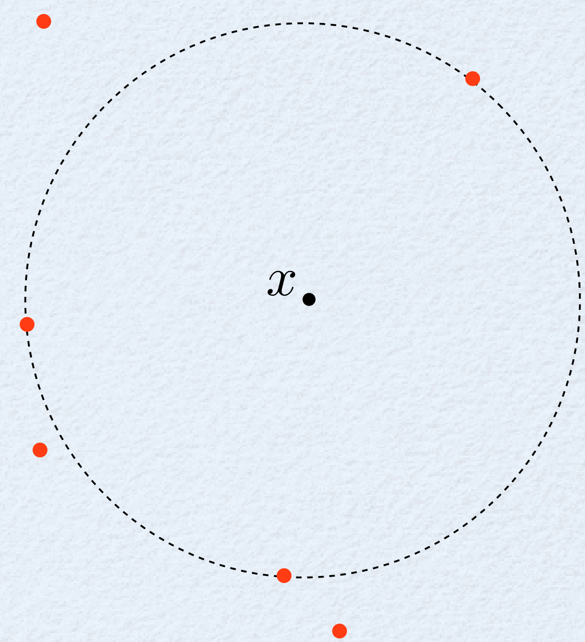

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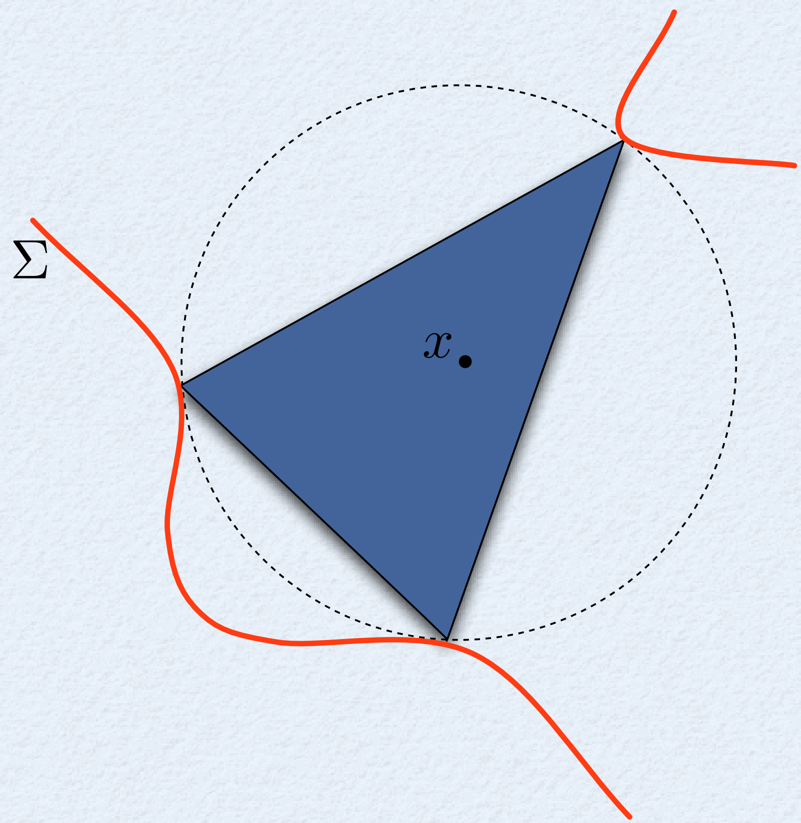

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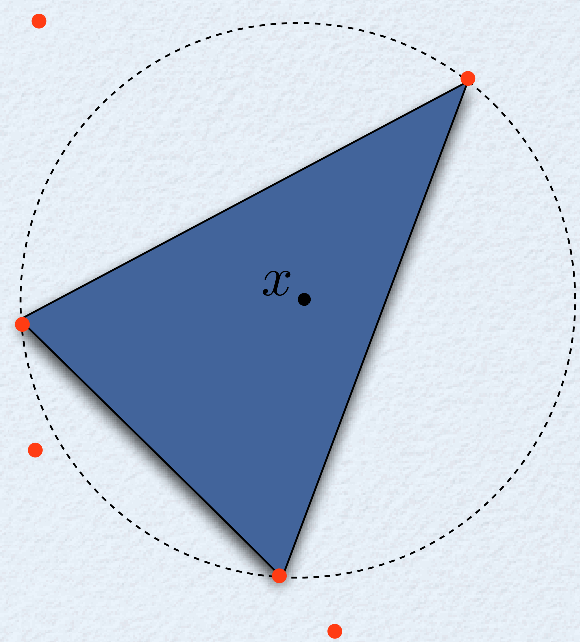
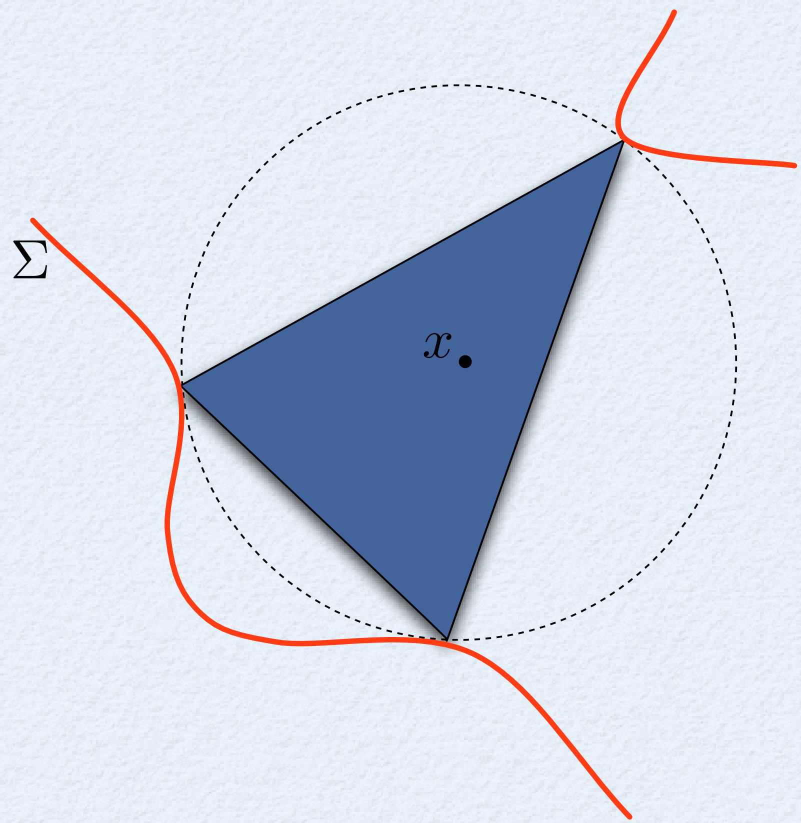


Critical Points of Distance Functions

Point x is a critical point of h iff $v(x) = 0$ or equivalently if $x = d(x)$.



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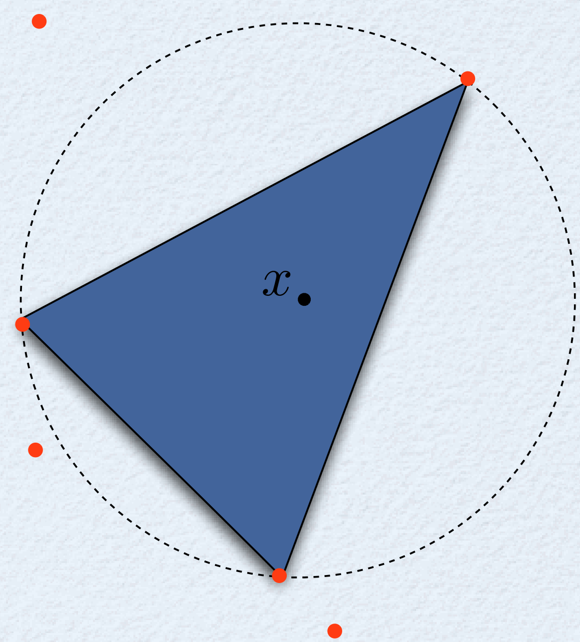


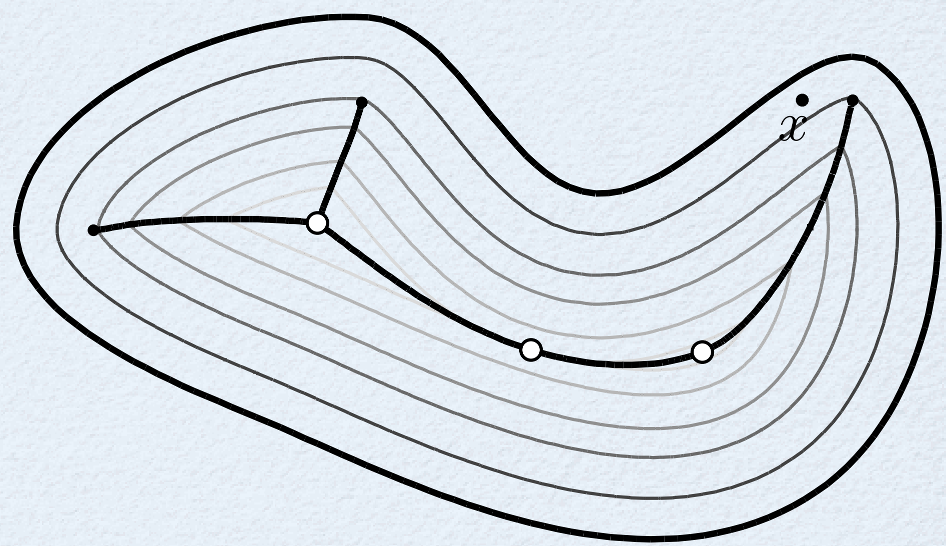
$V(x) \cap D(x) \neq \emptyset$

Integrating the Steepest Ascent Vector Field

[Lieutier '04]

Although vector field v is not continuous, Euler schemes on v converge uniformly to a continuous flow map

$$\phi : [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n.$$



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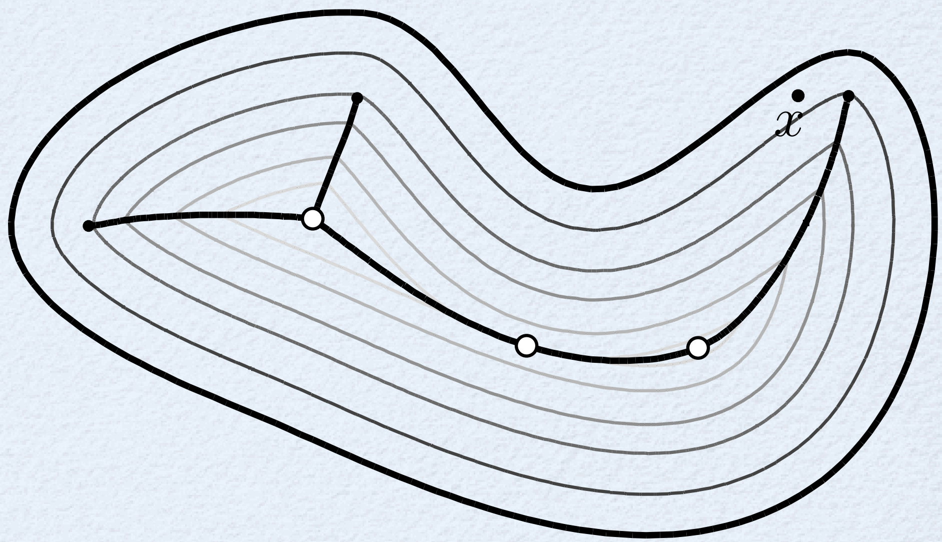

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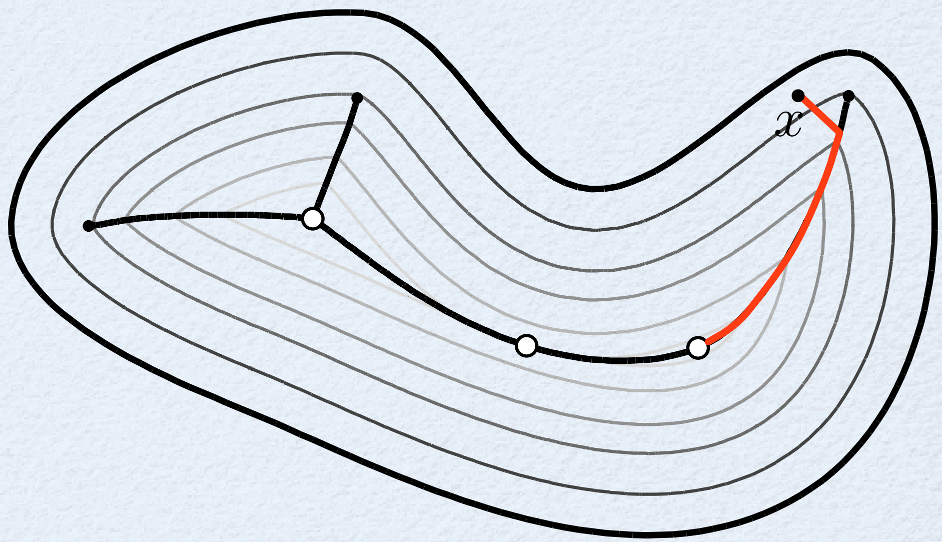

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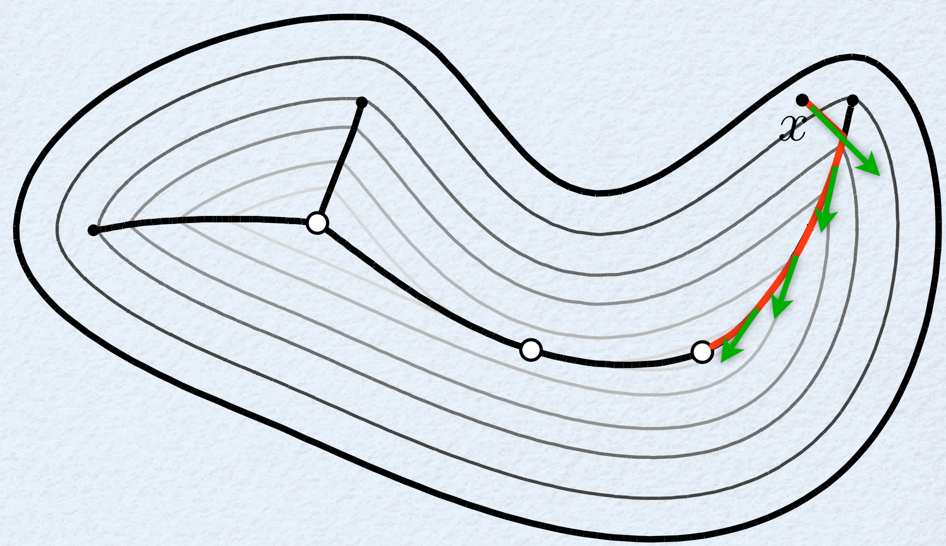

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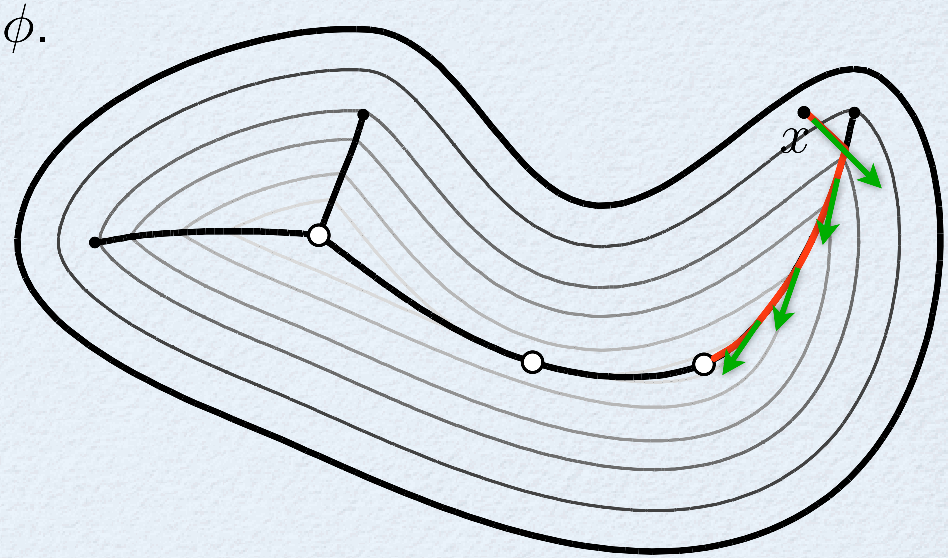

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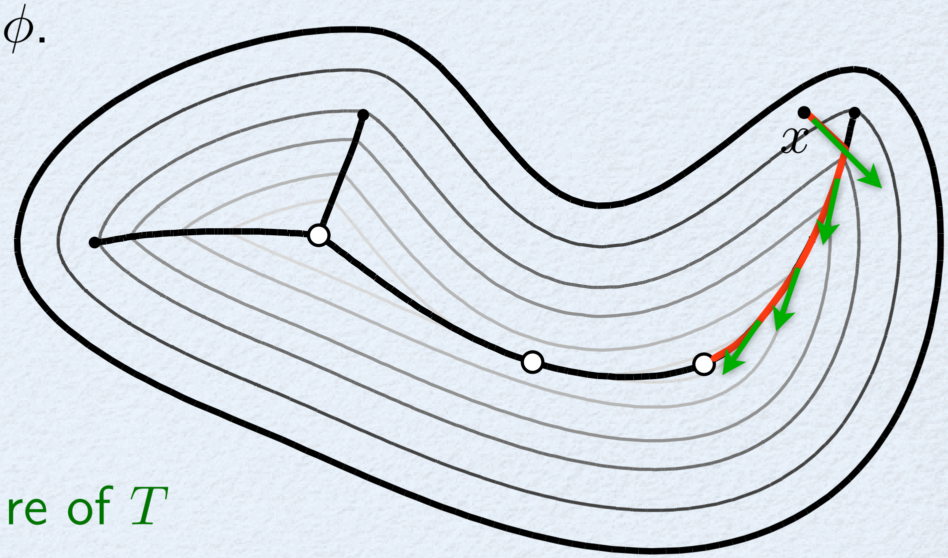

Notation:

$$\phi(x) = \bigcup_{t \geq 0} \phi(t, x)$$

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Integrating the Steepest Ascent Vector Field

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Although vector field $\tilde{v}$ is not continuous, Euler schemes on $\tilde{v}$ converge uniformly to a continuous flow map

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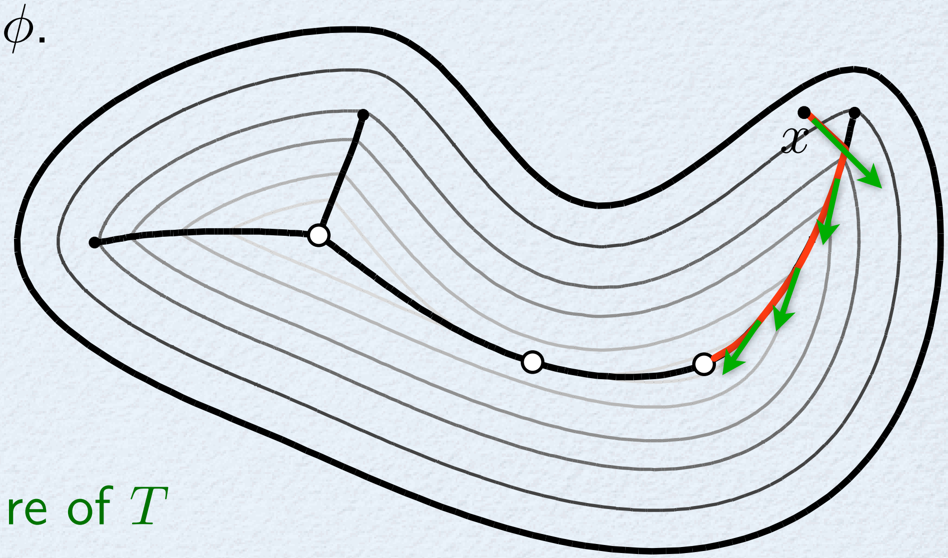
◇ Critical points are fixed points of $\tilde{\phi}$.



