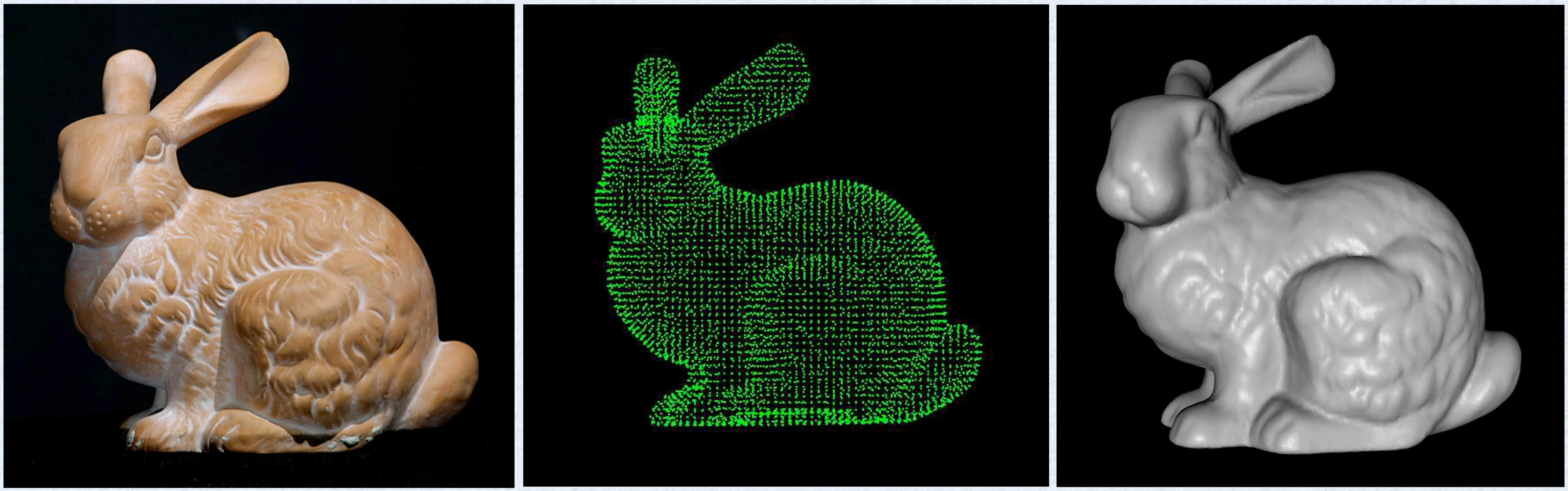
Geometric and Topological Guarantees for the WRAP Reconstruction Algorithm

Bardia Sadri

Joint Work Edgar Ramos

The Surface Reconstruction Problem



Given a point cloud sampled from a surface Σ, we want to compute a surface $\hat{\Sigma}$ that has the same topology as Σ and closely approximates it geometrically.

Shapes, Surfaces, and their Medial Axes

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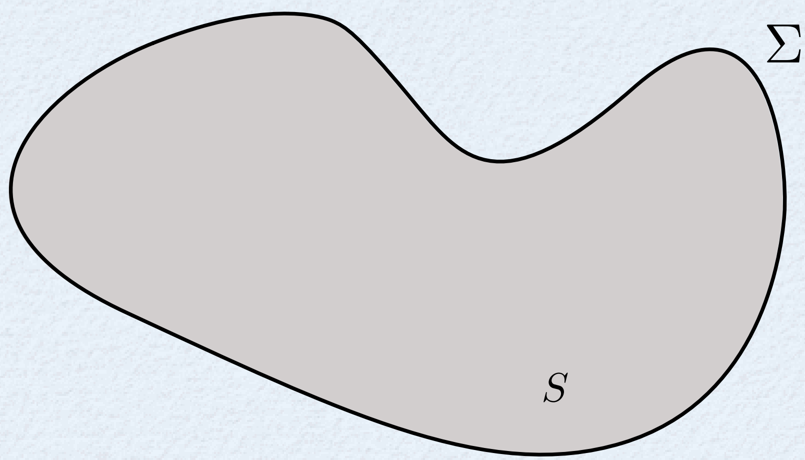

