Geometric and Topological Guarantees for the WRAP Reconstruction Algorithm

Bardia Sadri

Joint Work Edgar Ramos
Given a point cloud sampled from a surface $\Sigma$, we want to compute a surface $\hat{\Sigma}$ that has the same topology as $\Sigma$ and closely approximates it geometrically.
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\begin{center}
\begin{tikzpicture}
\draw[thick,black] (-3,0) .. controls (-1,1) and (1,-1) .. (3,0);
\draw[thick,black] (-3,-2) .. controls (-1,-1) and (1,-1) .. (3,-2);
\draw[thick,red] (-2,0) .. controls (-1.5,1) and (1.5,-1) .. (2,0);
\draw[thick,red] (-2,-2) .. controls (-1.5,-1) and (1.5,-1) .. (2,-2);
\node at (0,0) {$S$};
\node at (0,-2) {$S^*$};
\node at (-1,1) {$M$};
\node at (1,1) {$M^*$};
\node at (0,0.5) {$\Sigma$};
\end{tikzpicture}
\end{center}
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The medial axis of a surface $\Sigma$ is the union of medial axes of all components of $\mathbb{R}^n \setminus \Sigma$. 
We use the (relative) \( \varepsilon \)-sampling framework of [Amenta-Bern’99].

For a point \( x \in \Sigma \), the \textbf{local feature size} of \( x \) is

\[
\text{lfs}(x) := d(x, M).
\]

\( P \subset \Sigma \) is an \( \varepsilon \)-\textbf{sample} if every \( x \in \Sigma \) has a sample within distance \( \varepsilon \text{lfs}(x) \).

Samples of Surfaces

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There are many surface reconstruction methods!

• As 0-set of an approximate signed distance function: [Hoppe et al’92, Curless et al’96]

• As other iso-surfaces:
  - NN Interpolation [Boissonnat-Cazals’02]
  - MLS [Levin’98, Alexa et al’01, Amenta-Kil’04, Kolluri’05, Dey et al’05]
  - SVM [Schölkopf et al’04]

• Delaunay Methods:
  - [Boissonnat’84, Amenta-Bern’99, Amenta et al’91, Amenta-Choi-Kolluri’01]

• Using distance functions:
  - [Edelsbrunner’04, Chaine’03, Giesen-John’03, Dey et al’05]
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A Sketch of the WRAP Algorithm
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The WRAP Algorithm [Edelsbrunner'04]
For every $\tau \in \text{Del } P$, if $\tau$ is reachable from no “centered” simplex other than $\omega$, then remove $\tau$.

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The Machinery
A discrete set of points $P$ induces a squared distance function

$$h(x) = \min_{p \in P} \|x - p\|^2$$
(Squared) Distance to Discrete Point Sets

\[ h(x) = \min_{p \in P} \| x - p \|^2 \]

\( P \) is a discrete set of points

The squared distance function induced by \( P \) is

**Observation.** \( h \) is smooth at points with a unique closest point in \( P \).
Generalized Gradient

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\[ \mathbf{x} \]

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\[ v(x) = 2(x - d(x)) \]

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Moving at point $x$ in with speed $v(x)$ results a flow map $\phi : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. 

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x(0) &= x_0 \\
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Integrating \( \mathcal{V} \)

Moving at point \( x \) in with speed \( v(x) \) results a flow map \( \phi : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n \).

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$$\phi(x) = \{ \phi(t, x) : t \geq 0 \}$$

$$\phi(X) = \bigcup_{x \in X} \phi(x)$$
Theorem. The flow map \( \phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is continuous on both variables.
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**Theorem.** For $y = \phi(t, x)$,

$$h(y) = h(x) + \int_0^t \|v(\phi(\tau, x))\|^2 d\tau.$$
A point $c$ with $v(c) = 0$ is called \textit{critical}.
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The \( \delta \)-tubular neighborhoods of \( \Sigma \) and \( M \):

\[
\Sigma_\delta = \{ x \in \mathbb{R}^n \setminus M : \| x - \hat{x} \| < \delta f(\hat{x}) \}
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M_\delta = \{ x \in \mathbb{R}^n \setminus \Sigma : \| x - \tilde{x} \| < \delta f(\tilde{x}) \}
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**Theorem [DGRS’05]**

If \( h \) is induced by an \( \varepsilon \)-sample of \( \Sigma \) with \( \varepsilon < 1/\sqrt{3} \), the all critical points of \( h \) are contained in either \( \Sigma_{\varepsilon^2} \) or \( M_{2\varepsilon^2} \).
Stable manifold of a critical point $c$ is everything that flows into $c$.

$$\text{Sm}(c) = \{ x : \phi(\infty, x) = c \}.$$
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Unstable manifold of a critical point $c$ is everything that flows “out of” $c$.

$$\text{Um}(c) = \bigcap_{\varepsilon > 0} \phi(B(x, \varepsilon)) = \phi(V(c)).$$
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Proposition. Let $X$ and $Y \subseteq X$ be arbitrary sets and

$$H : [0, 1] \times X \to X$$

be a \textit{continuous} function (on both variables) satisfying

1. $\forall x \in X : H(0, x) = x$
2. $\forall y \in Y, \forall t \in [0, 1] : H(t, y) \in Y$
3. $\forall x \in X : H(1, x) \in Y$