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Integrating the Steepest Ascent Vector Field

Lemma. [Lieutier'04] if $h(t) = h(\phi(t,x))$ and $v(t) = v(\phi(t,x))$, then

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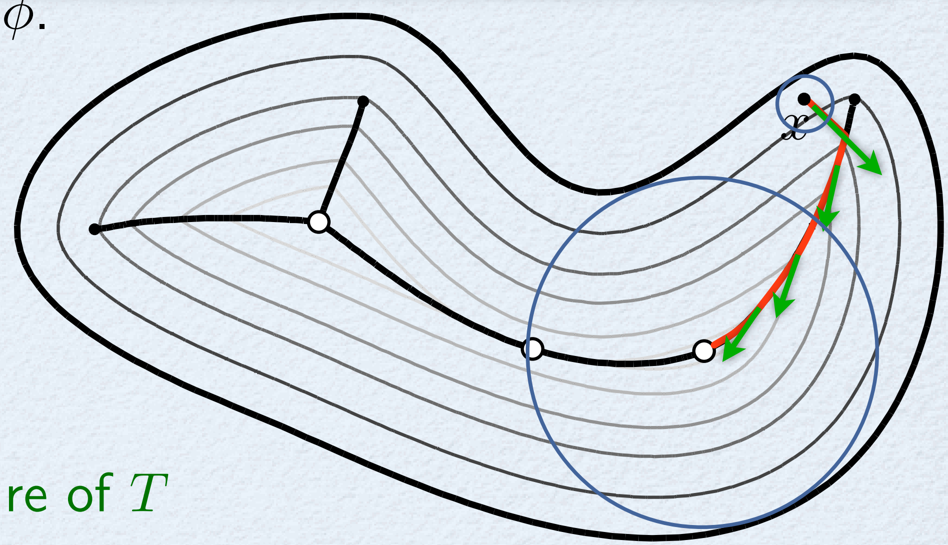

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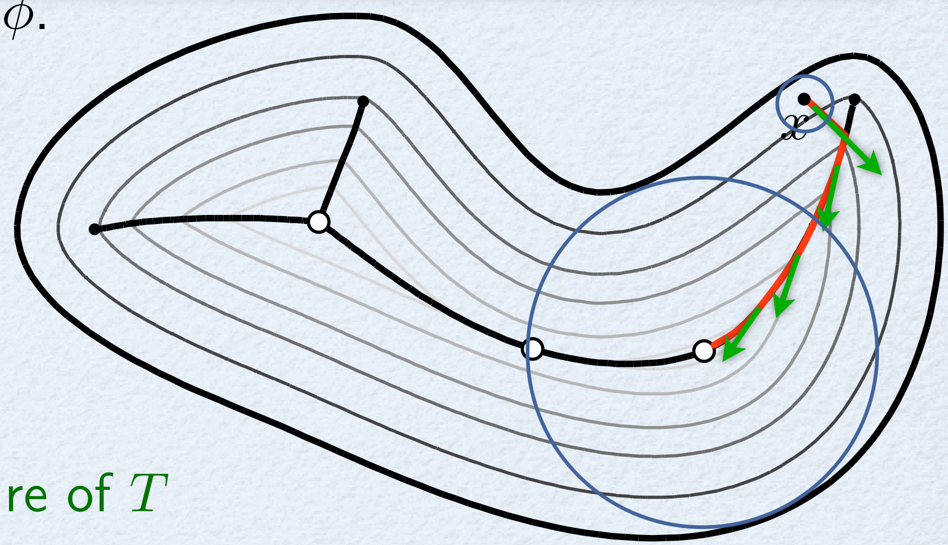

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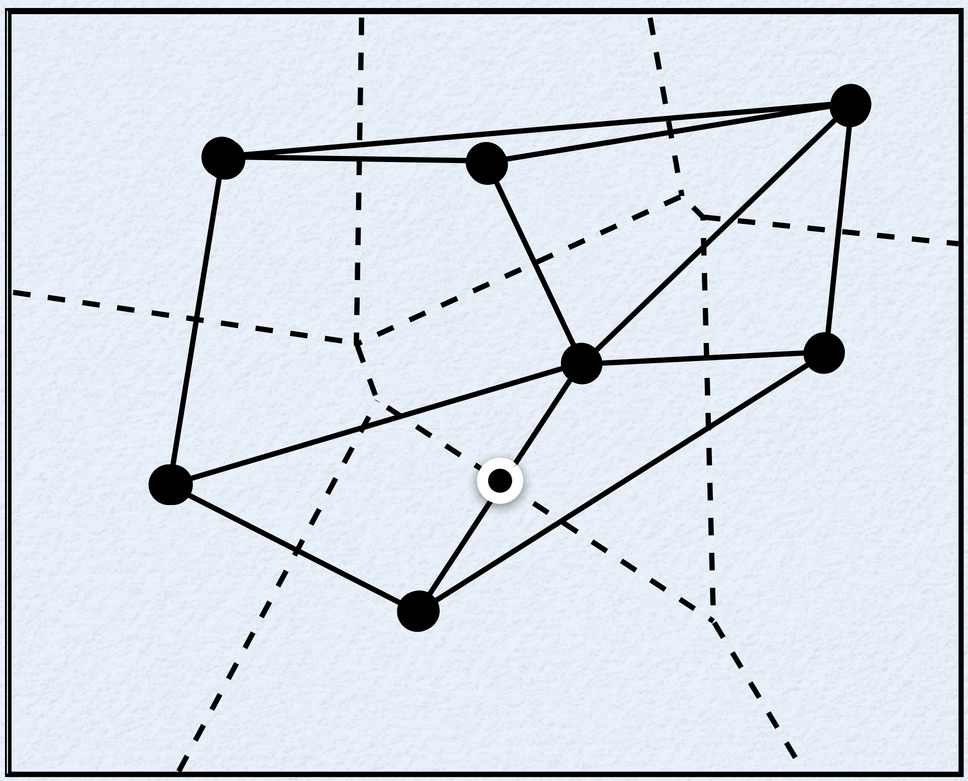
Unstable Manifolds

Unstable manifold of a critical point c is the set of all points that "flow out of" c.

$$U(c) = \bigcap_{\varepsilon > 0} \phi(N_\varepsilon(c))$$



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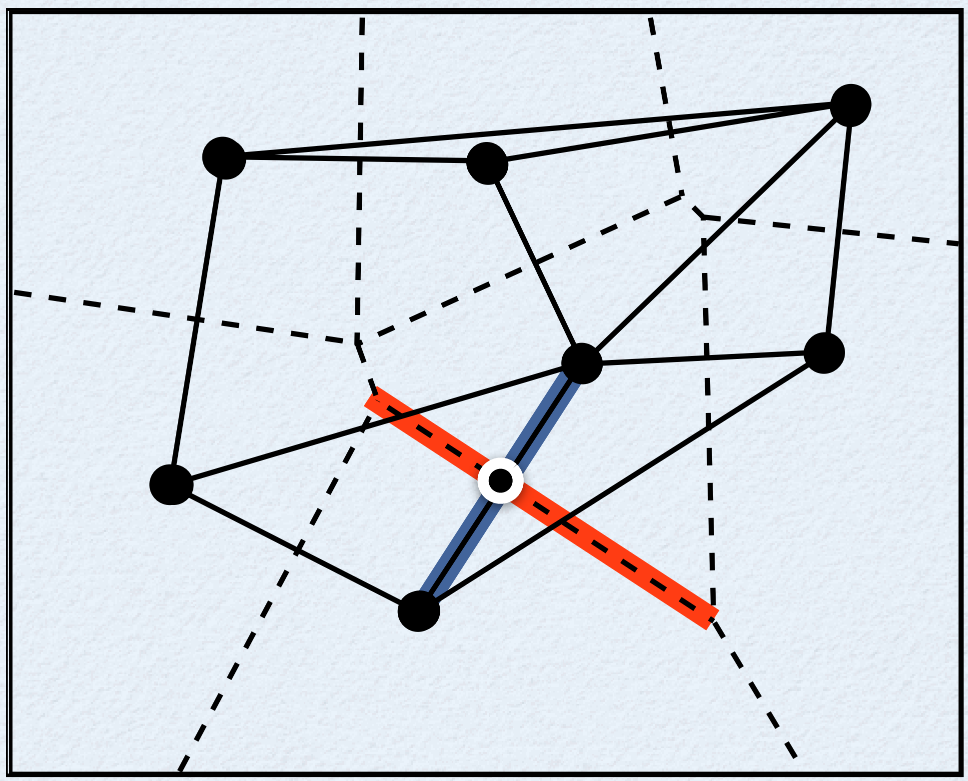

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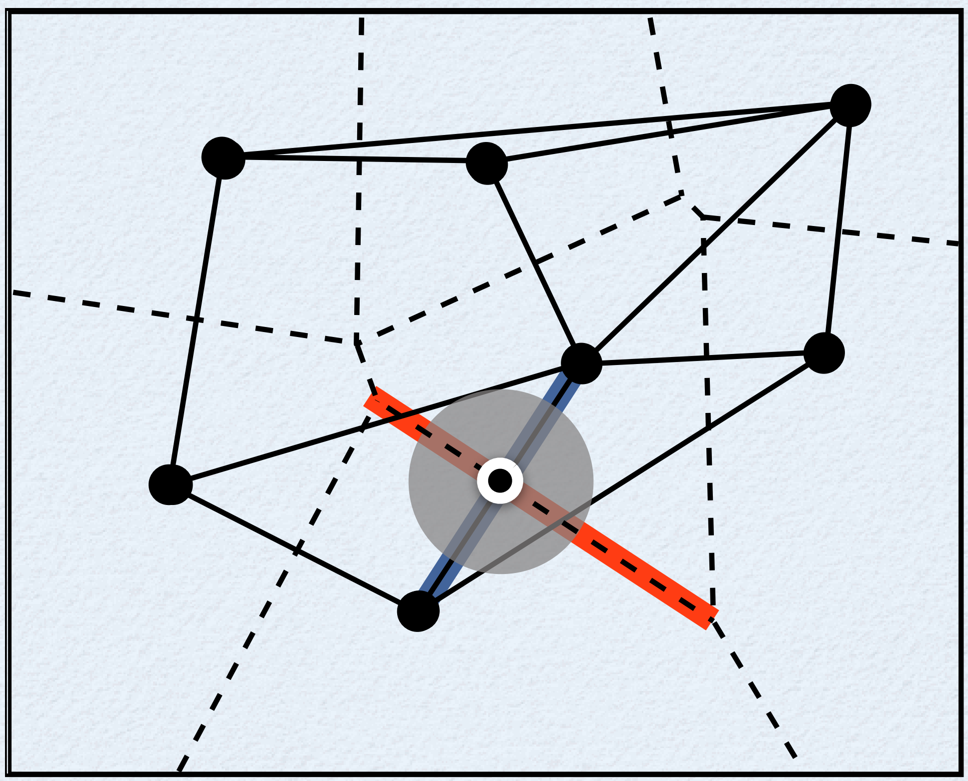

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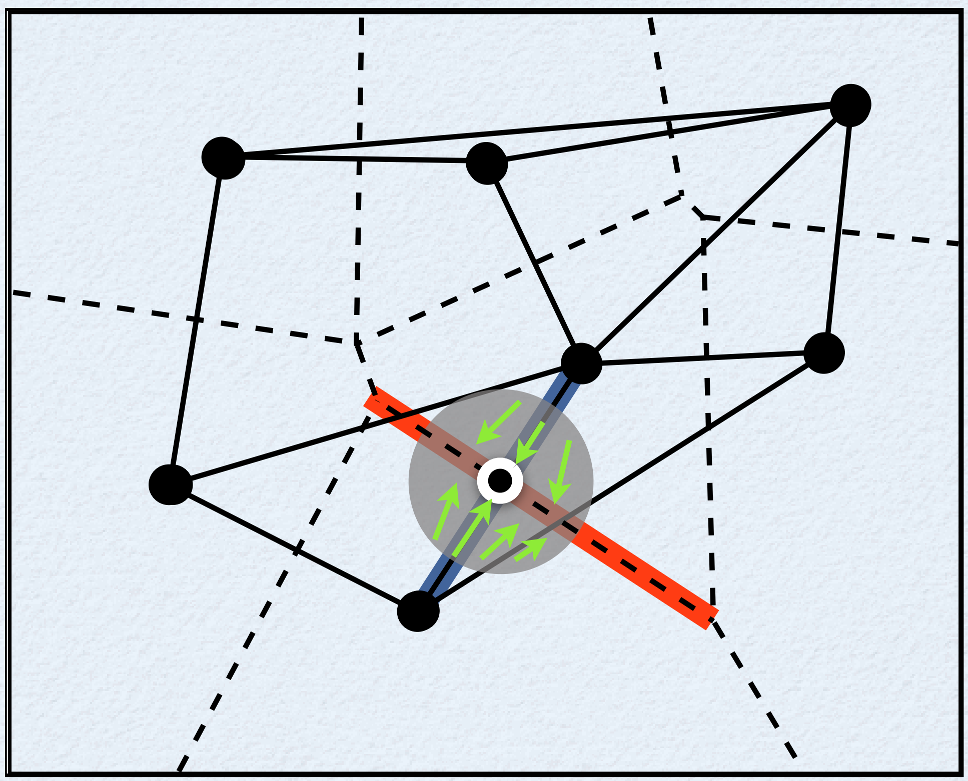

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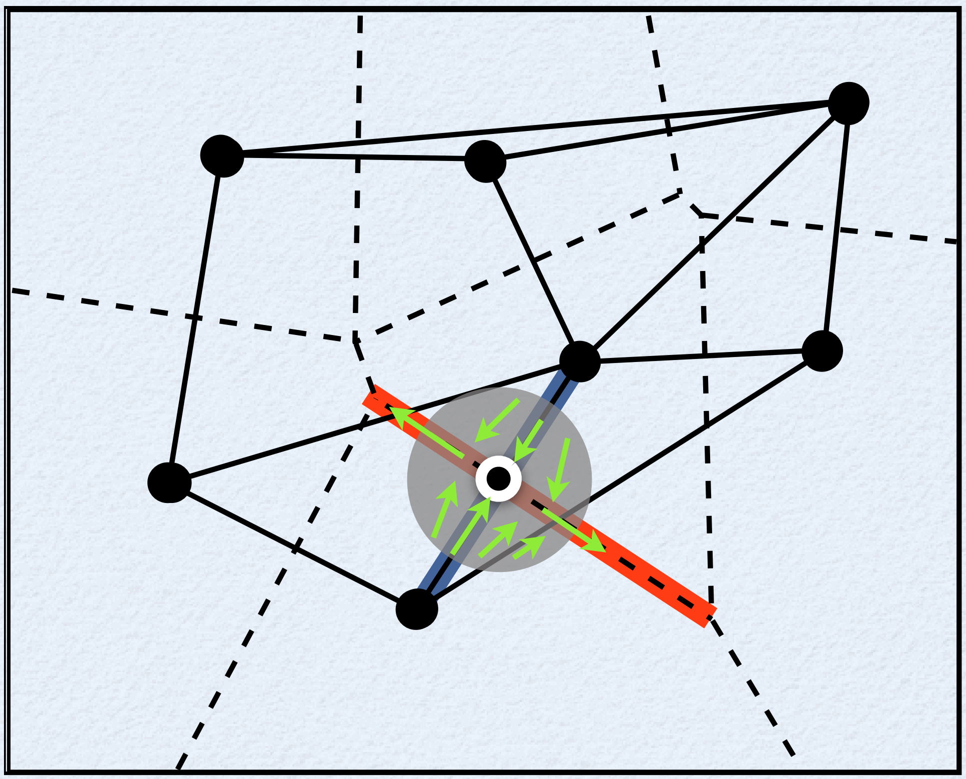

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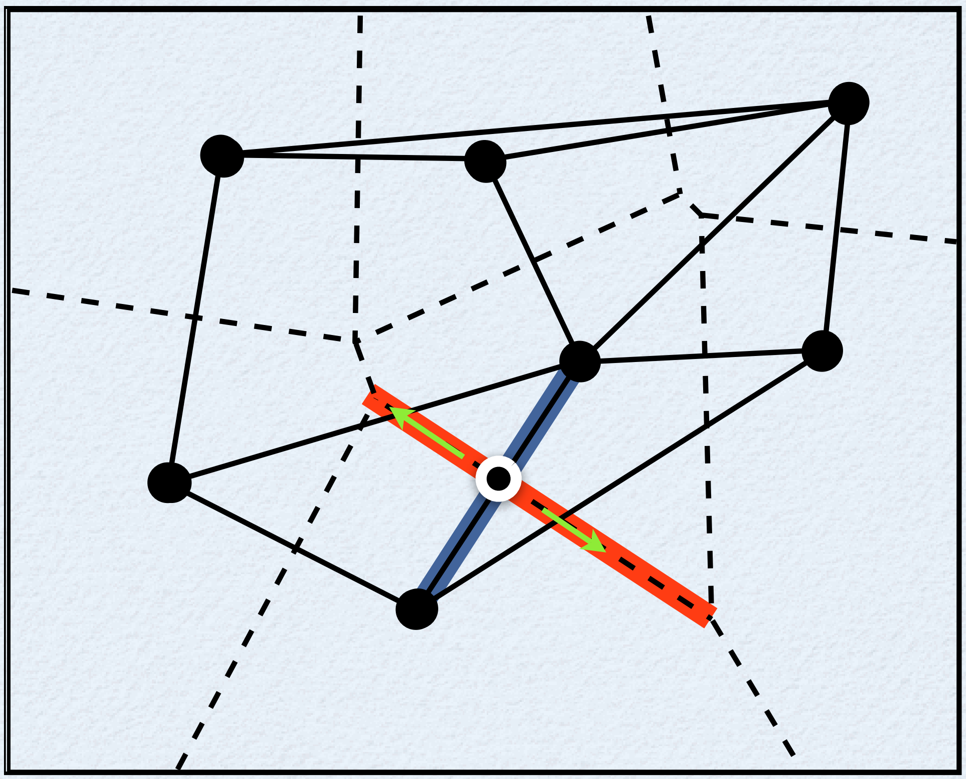