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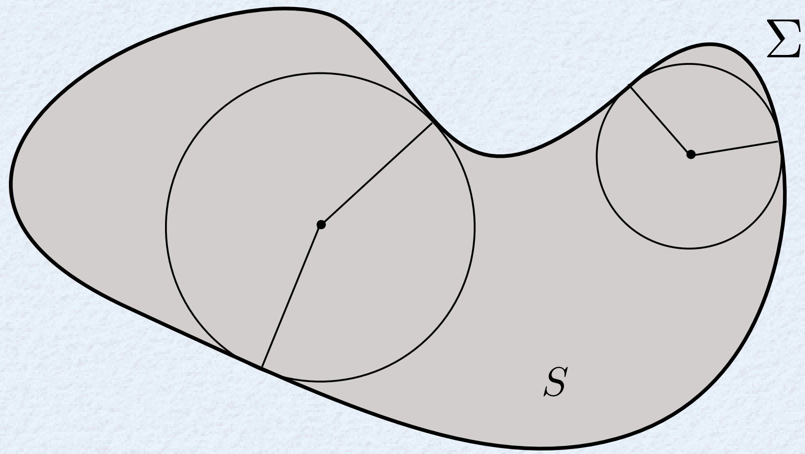
The medial axis (MA) of a shape S is the set of points in S that have ≥ 2 closest points in Σ.



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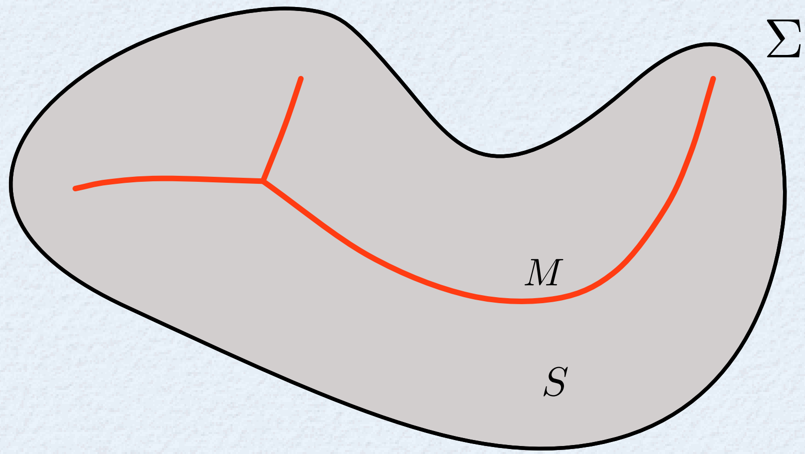

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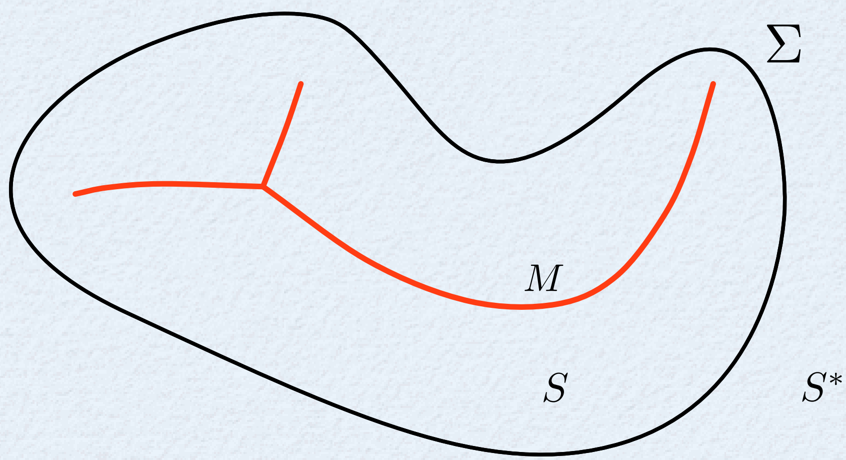