Then $X$ and $Y$ have the same homotopy type.
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   **Nothing leaves $Y$**

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A Criterion for Homotopy Equivalence

**Proposition.** Let $X$ and $Y \subseteq X$ be arbitrary sets and

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**Proposition.** Let $X$ and $Y \subseteq X$ be arbitrary sets and

$$H : [0, T] \times X \rightarrow X$$

be a **continuous** function (on both variables) satisfying

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Proposition. Let \( X \) and \( Y \subseteq X \) be arbitrary sets and
\[
\phi : [0, T] \times X \to X
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Then $X$ and $Y$ have the same homotopy type.

This is the idea Lieutier used in [Lieuteir'04] to show $M(S) \simeq S$. 

Always true

$\phi(X) = X$

$\phi(Y) = Y$

Everything in $Y$ by time 1
**Key Theorem.** If $Y \subset X$ are bounded and

1. $\phi(X) = X$ and $\phi(Y) = Y$, and
2. $\|v(x)\| \geq c > 0$ for $x \in X \setminus Y$,

then $X$ and $Y$ are homotopy equivalent.
Key Theorem. If $Y \subset X$ are bounded and

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Key Theorem. If \( Y \subset X \) are bounded and

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Proof. If \( \phi(t, x) \notin Y \), then

\[
 h(\phi(t, x)) = h(x) + \int_0^t \|v(\phi(\tau, x))\|^2 d\tau \\
\geq h(x) + \int_0^t c^2 d\tau \\
= h(x) + tc^2 \\
< d_H(X, P)^2.
\]
If $V(x) \cap D(x) = \emptyset$ then

$$
\|v(x)\| = 2 \cdot \|x - d(x)\| \\
\geq 2 \cdot \text{dist}(V(x), D(x)).
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If $V(x) \cap D(x) = \emptyset$ then

$$\|v(x)\| = 2 \cdot \|x - d(x)\| \geq 2 \cdot \text{dist}(V(x), D(x)).$$

If $V(x) \cap D(x) = \{c\}$ then $x \in \text{Um}(c)$. 
A Handy Lower Bound for Speed

If \( V(x) \cap D(x) = \emptyset \) then

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\|v(x)\| = 2 \cdot \|x - d(x)\| \geq 2 \cdot \text{dist}(V(x), D(x)).
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If \( V(x) \cap D(x) = \{c\} \) then \( x \in \text{Um}(c) \).

So, if \( \text{Um}(c) \subset Y \) we are fine!
Flow Induced by Weighted Points

Squared distance to $p$ with weight $w_p$ is $\|x - p\|^2 - w_p$.

The squared distance to a set $P$ of weighted points is

$$h(x) = \min_{p \in P} \|x - p\|^2 - w_p.$$
For every set $P$ of weighted points there is a set $Q$ of weighted points such that

\[ \text{Vor } P = \text{Del } Q \quad \text{and} \quad \text{Del } P = \text{Vor } Q \]
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\[ \text{Vor } P = \text{Del } Q \quad \text{and} \quad \text{Del } P = \text{Vor } Q \]
For unweighted $P$, $Q$ is the Voronoi vertices of $P$ and for $q \in Q$:

$$w_q = \text{dist}(q, P)^2.$$
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Critical Points of $h^*$

**Observation.** critical points of $h^*$ and $h$ are the same.

A simplex $\tau \in \text{Del } P$ that contains a critical point is called a **centered simplex**.
Observation. critical points of $h^*$ and $h$ are the same.

A simplex $\tau \in \text{Del } P$ that contains a critical point is called a centered simplex. We treat $\mathbb{R}^n \setminus \text{conv } P$ as an abstract critical simplex $\omega$. 
A Partial Order on Delaunay Simplices

$\tau \prec \sigma$: some flow line of $\phi^*$ visits relative interiors of $\sigma$ and $\tau$ consecutively.

$\tau \prec^* \sigma$: there is a sequence $\tau = \tau_0 \prec \cdots \prec \tau_k = \sigma$. 
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A Partial Order on Delaunay Simplices

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The WRAP Algorithm [Edelsbrunner’04]

1. For every \( \tau \in \text{Del } P \), if the only critical simplex that precedes \( \tau \) is the abstract critical simplex \( \omega \), then remove \( \tau \).

2. Return what is left as WRAP.
The WRAP Algorithm [Edelsbrunner’04]

1. For every $\tau \in \text{Del } P$, if “every” critical simplex that precedes $\tau$ is an outer medial axis critical simplex, then remove $\tau$.

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The Guarantees
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Some Open Questions

- Can the geometric guarantee (and therefore the topological one) be extended to higher dimensions?
- Can WRAP be generalized for reconstruction of shapes with non-smooth boundaries? How should the sampling condition be defined? (some work done in [Lieutier-Chazal'06])
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Thank You!