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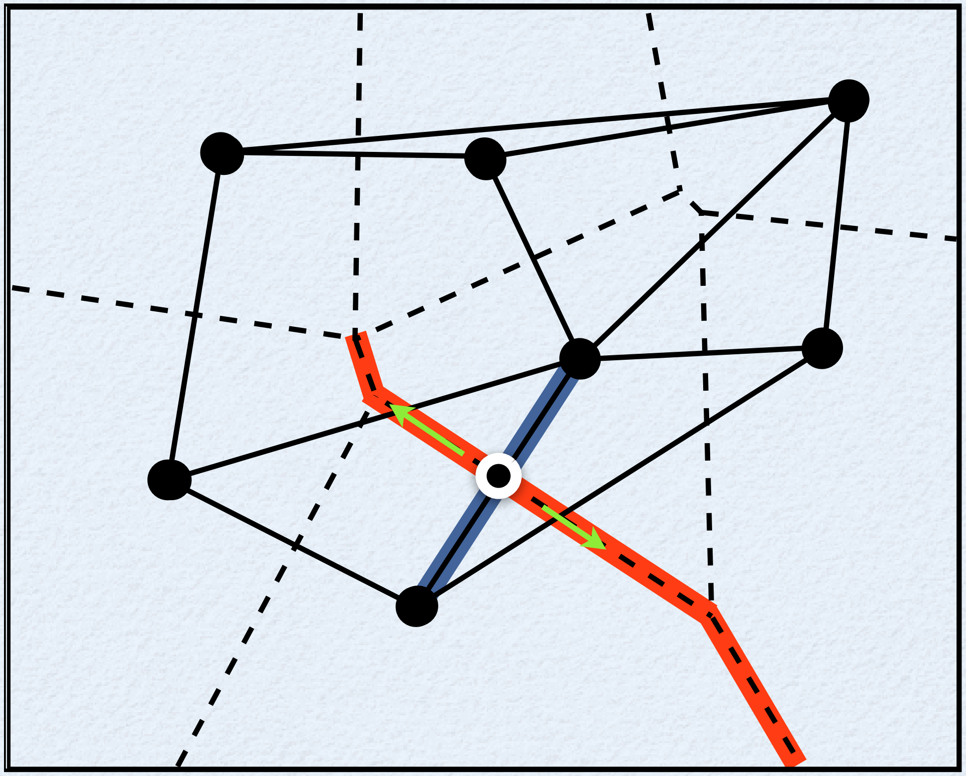

Unstable Manifolds for Discrete Sets

In general when dealing with discrete sets

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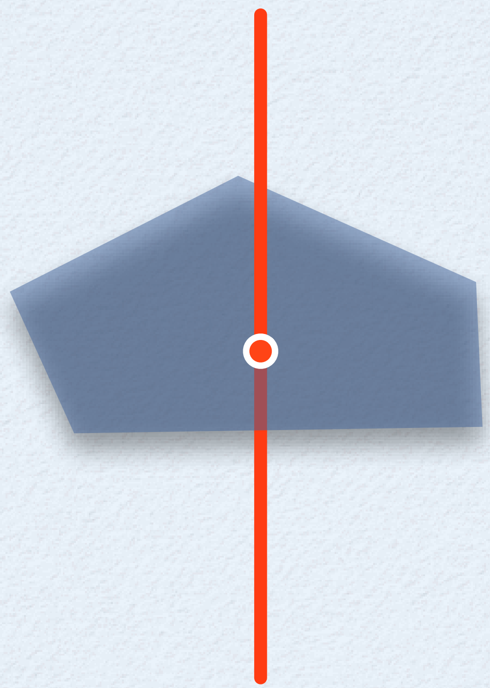

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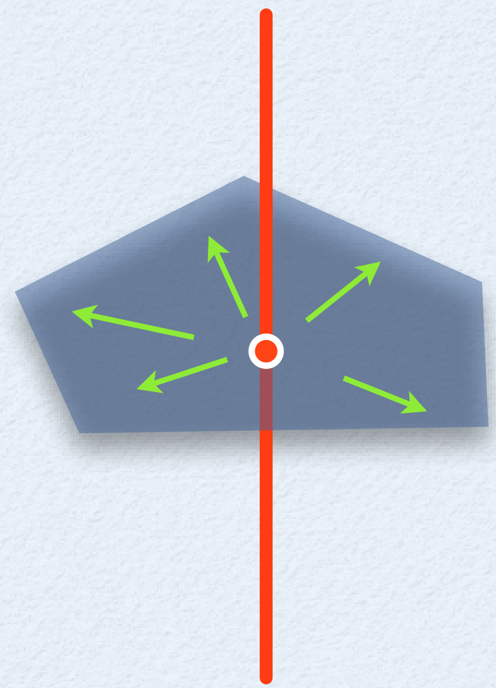

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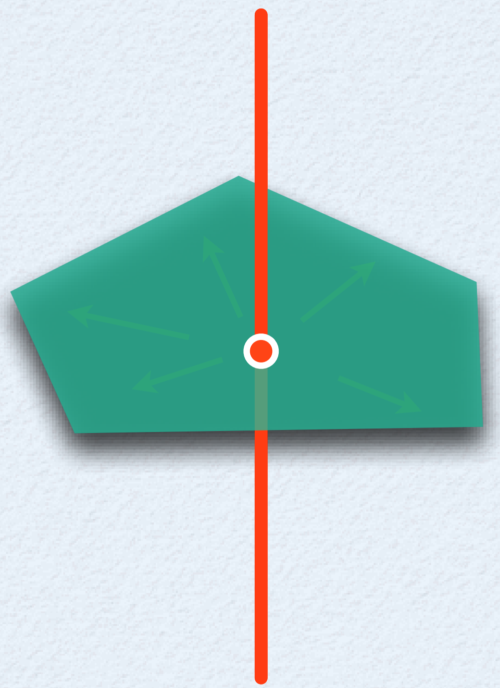

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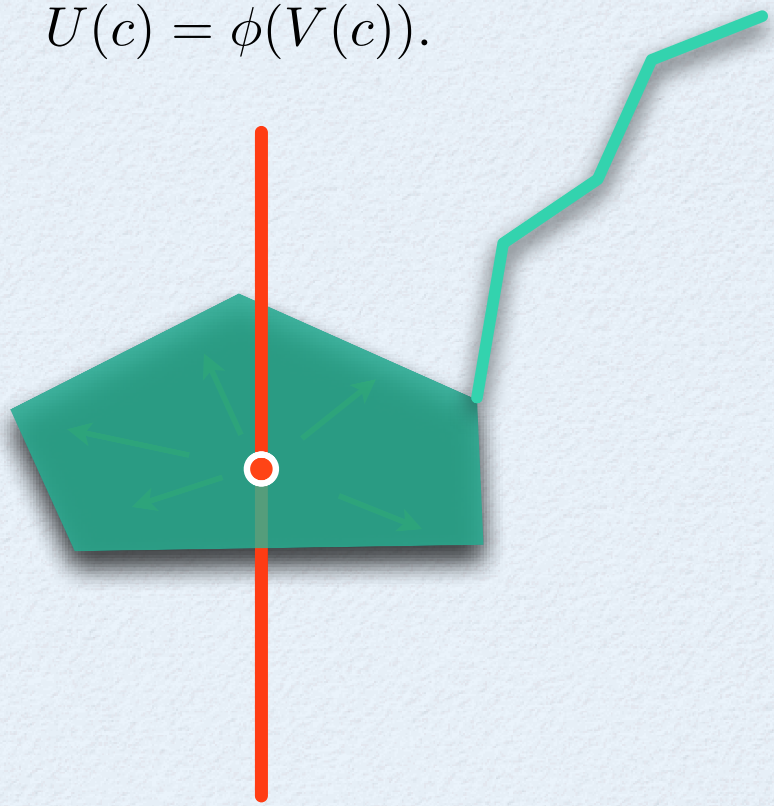

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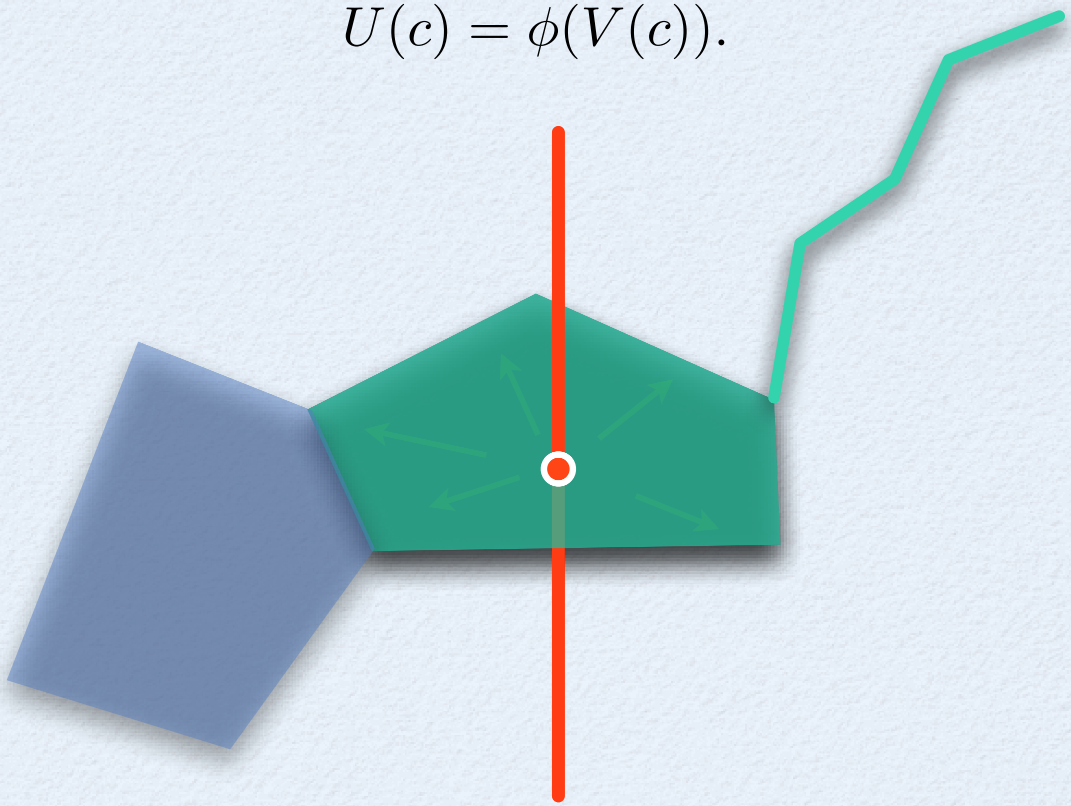

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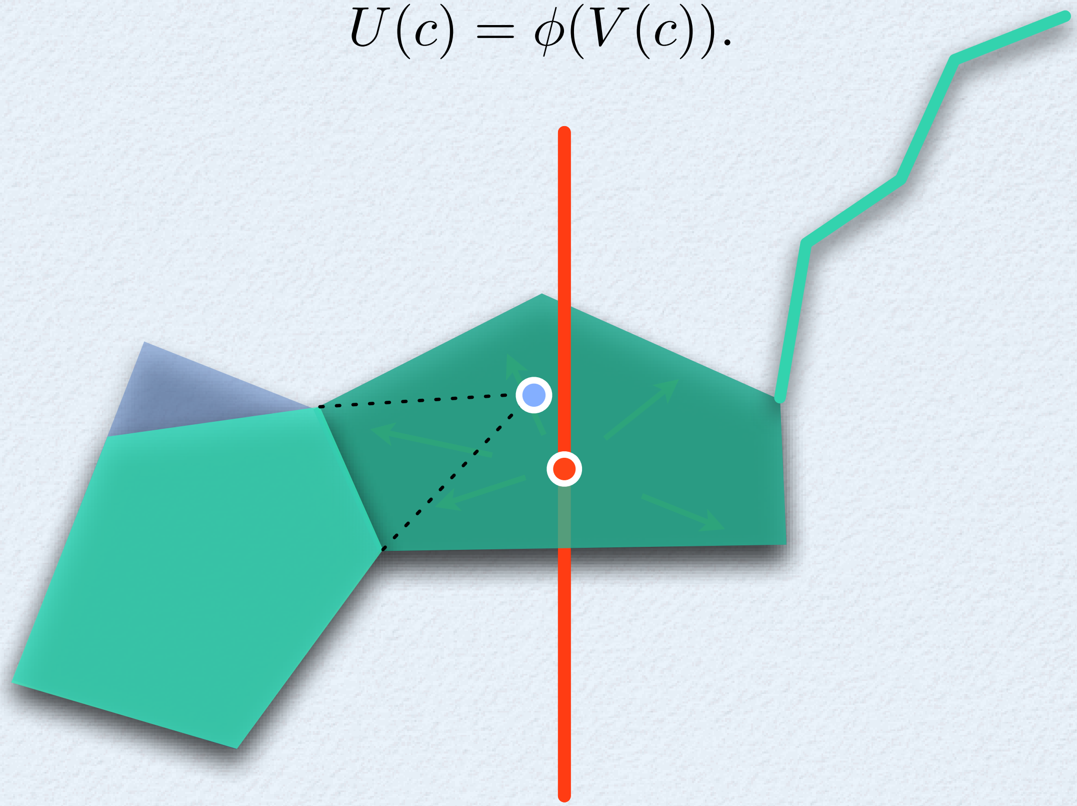
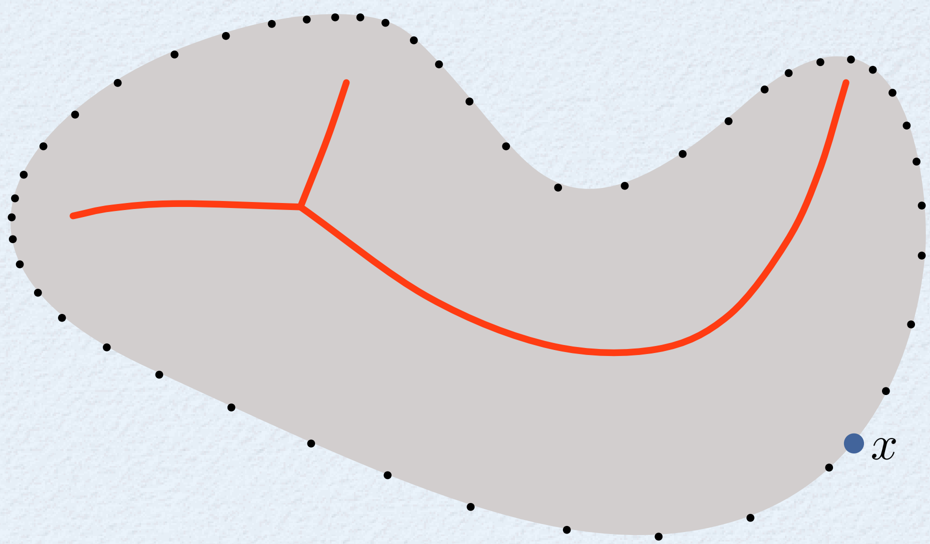


Sampling Assumption

For a point $x \in \Sigma$, the local feature size of x is

$$\mathrm{lfs}(x) := d(x, M).$$



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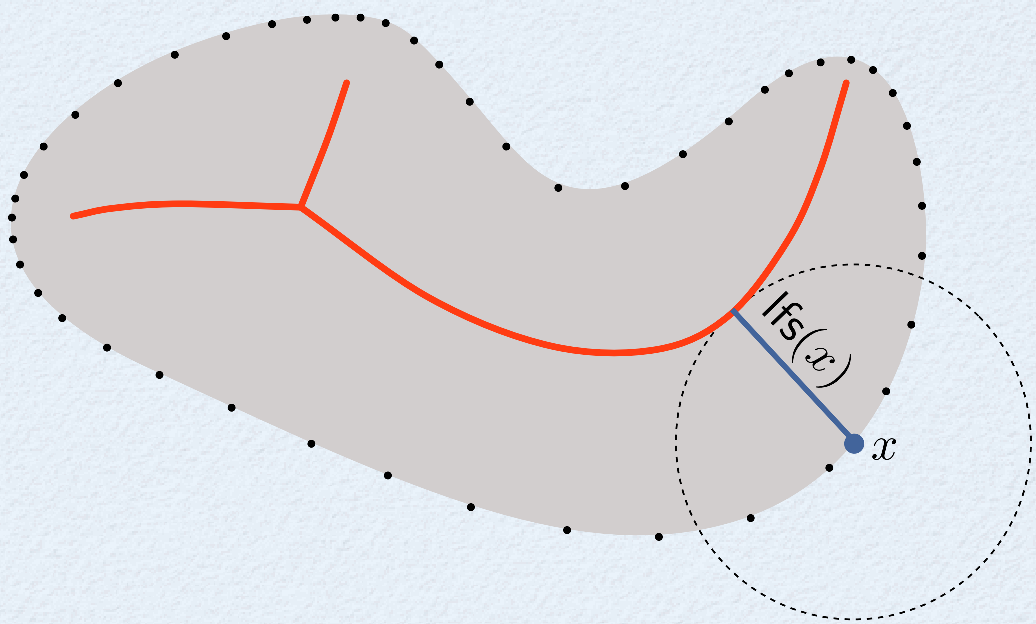

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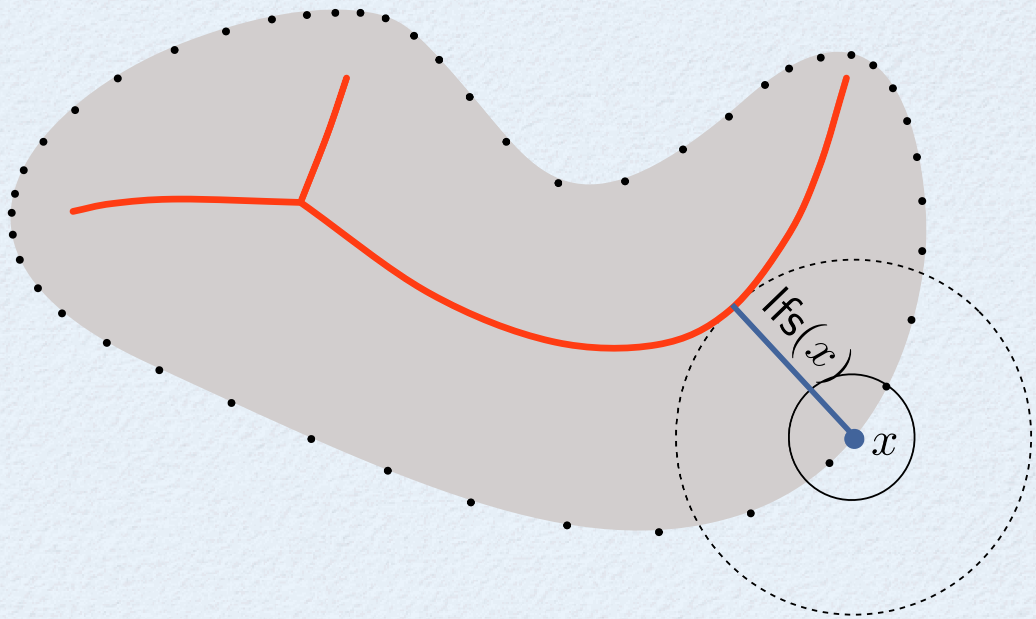