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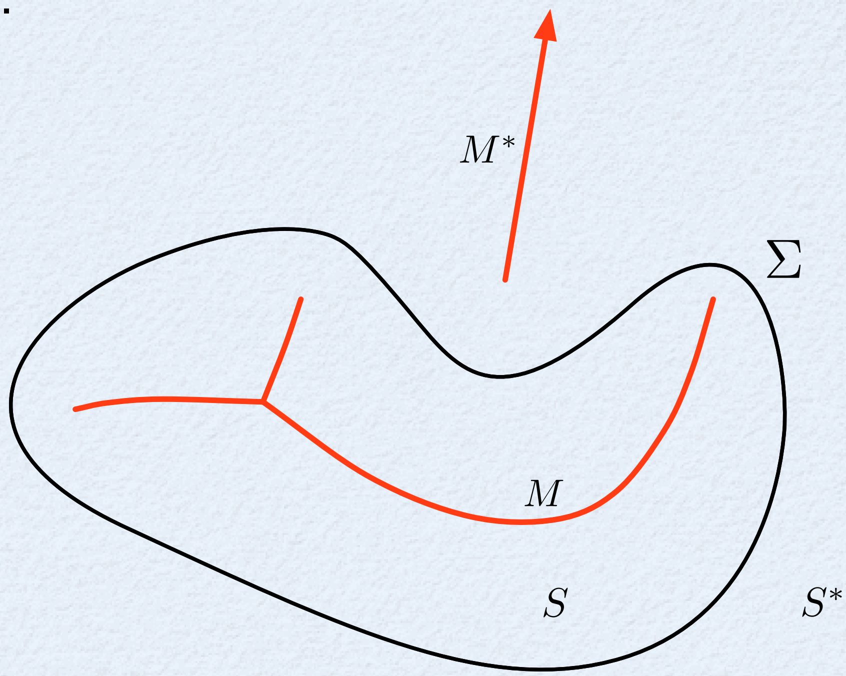

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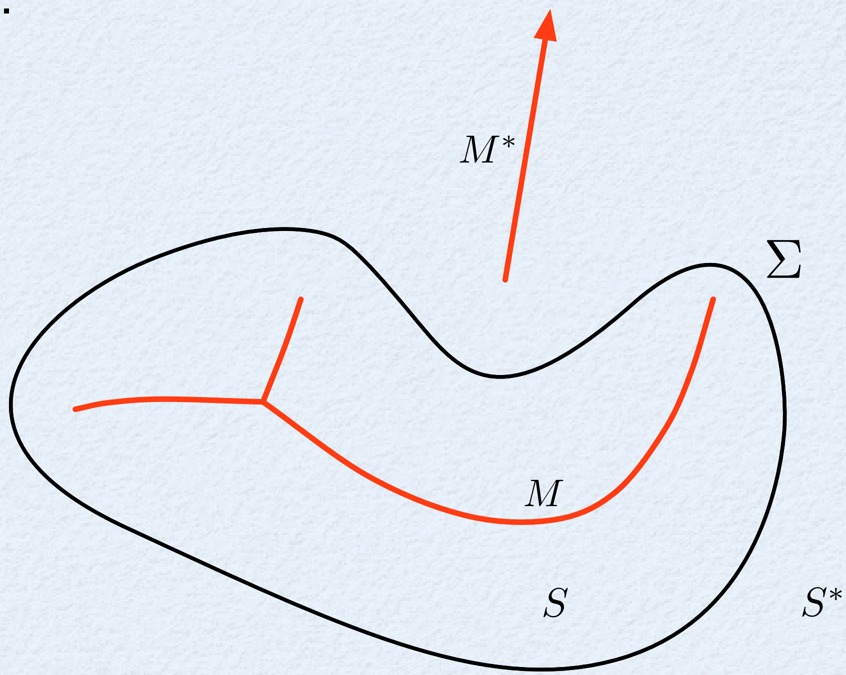