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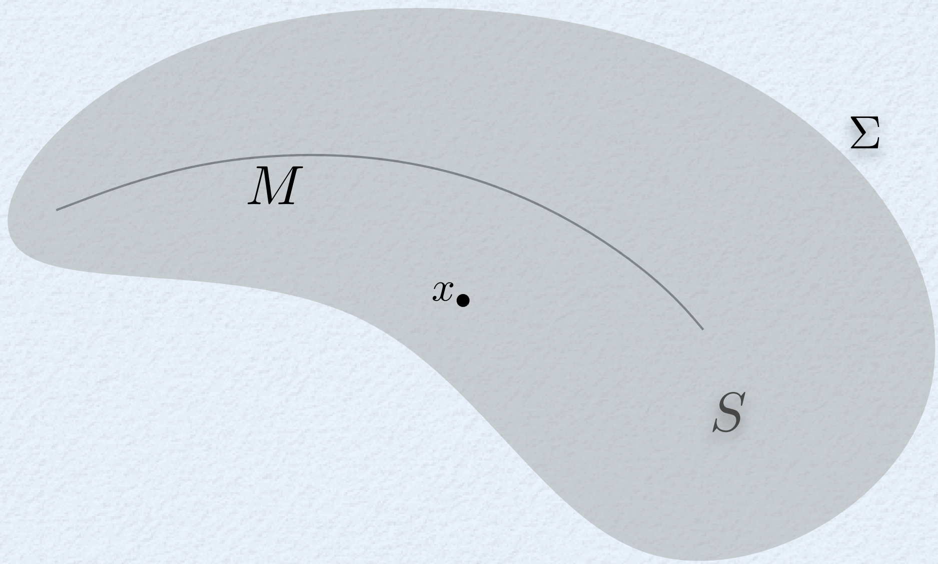
Reduced Objects and Tubular Neighborhoods
and
Separation of Critical Points



◇ The δ-tubular neighborhood of Σ:

$$\Sigma_\delta := \left\{ x \in \mathbb{R}^n \setminus M : \|x - \hat{x}\| < \delta f(\hat{x}) \right\}$$

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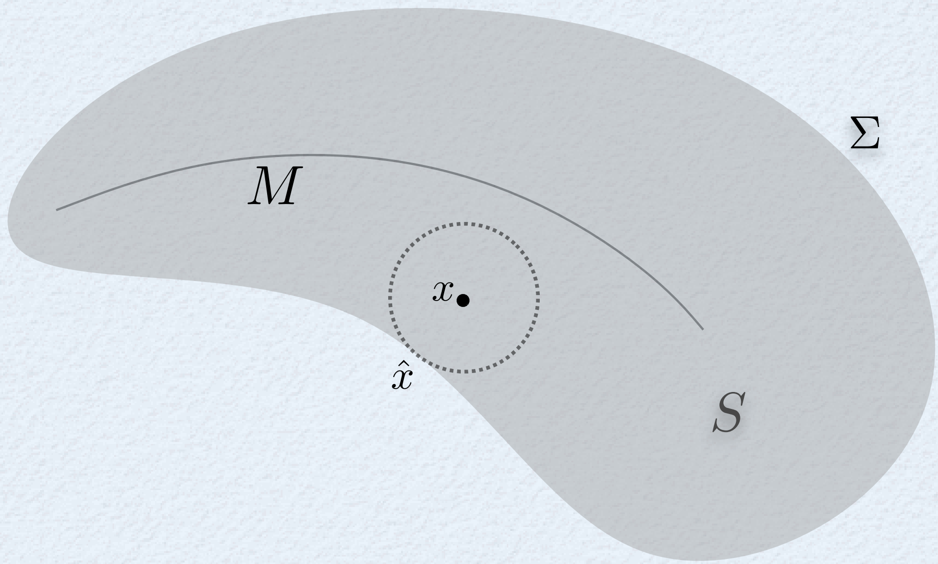

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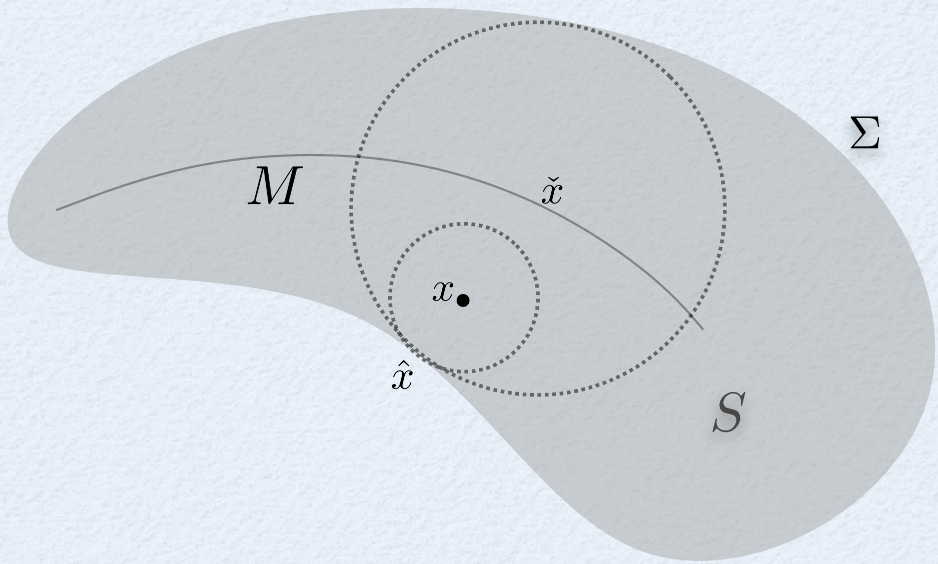

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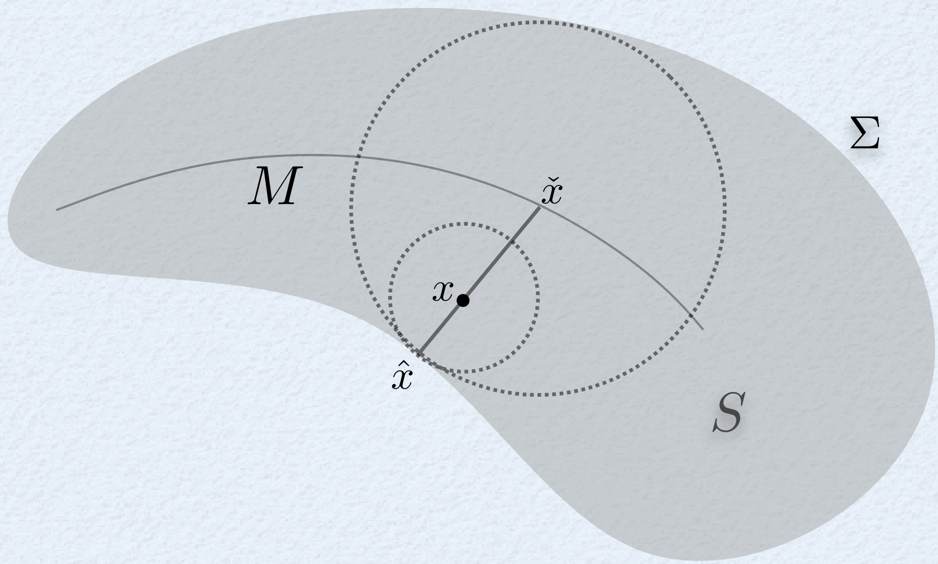

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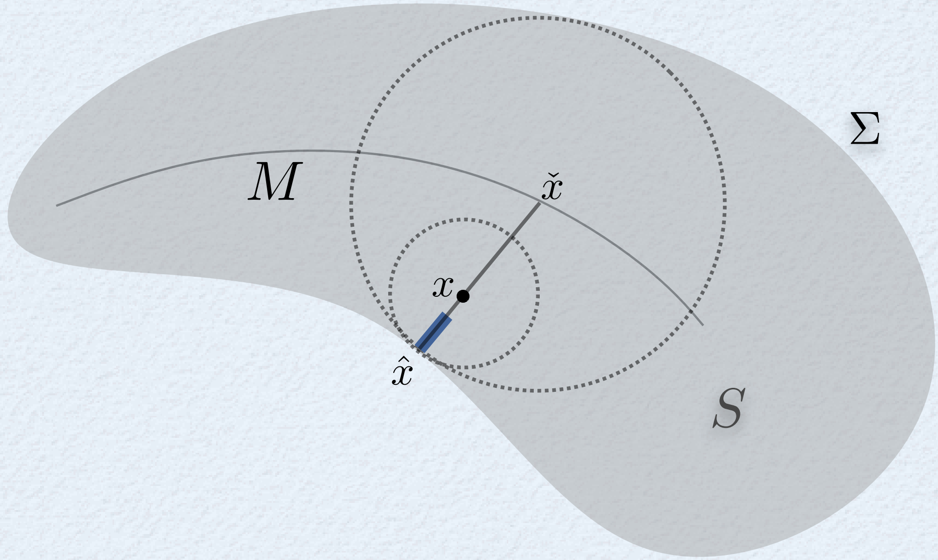

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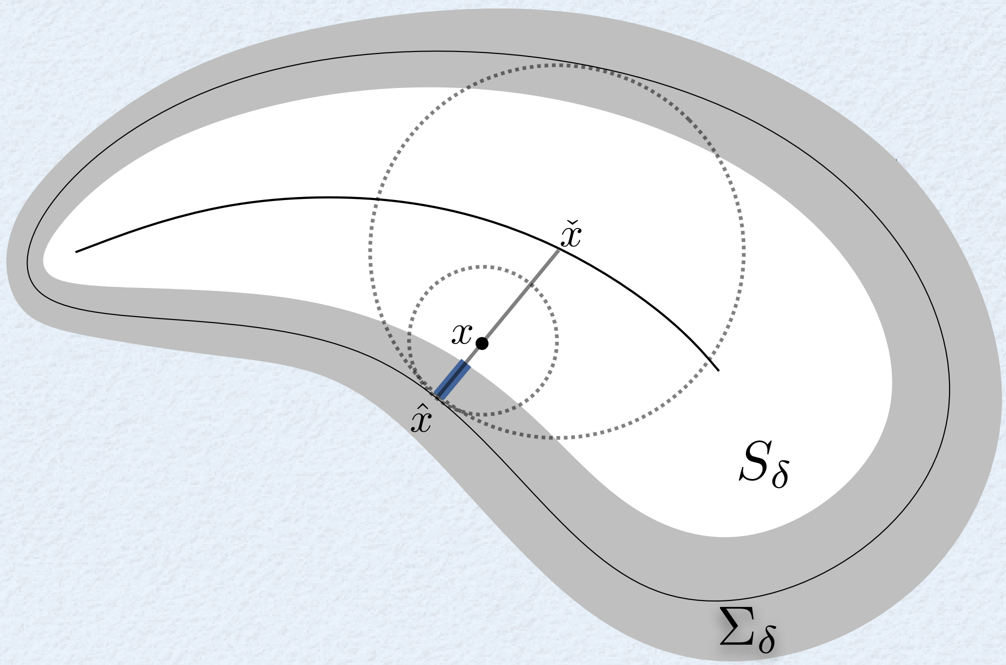

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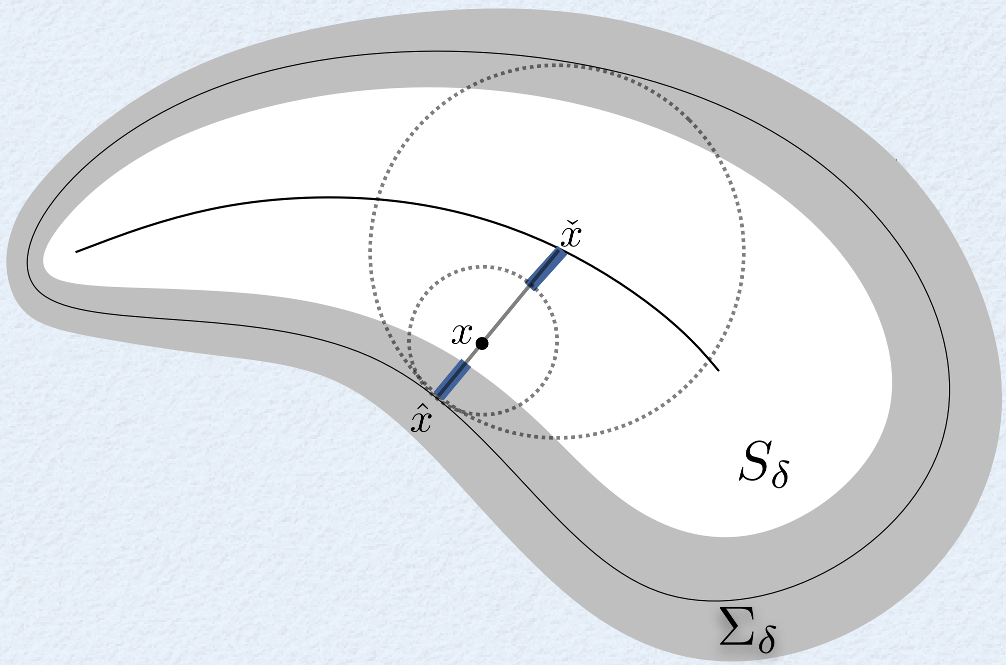

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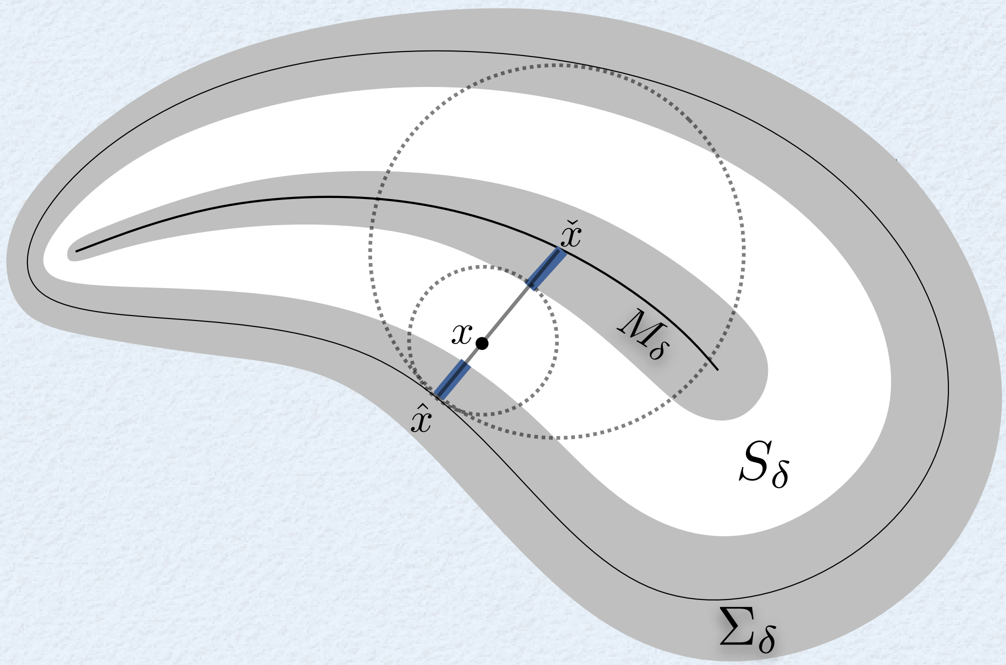

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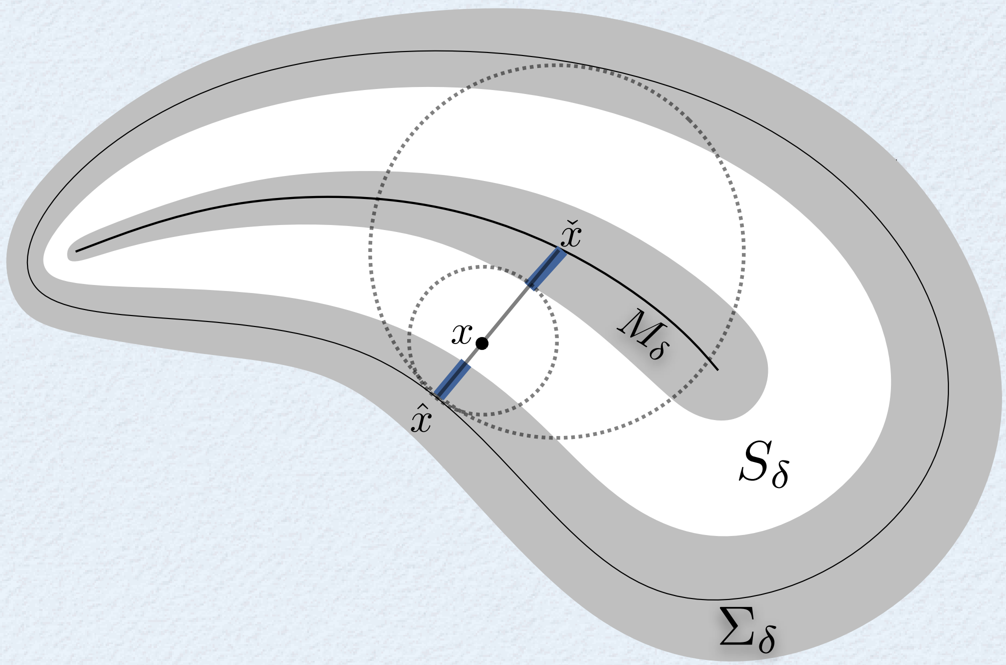

Theorem [Dey, Giesen, Ramos, S '05]
For an ε-sample of Σ with $\varepsilon < 1/3$, all critical points of $\tilde{h}$ are contained in either Σ_{ε^2} or $M_{2\varepsilon}$.