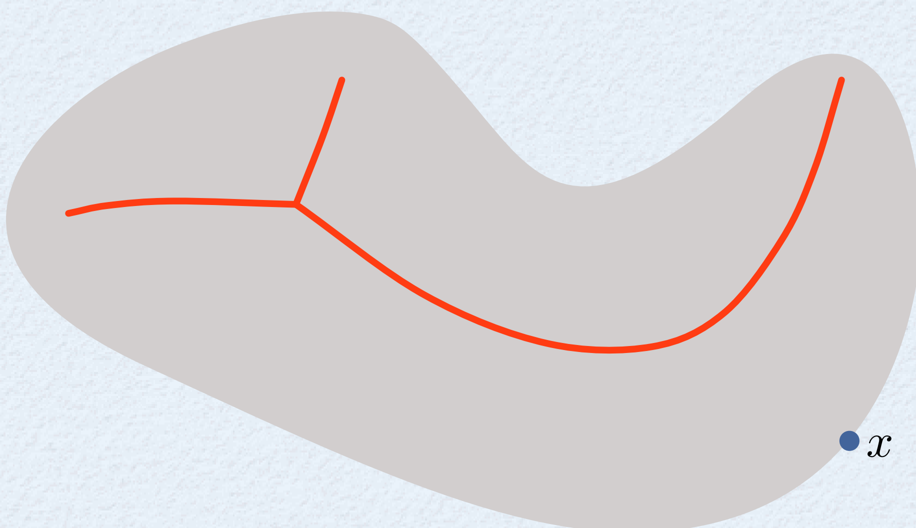
The medial axis of a surface Σ is the union of medial axes of all components of $\mathbb{R}^n \setminus \Sigma$.

Samples of Surfaces

We use the (relative) ε-sampling framework of [Amenta-Bern'99].

For a point $x \in \Sigma$, the local feature size of x is

$$\mathsf{lfs}(x) := d(x, M).$$



$P \subset \Sigma$ is an ε-sample if every $x \in \Sigma$ has a sample within distance $\varepsilon\,\mathsf{lfs}(x)$.

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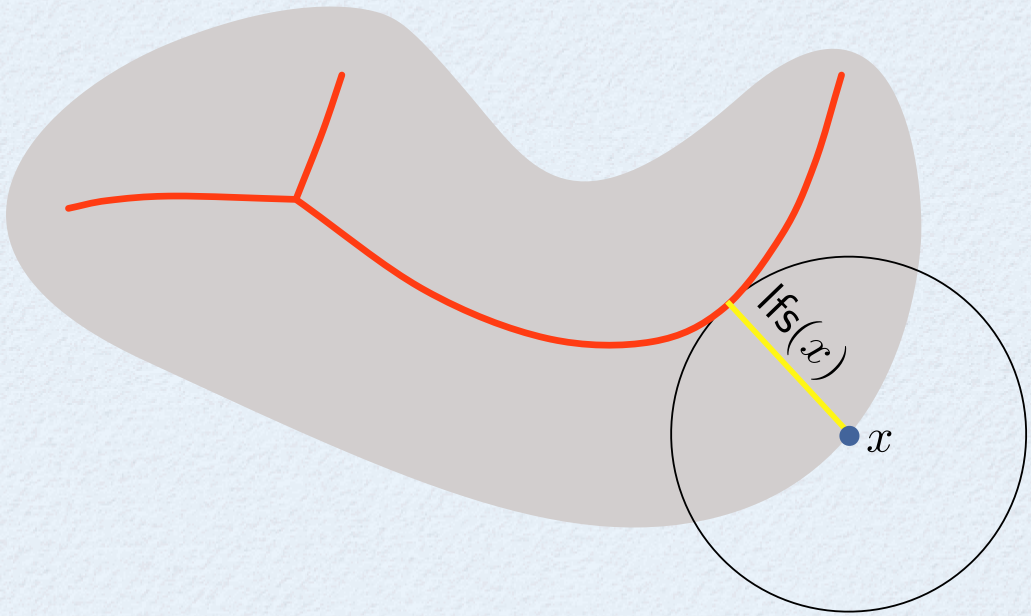

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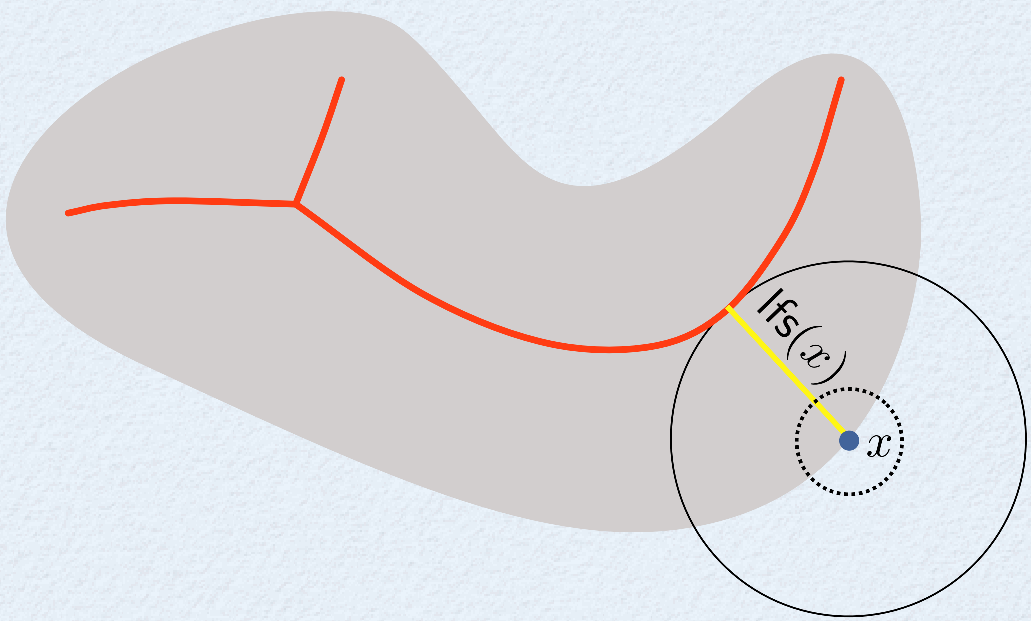

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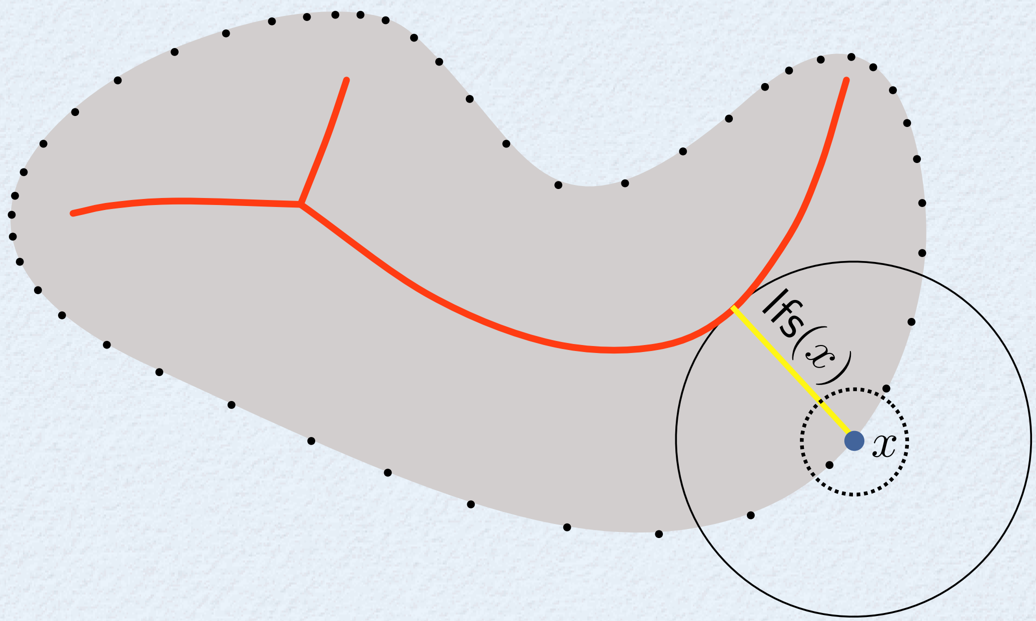

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There are many surface reconstruction methods!