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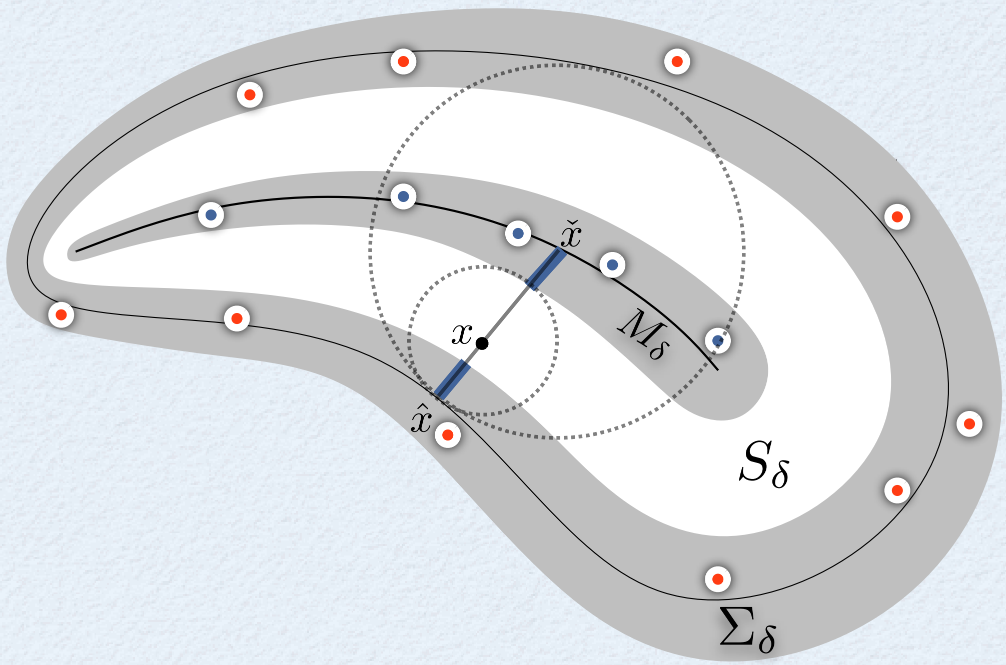

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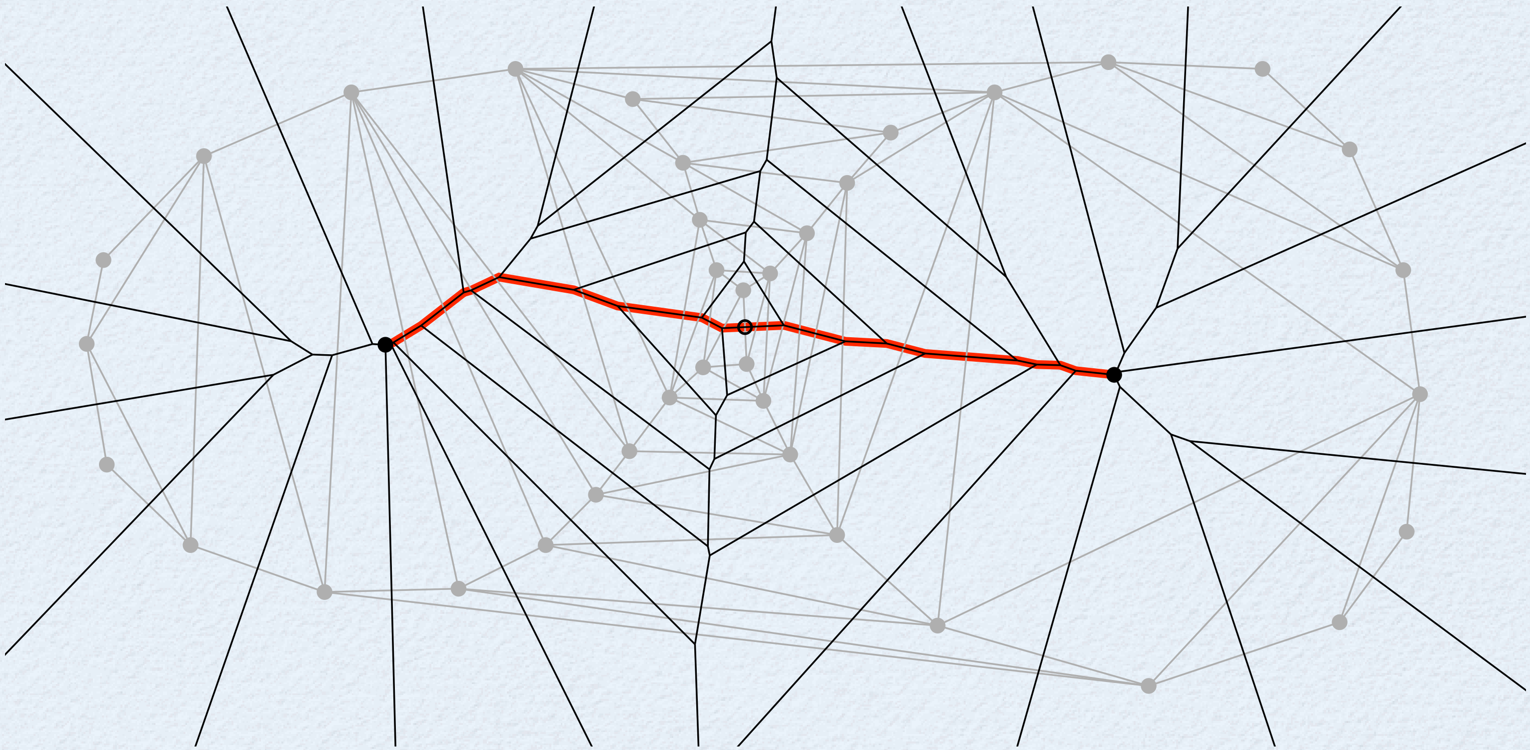
Computing the MA Core (Capturing the Topology)

Definition. Let N be the set of all medial axis critical points of $\tilde{h}$ inside S. The core for approximating $M(S)$ is

$$\mathcal{C} = \bigcup_{c \in N} U(c)$$



A Criterion for Homotopy Equivalence

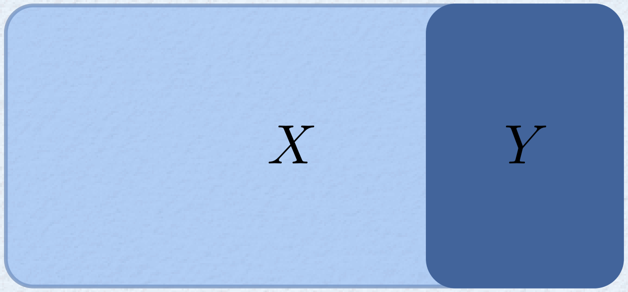
Proposition. Let X and $Y \subseteq X$ be arbitrary sets and

$$H : [0,1] \times X \to X$$

be a continuous function (on both variables) satisfying

1. $\forall x \in X : H(0,x) = x$
2. $\forall y \in Y, \forall t \in [0,1] : H(t,y) \in Y$
3. $\forall x \in X : H(1,x) \in Y$

Then X and Y have the same homotopy type.



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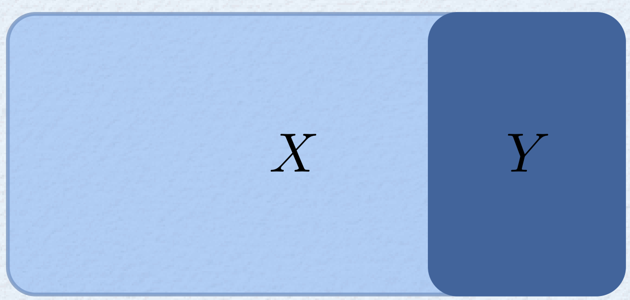

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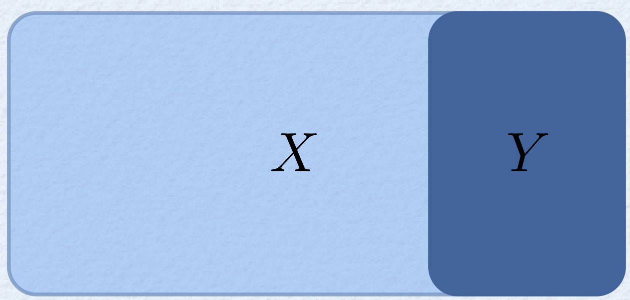
2. $\forall y \in Y, \forall t \in [0,1] : H(t,y) \in Y$
3. $\forall x \in X : H(1,x) \in Y$

Then X and Y have the same homotopy type.



A Criterion for Homotopy Equivalence

Proposition. Let X and $Y \subseteq X$ be arbitrary sets and

$$H : \underbrace{[0,1]}_{\text{time}} \times X \to X$$

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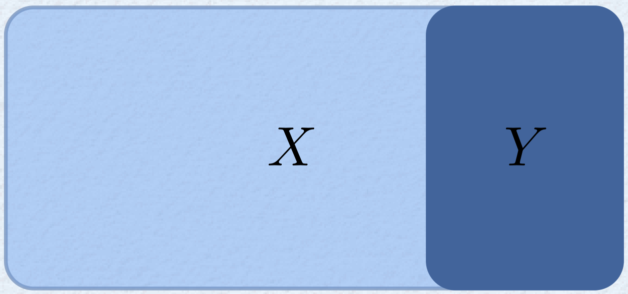

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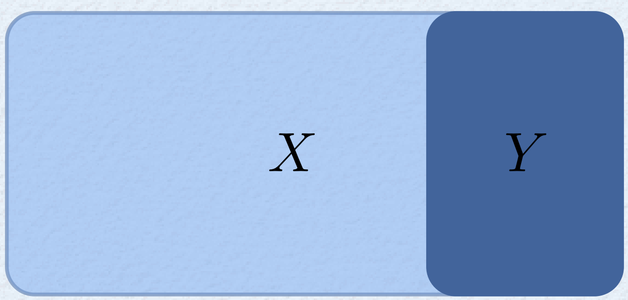

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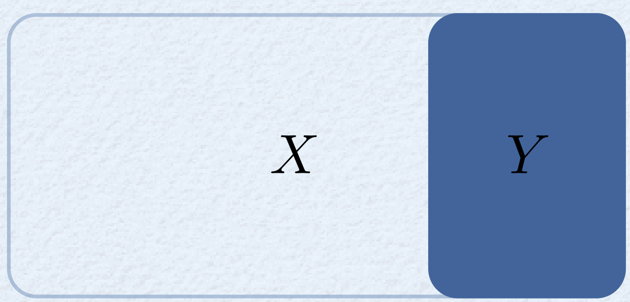

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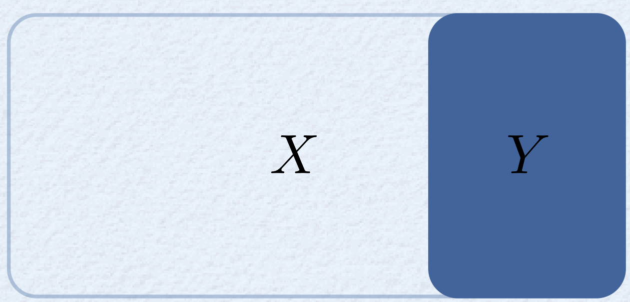

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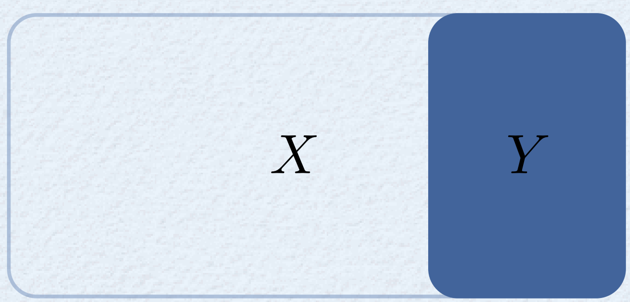

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$$\phi : \underbrace{[0,T]}_{\text{time}} \times X \to X$$

be a continuous function (on both variables) satisfying

1. $\forall x \in X : \phi(0,x) = x$ — Always true
2. $\forall y \in Y, \forall t \in [0,T] : \phi(t,y) \in Y$ — Nothing leaves Y
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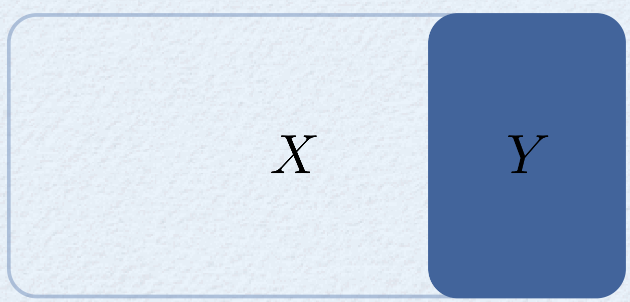

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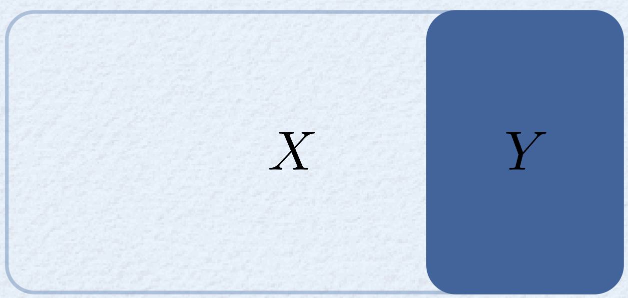

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Always true

$\phi(Y) = Y$

Enough if $\|v(x)\| > c, \forall x \in X \setminus Y$

since

$$\begin{aligned} h(t) &= h(0) + \int_0^t \|v(\tau)\|^2 d\tau \\ &< \operatorname{diam} X \end{aligned}$$

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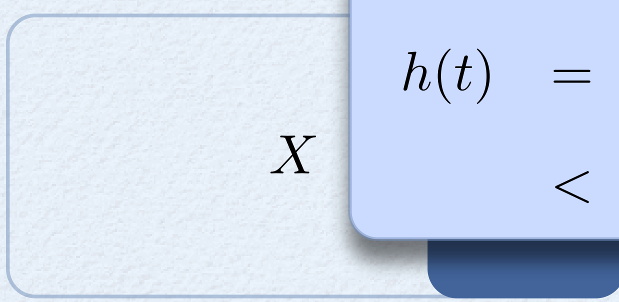
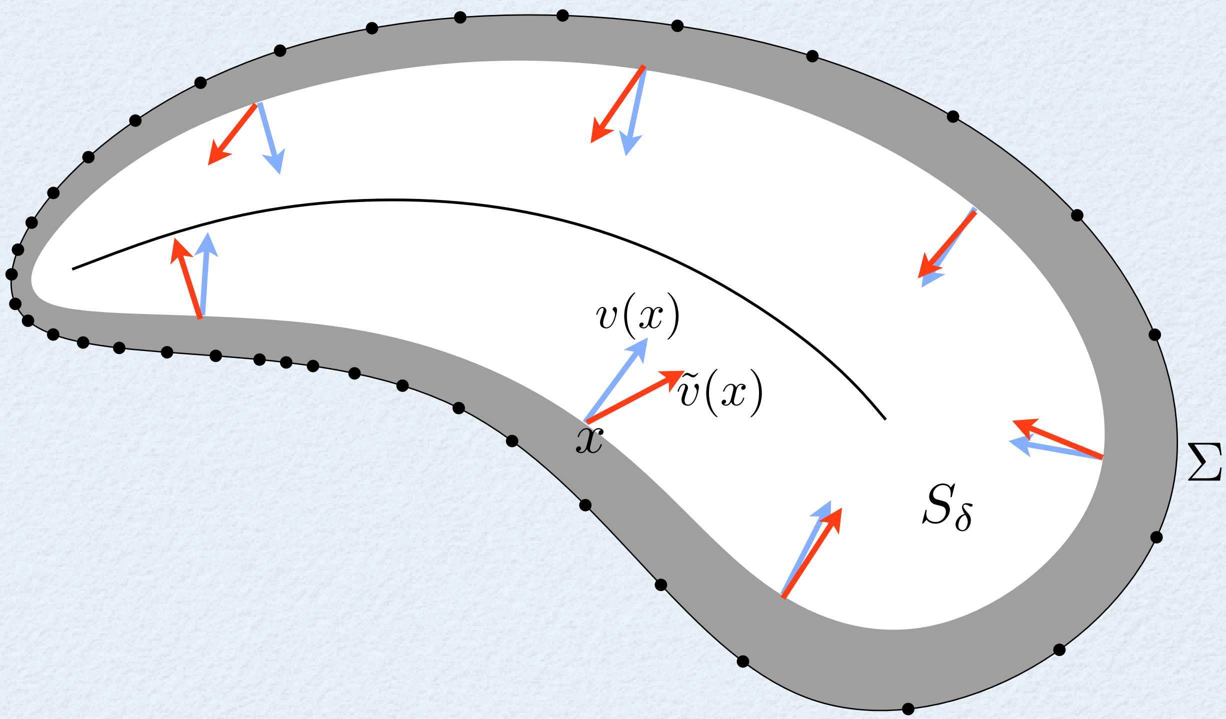


Reduced Shapes are Closed under Discrete and Continuous Flows

Lemma. If $\varepsilon < 0.14$ and let $\varepsilon^2 \leq \delta < 10\varepsilon^2$. Then S_δ is closed under both ϕ and $\tilde{\phi}$.

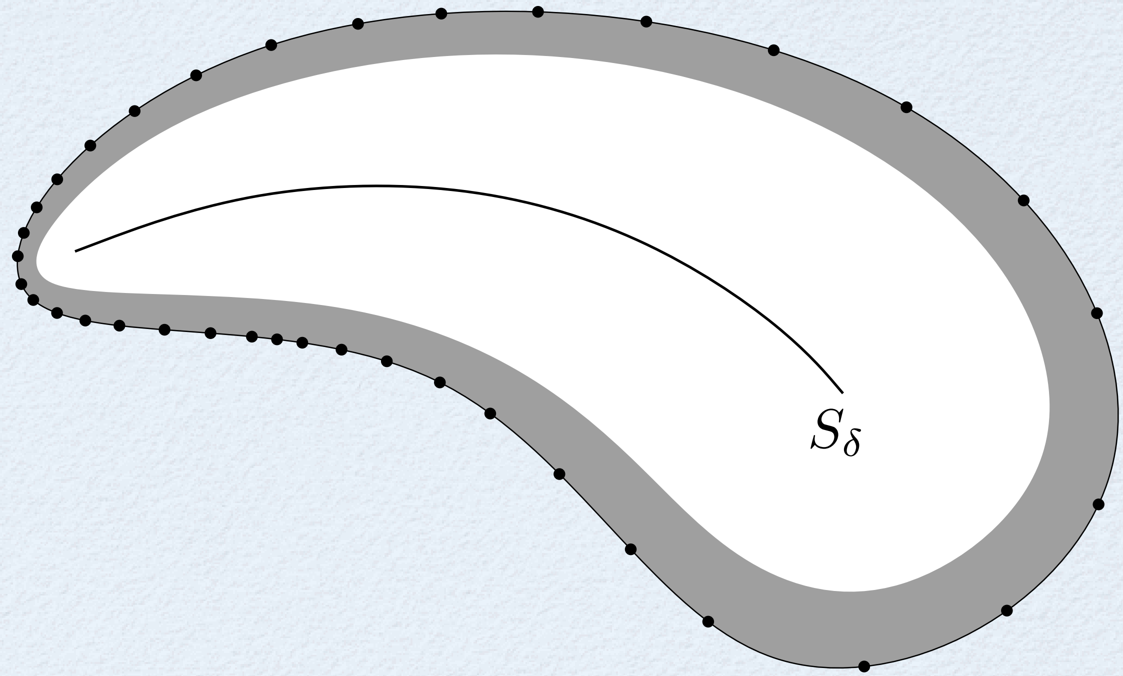
$$\phi(S_\delta) = S_\delta$$



Shape and Reduced Shape are Homotopy Equivalent

Step 1. $S \simeq S_\delta$

Proof. Using ϕ:



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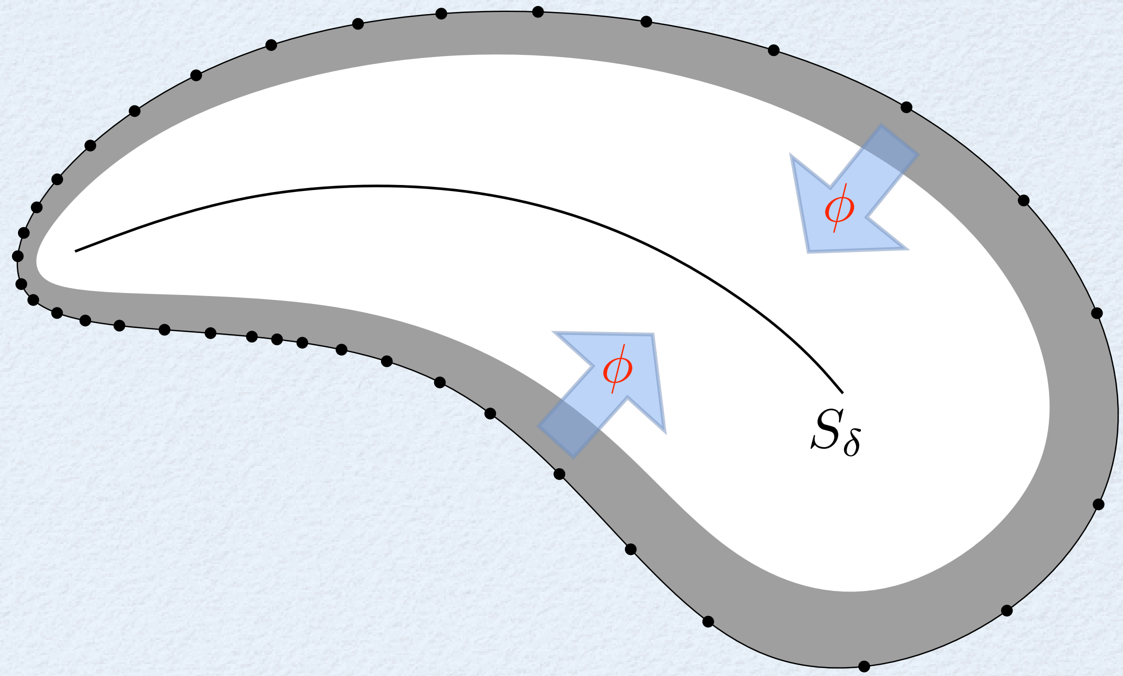

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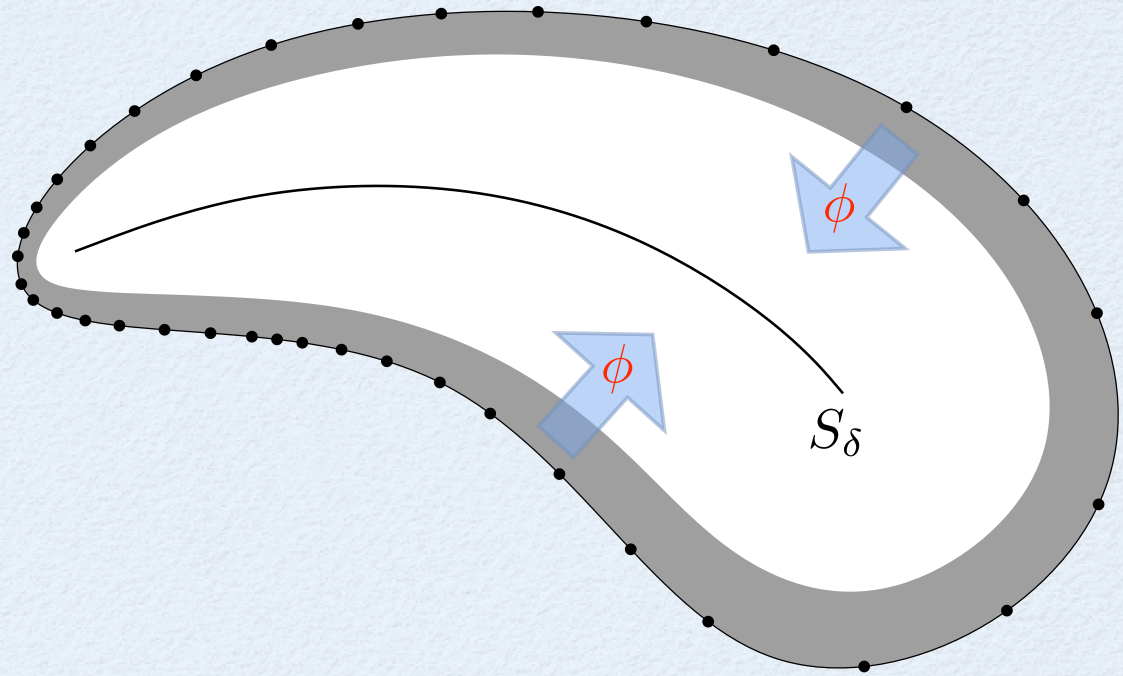

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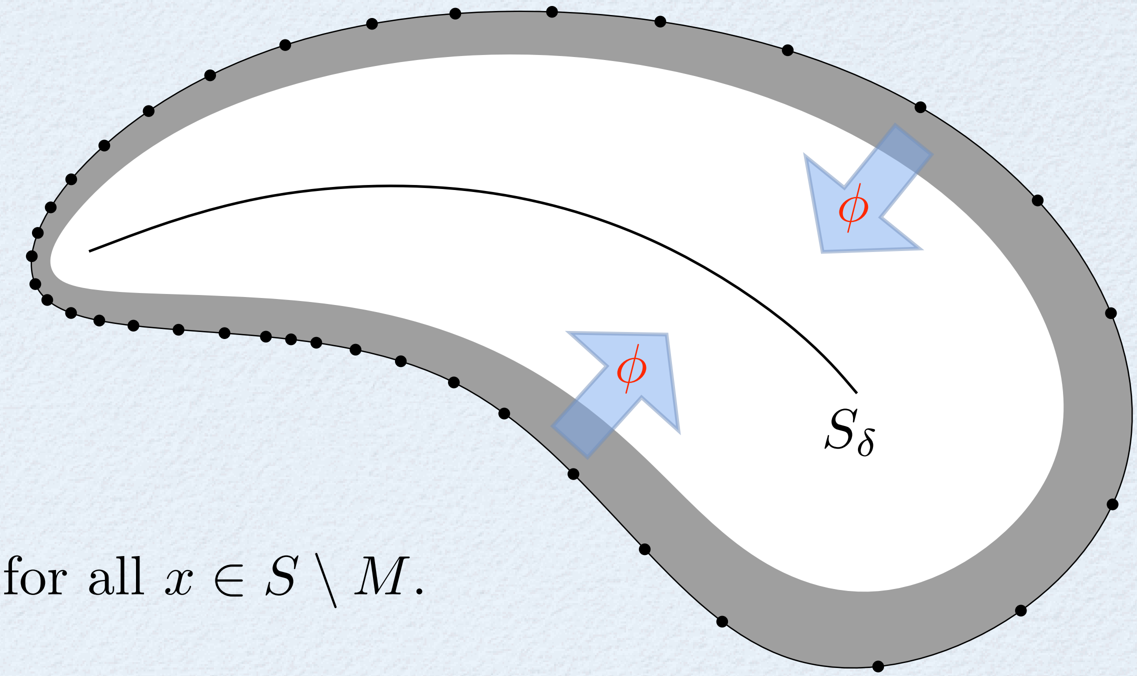
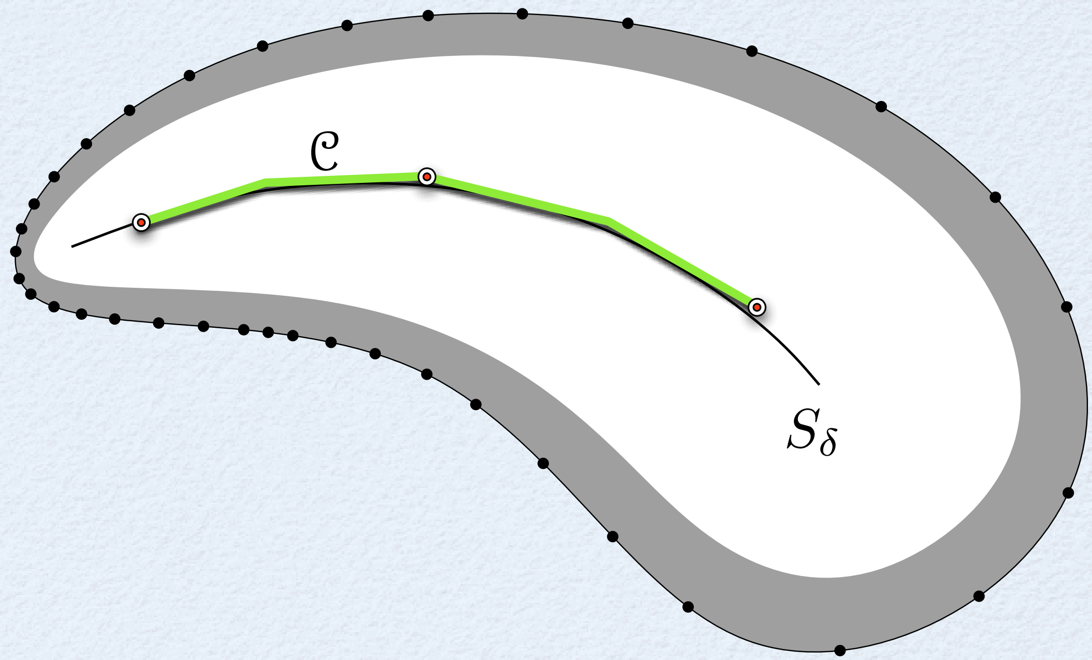


Reduced Shape and Core are Homotopy Equivalent

Step 2. $S_\delta \simeq \mathcal{C}$

Proof. Using $\tilde{\phi}$:



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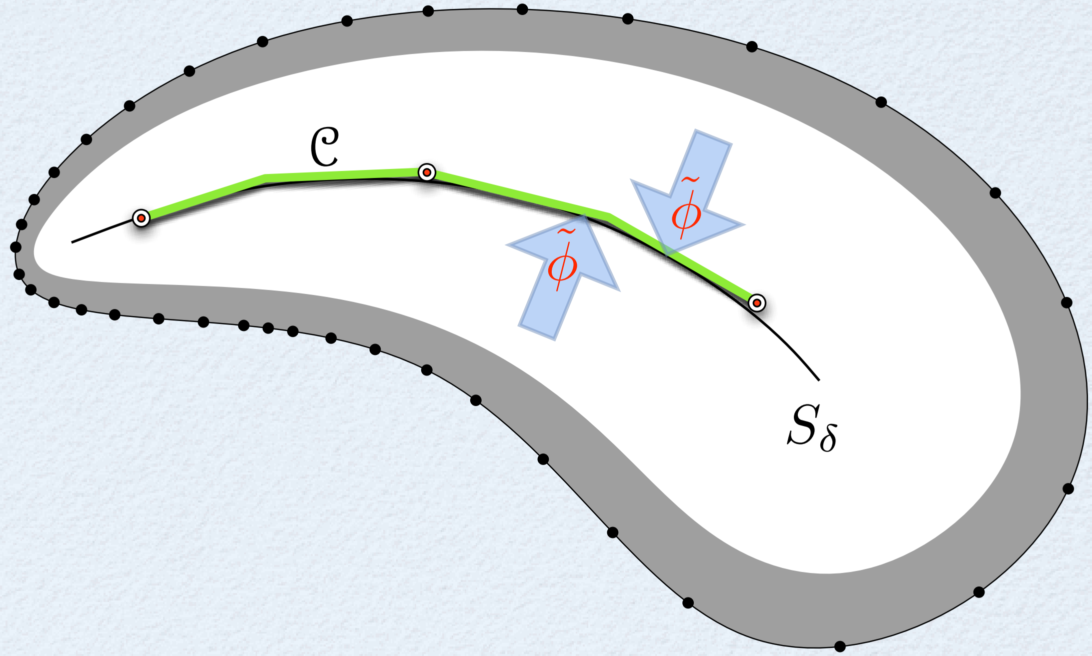

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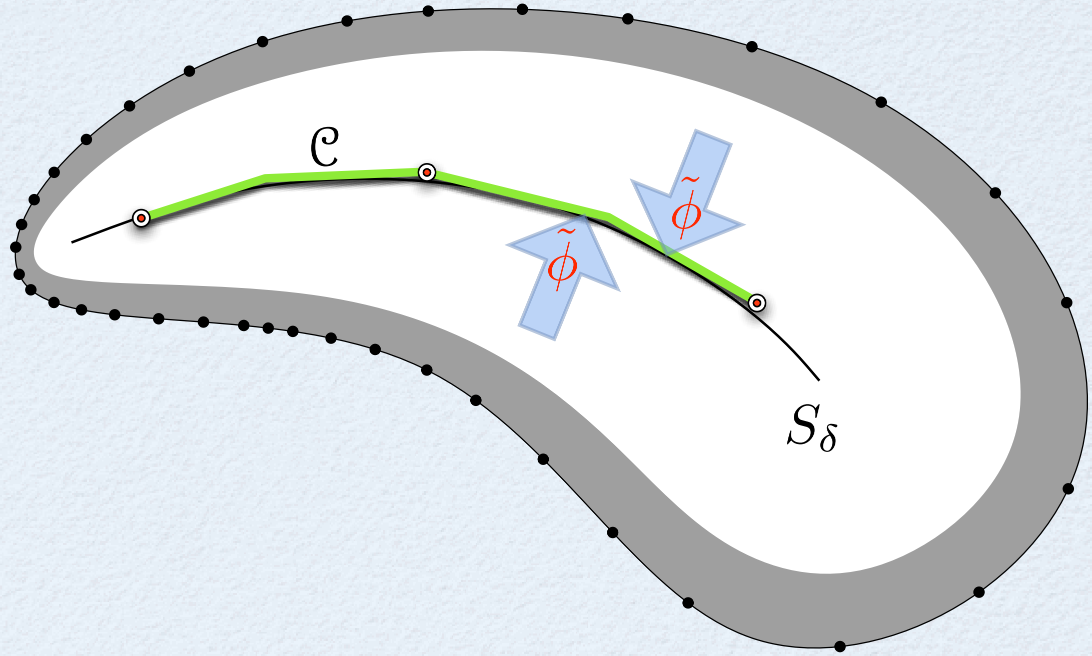

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Step 2. $S_\delta \simeq \mathcal{C}$

Proof. Using $\tilde{\phi}$:

- $\tilde{\phi}(\mathcal{C}) = \mathcal{C}$
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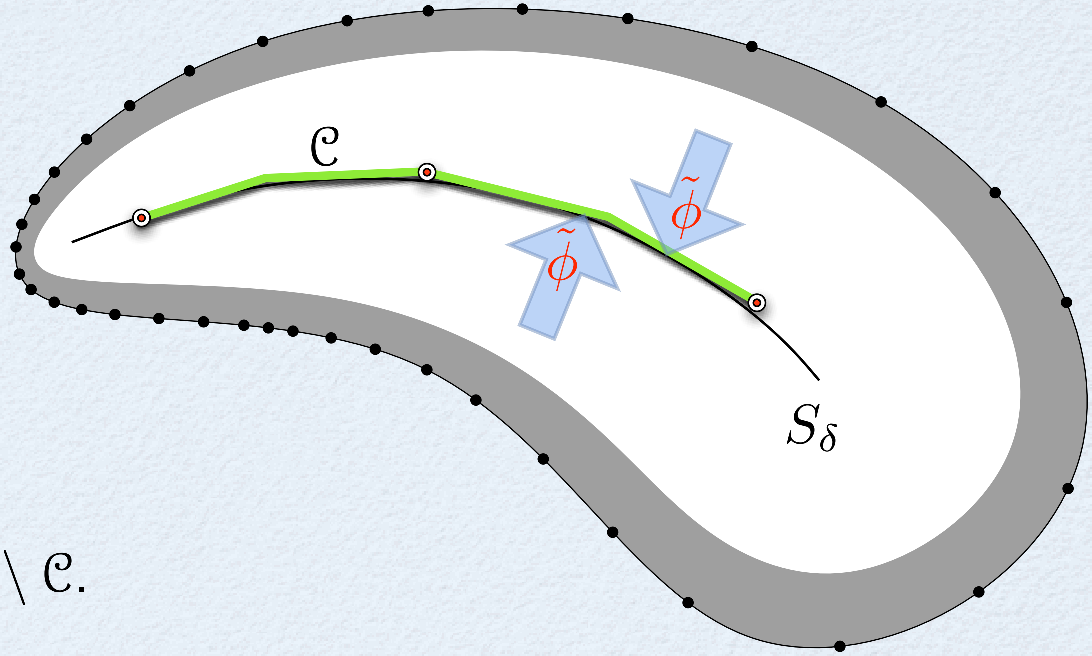

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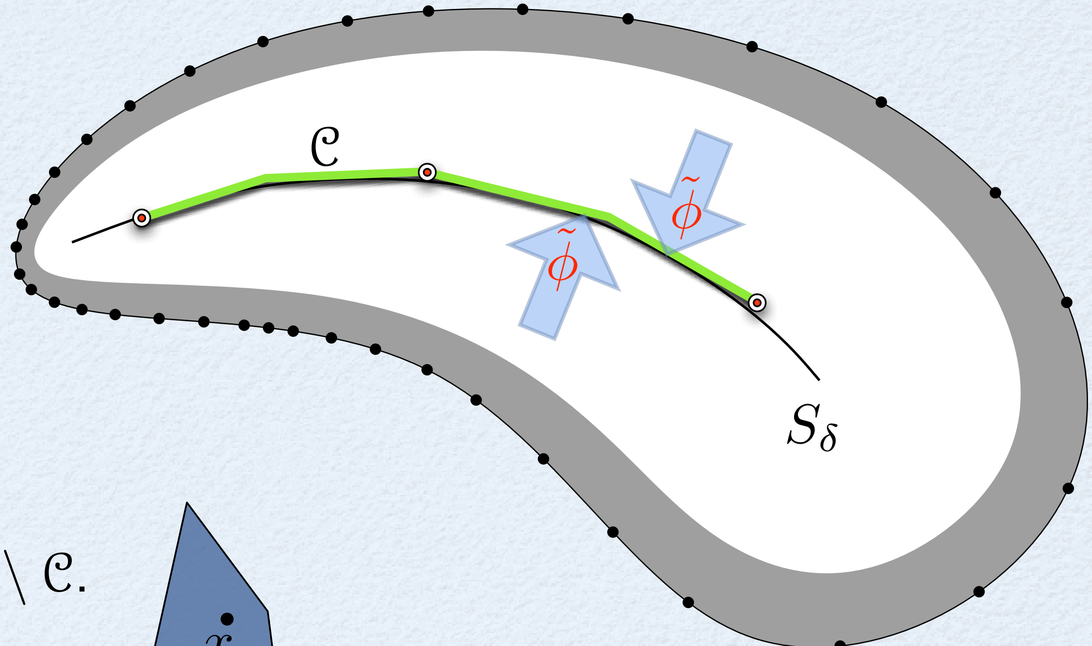

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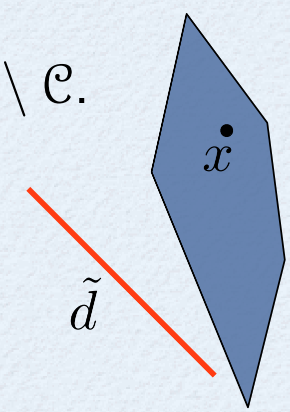
$$\tilde{v}(x) = \frac{x - \tilde{d}(x)}{\tilde{h}(x)}$$