- As 0-set of an approximate signed distance function: [Hoppe et al'92, Curless et al'96]

- As other iso-surfaces:
  NN Interpolation [Boissonnat-Cazals'02]
  MLS [Levin'98, Alexa et al'01, Amenta-Kil'04, Kolluri'05, Dey et al'05]
  SVM [Schölkopf et al'04]

- Delaunay Methods:
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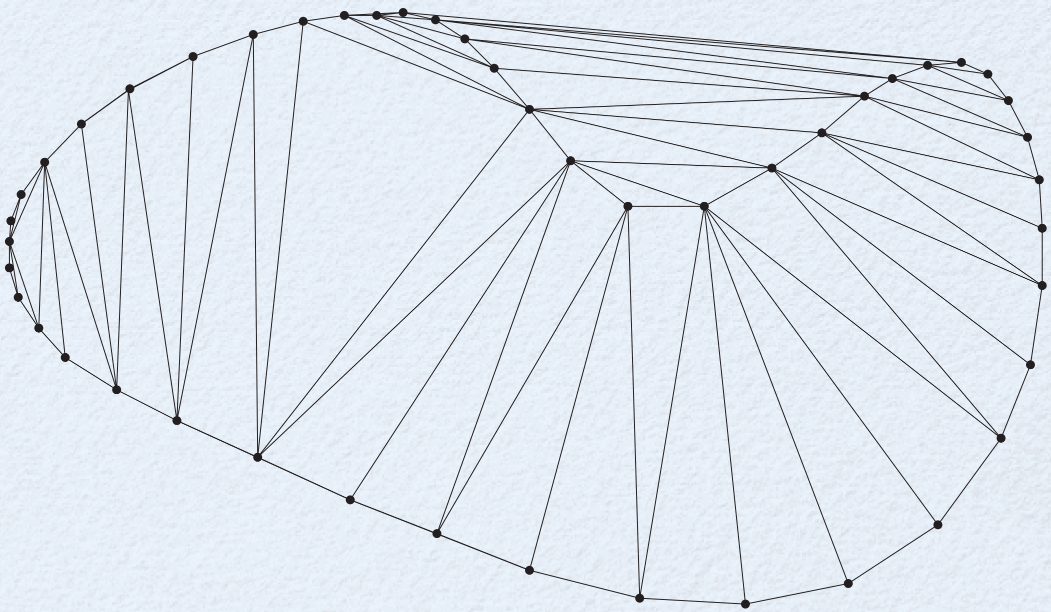

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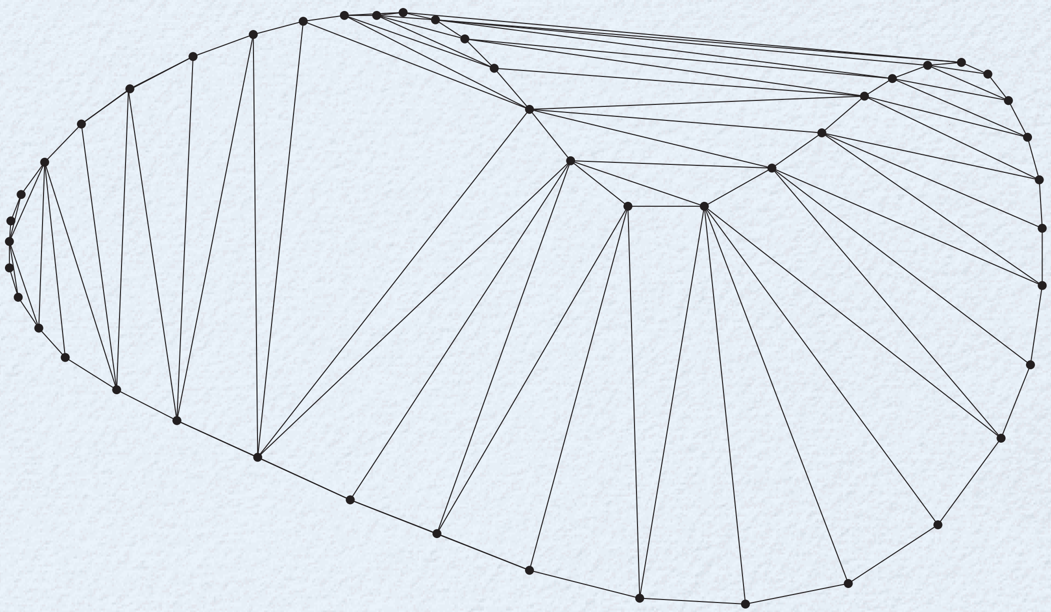