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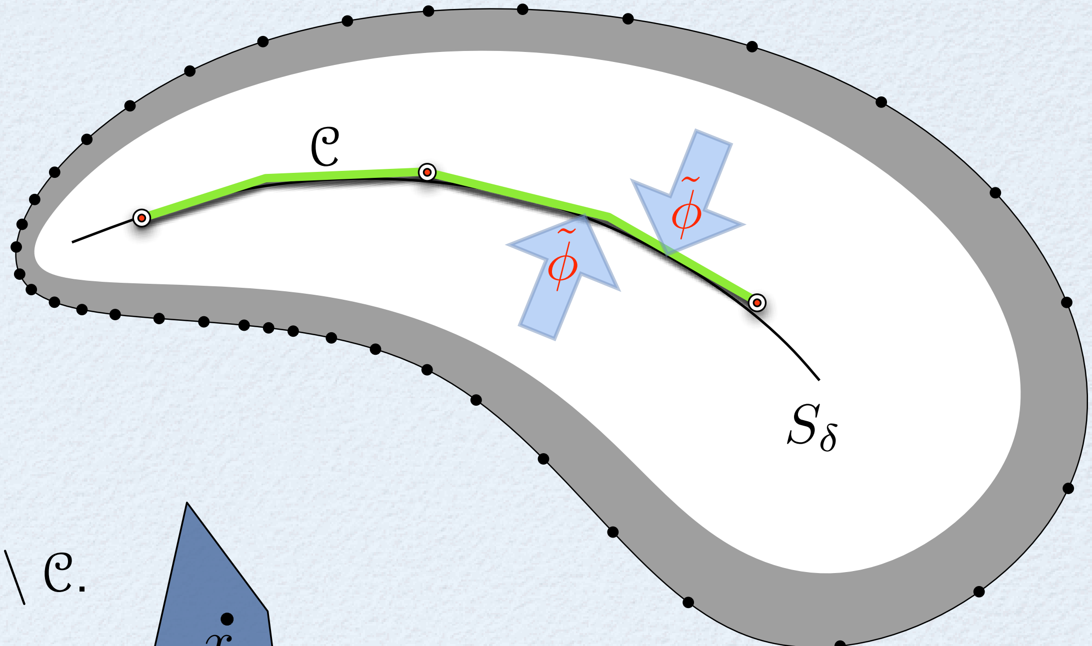

Proof. Using $\tilde\phi$:

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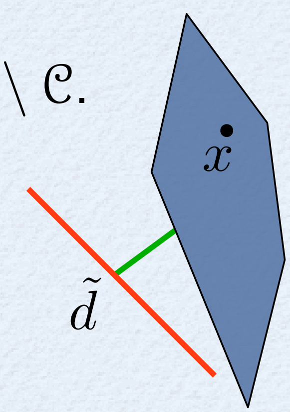

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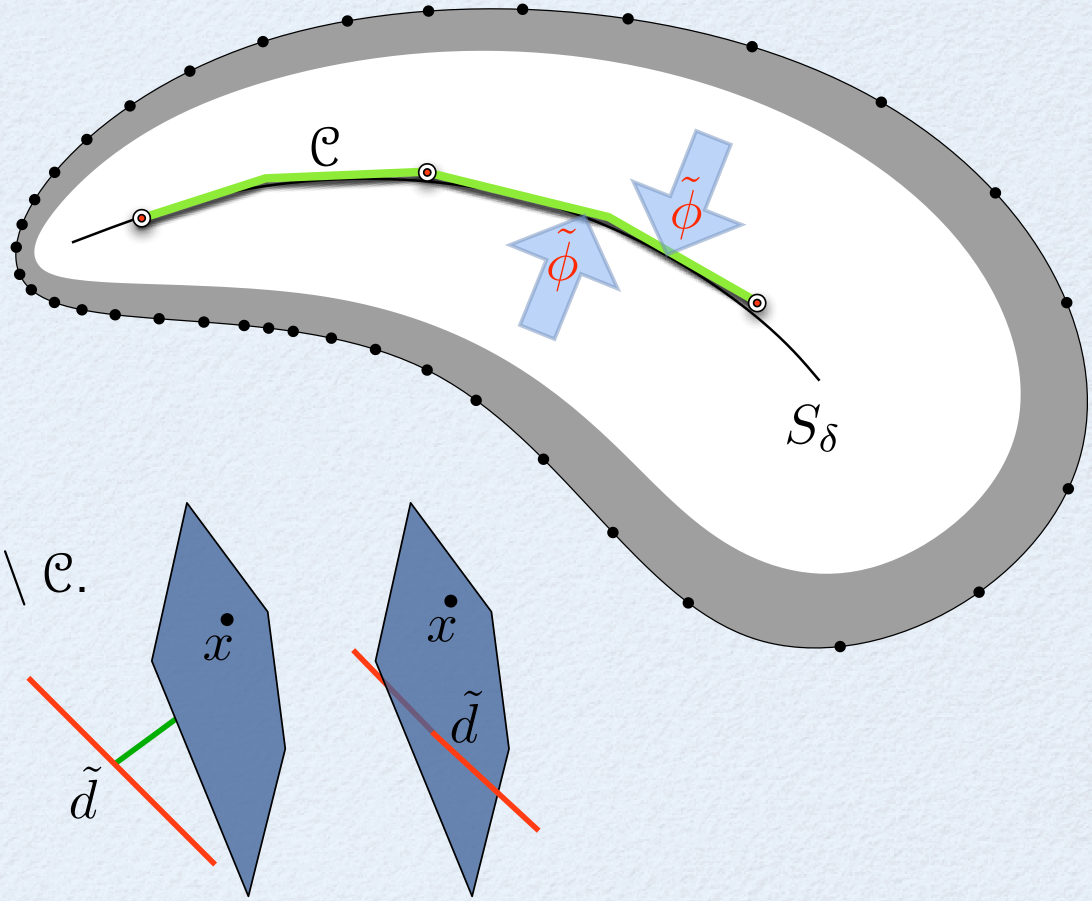

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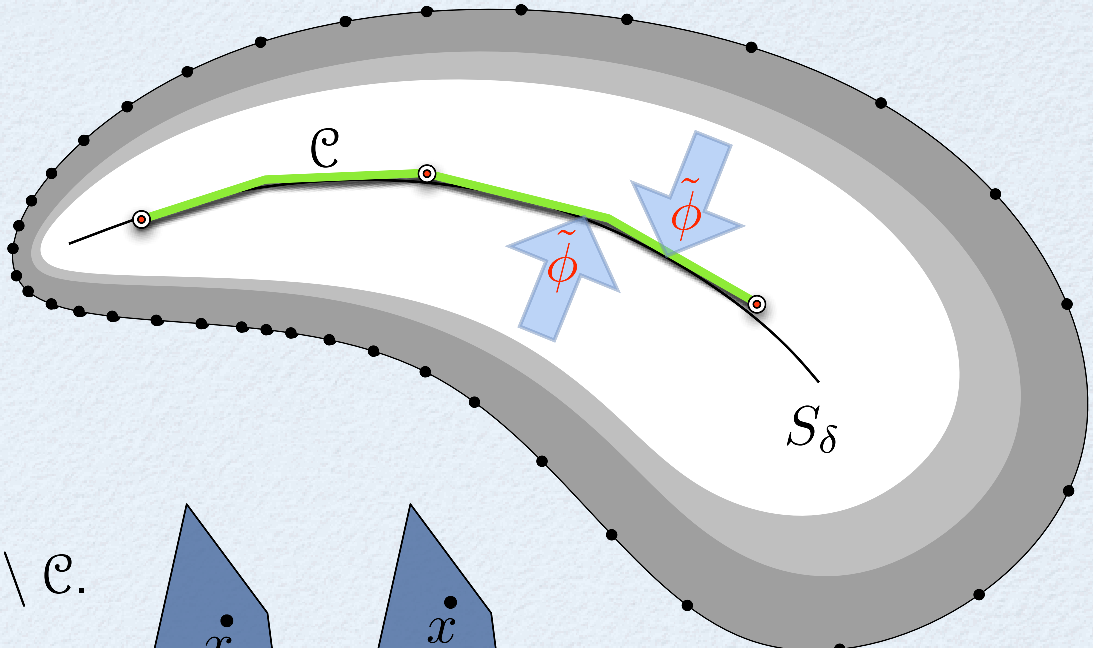

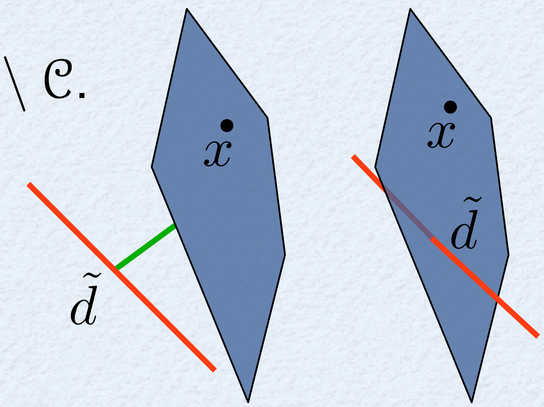

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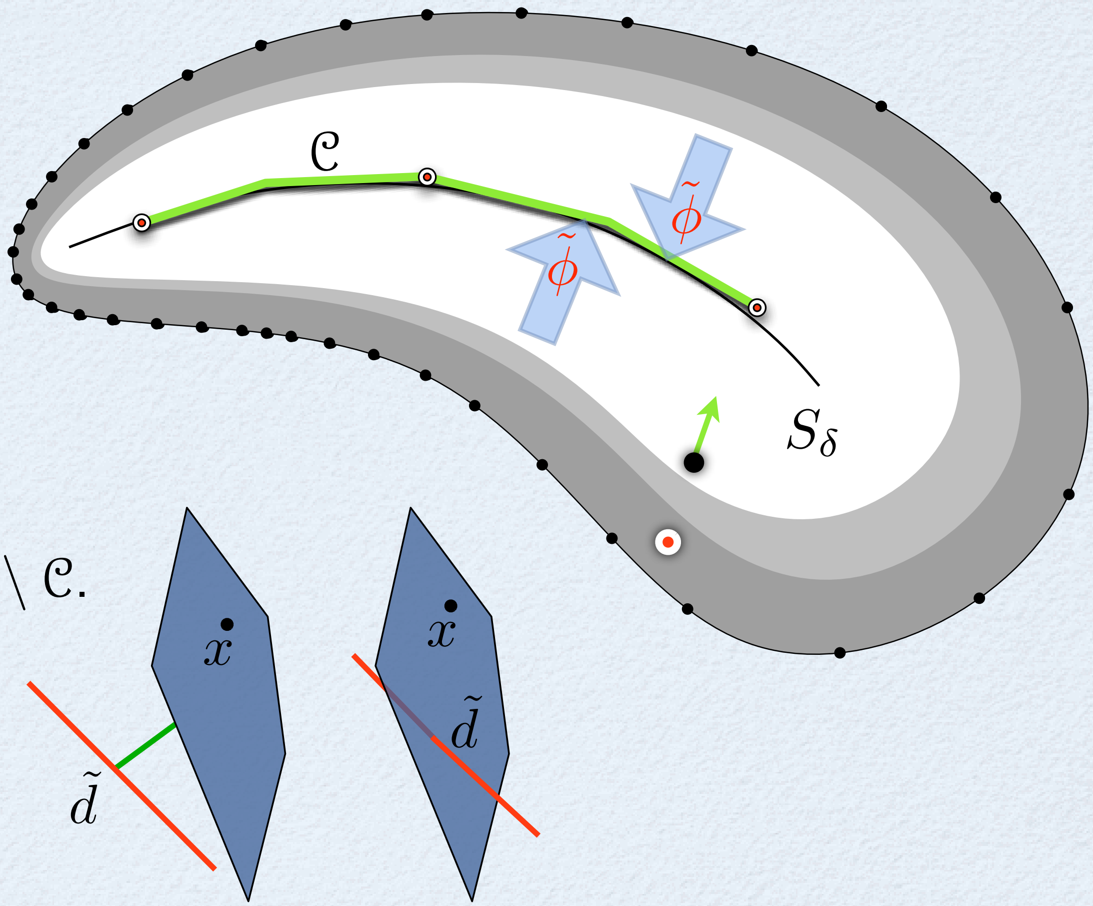

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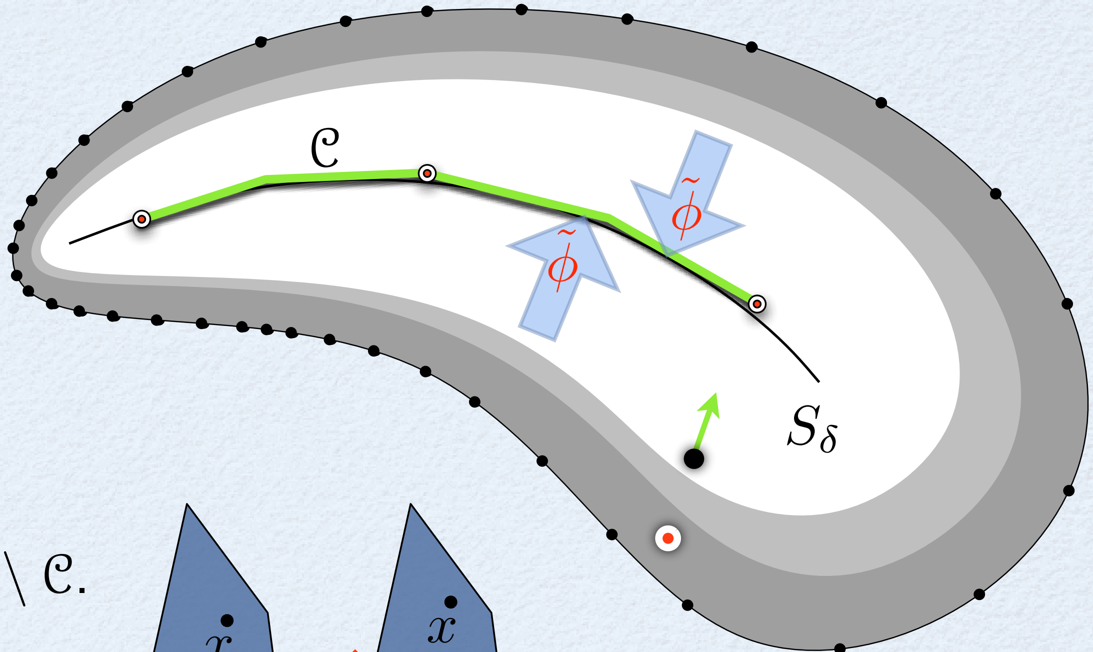
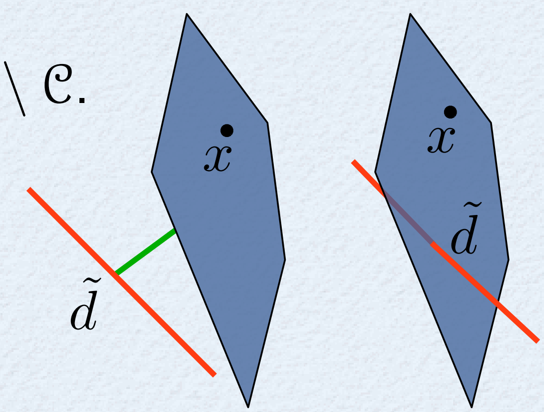
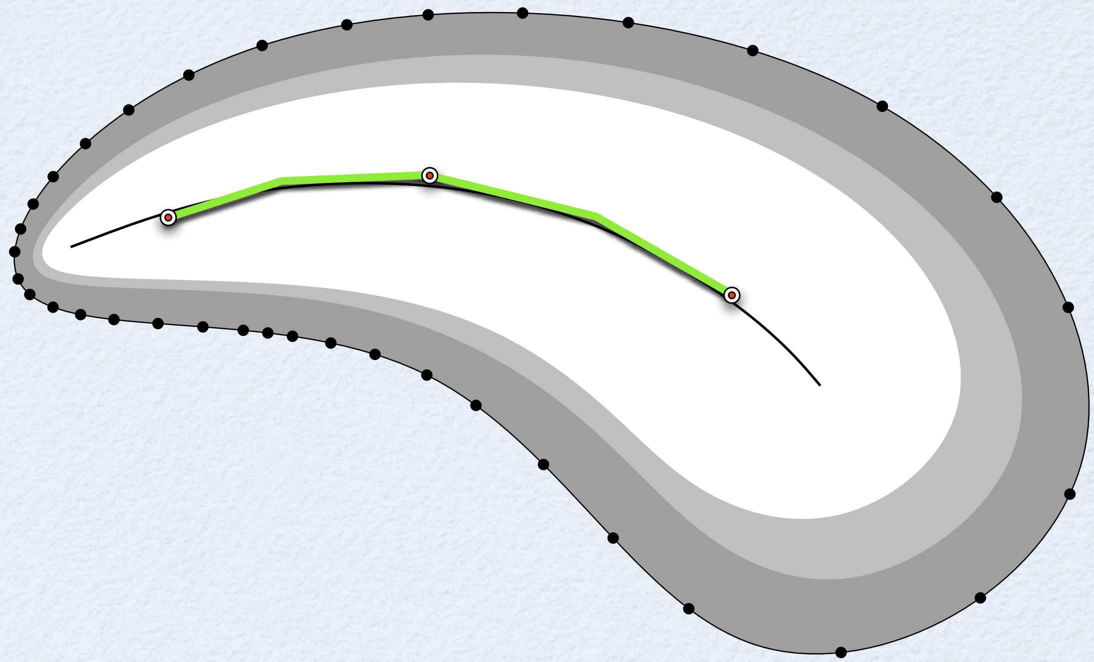




Thus core and shape (and MA) are homotopy equivalent.

Improving Geometric Quality
or
Fattening the Core



Corollary. For any $T \subset S_\delta$,

$$\phi(T) \cup \mathcal{C}$$

is homotopy equivalent to $M(S)$.

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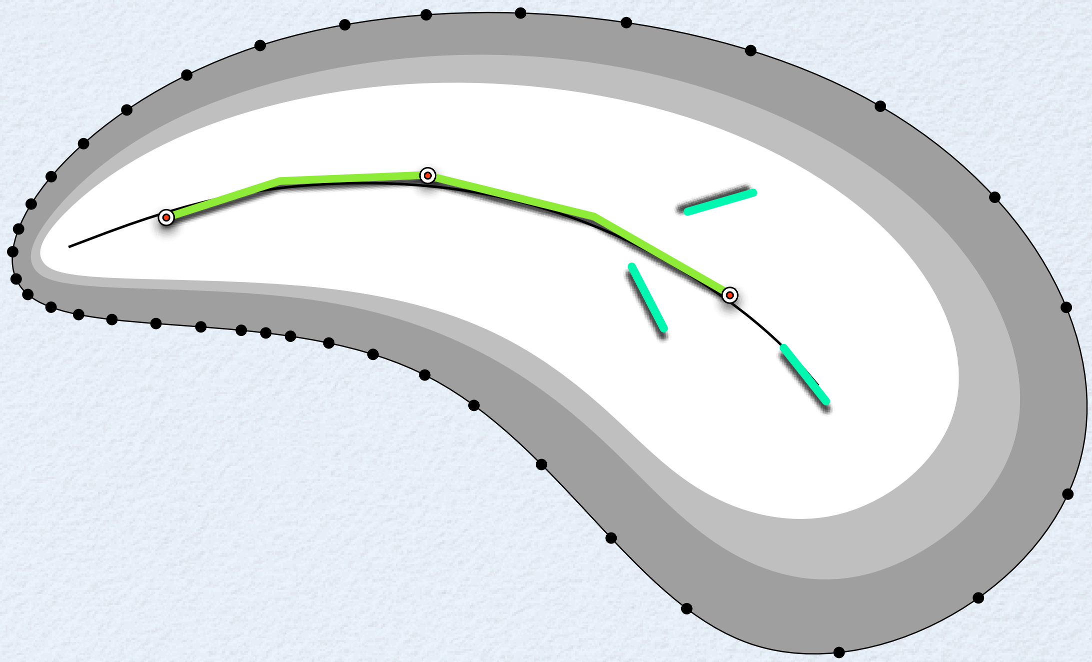

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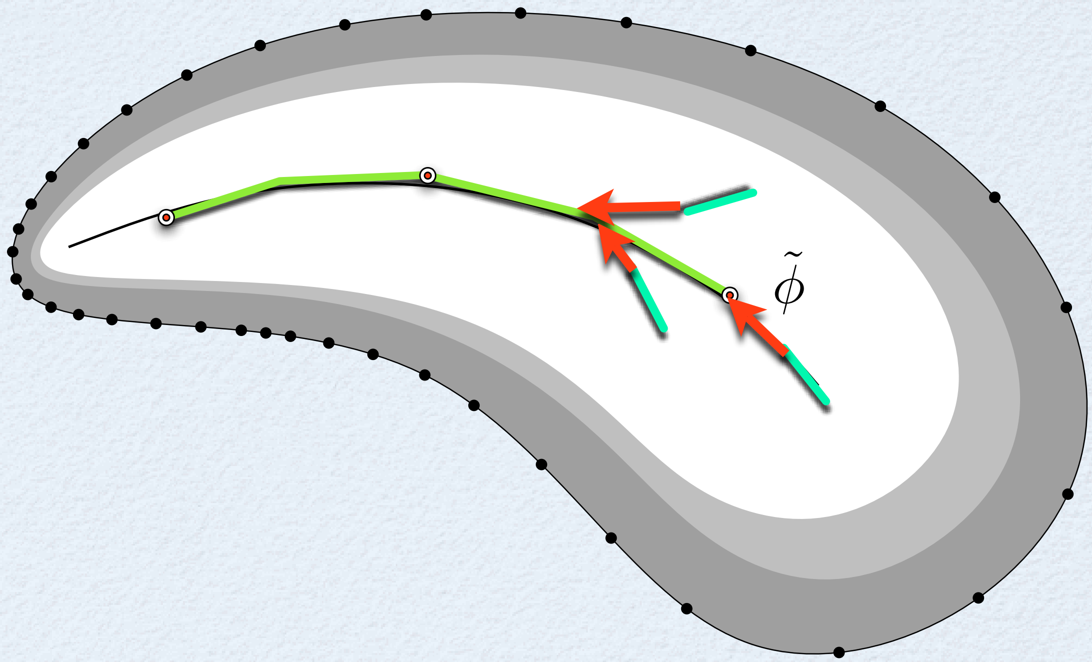

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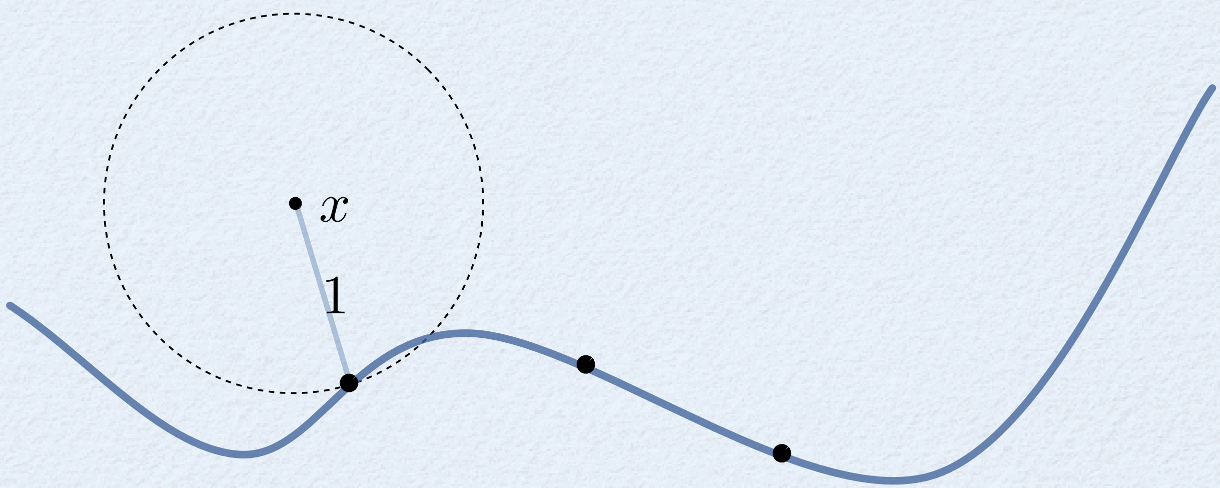
Geometric Quality and Flow Closure

Can we diverge too far away from MA when taking flow closure?



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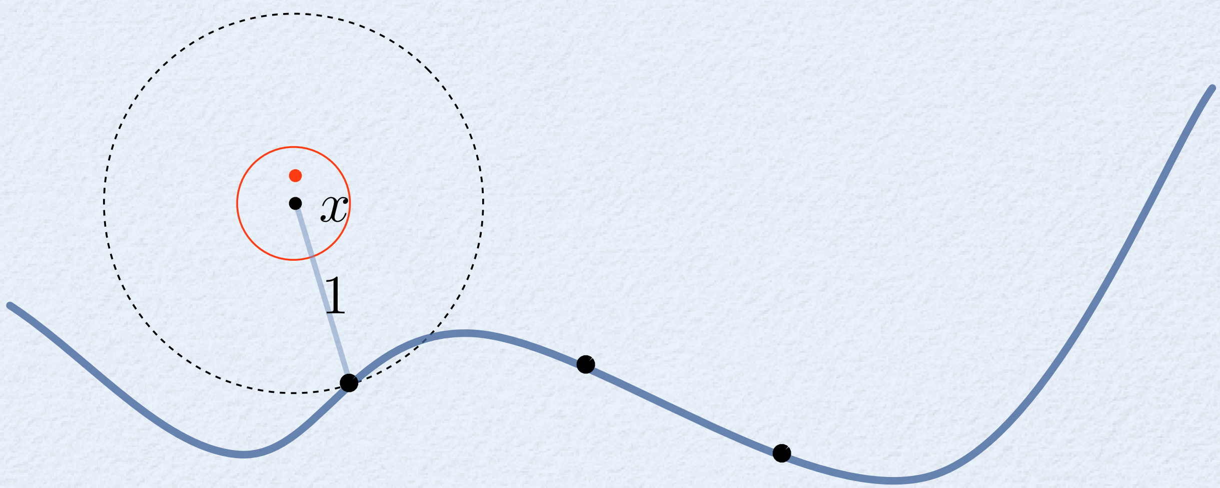

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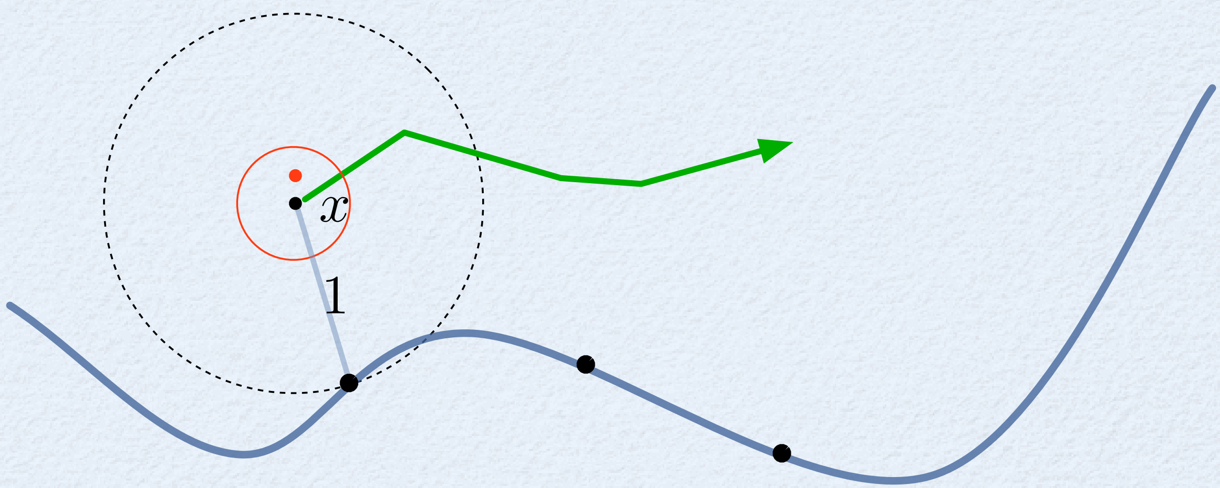

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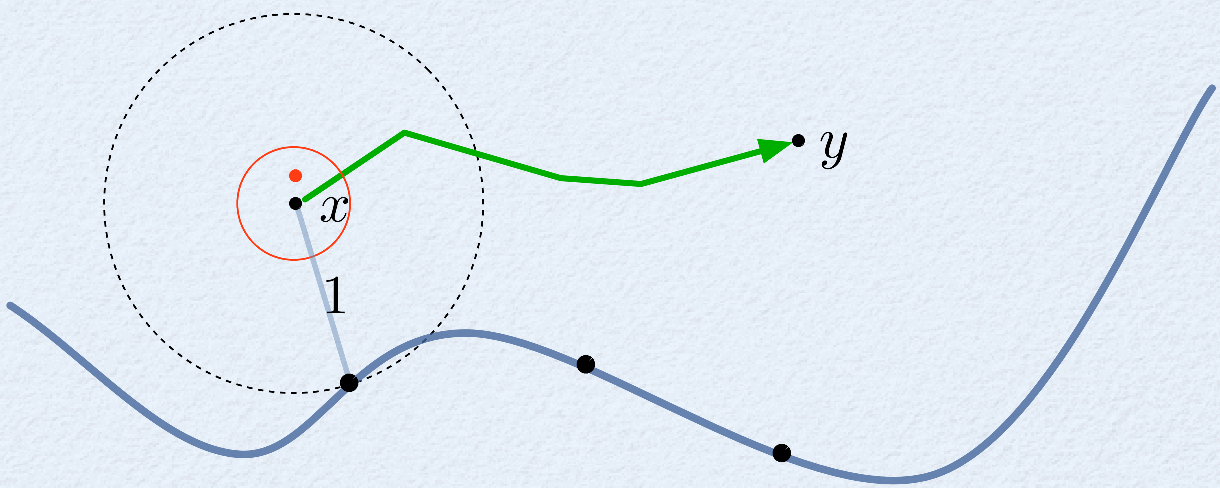

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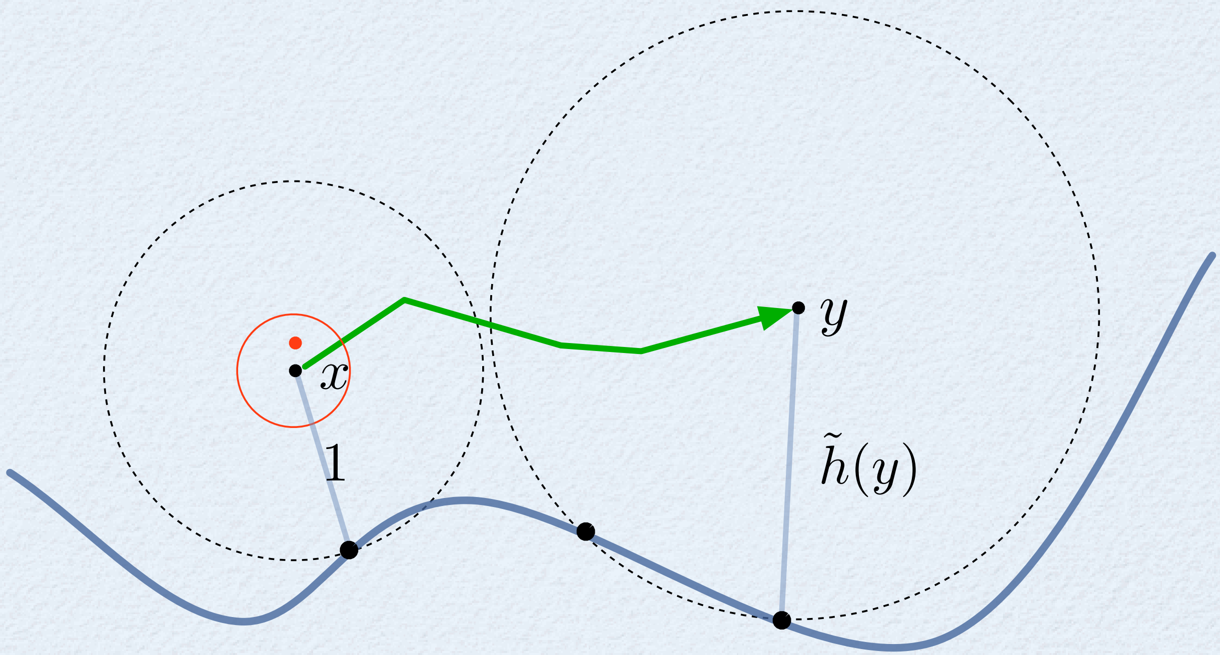

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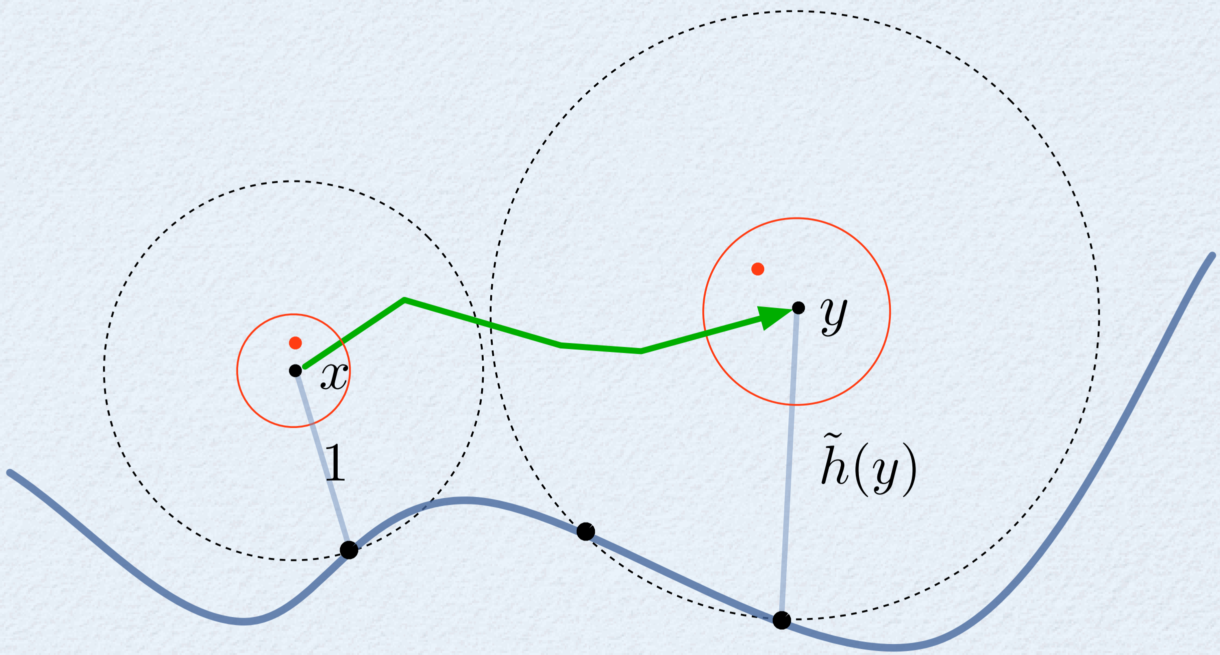

Geometric Quality and Flow Closure

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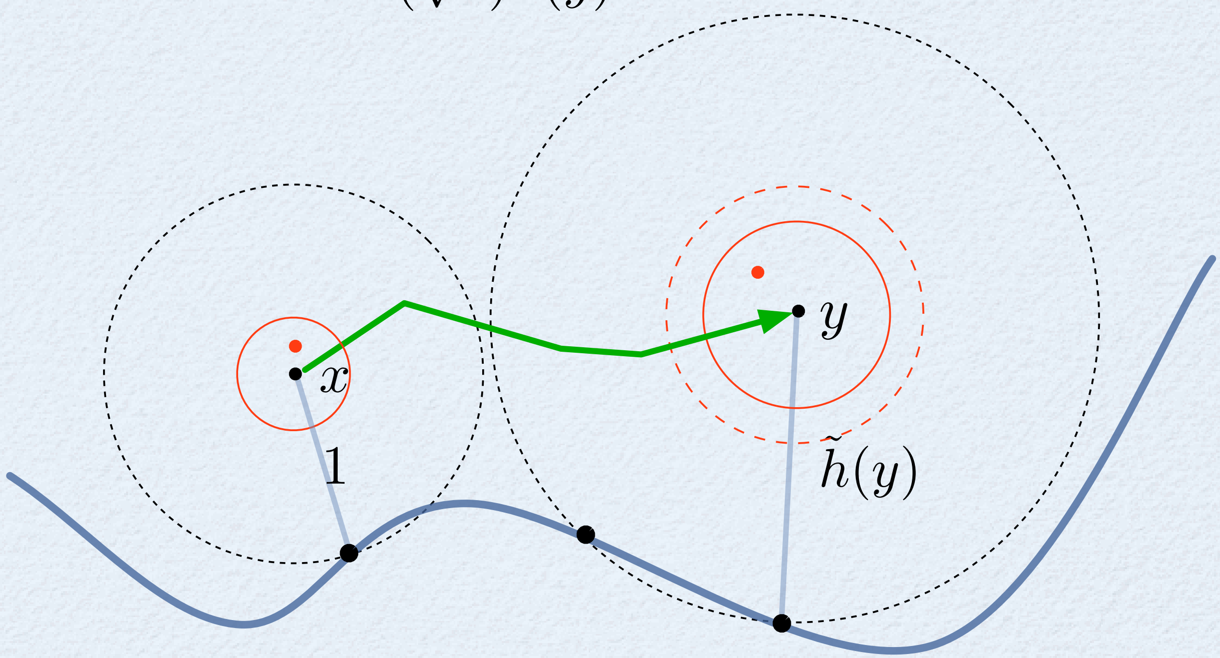
Theorem. If $\tilde{h}(x) = 1$ and x has a medial axis point within distance $O(\sqrt{\varepsilon})$, then for any $t \geq 0$, $y = \tilde{\phi}(t, x)$ has a medial axis point within distance

$$O(\sqrt{\varepsilon})\tilde{h}(y)^{1+O(\sqrt{\varepsilon})}.$$



Conclusions and Open Problems

1. Core is defined as the union of unstable manifolds of (inner) medial axis critical points.
2. Core has the same homotopy type as MA.
3. Core plus the flow closure of any approximation has the right homotopy.

Some Open Questions:

- Understanding the properties of the core in the limit.
- Can the core be thinned down further to a sub-complex of Vor(P)?
- Can flow closure of Voronoi facets be approximated safely with whole Voronoi facets?
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Thank You!