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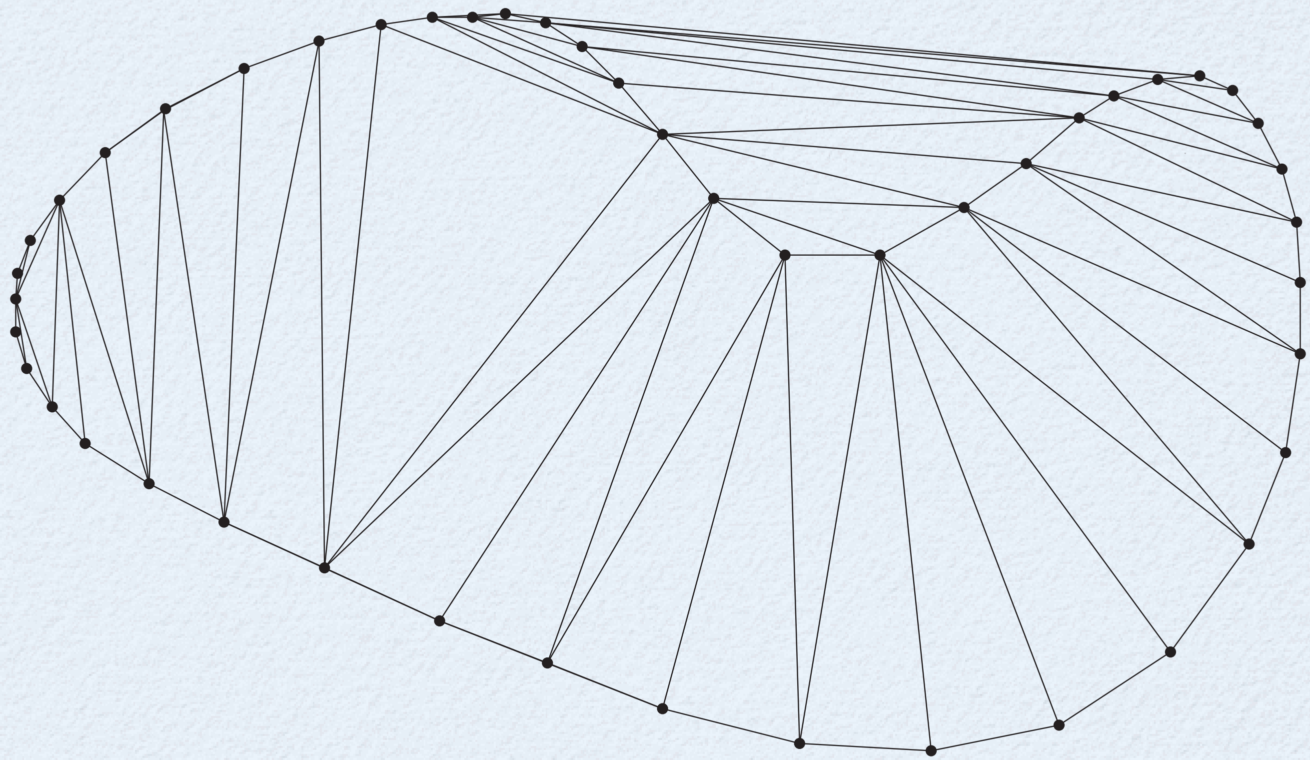

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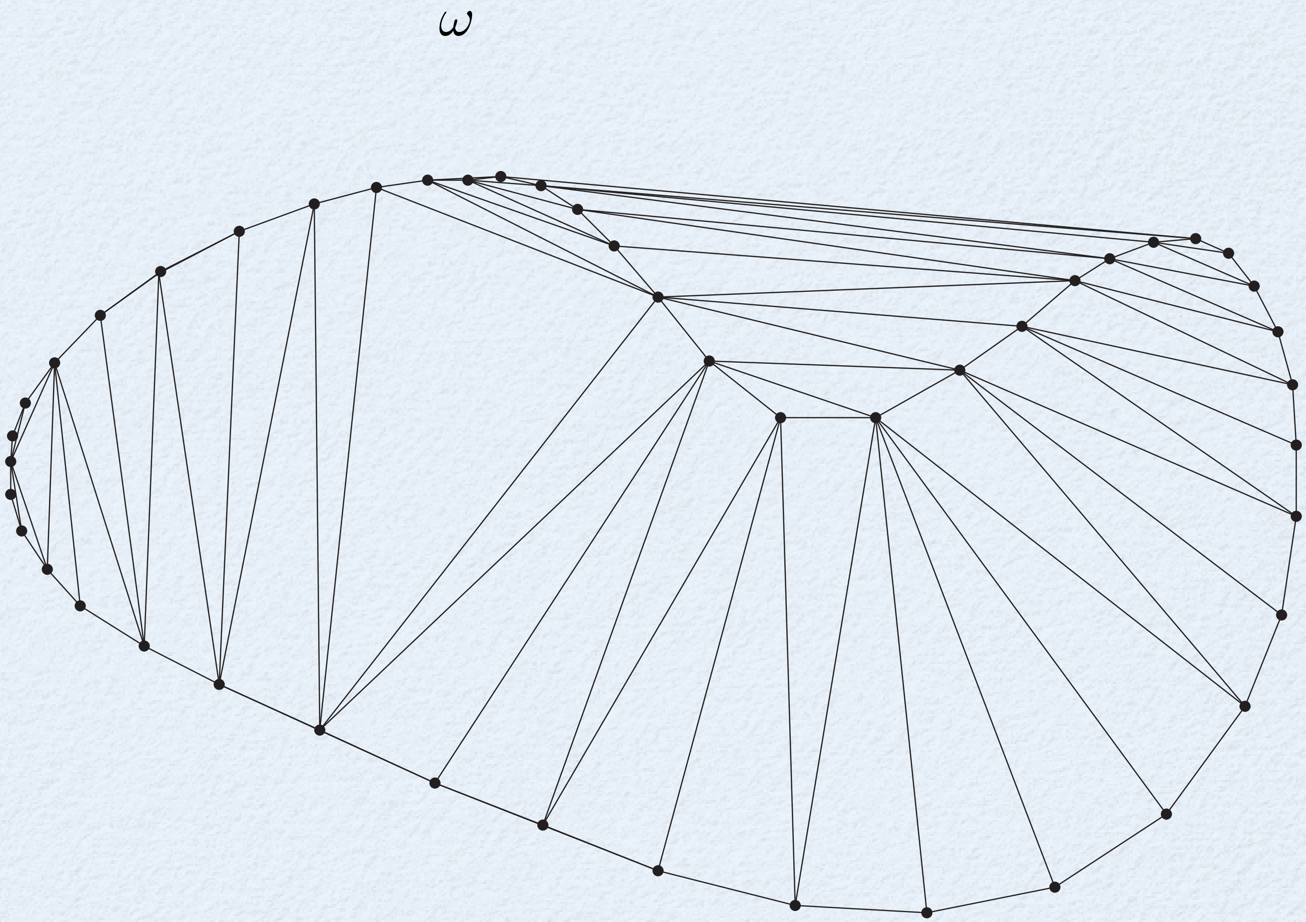

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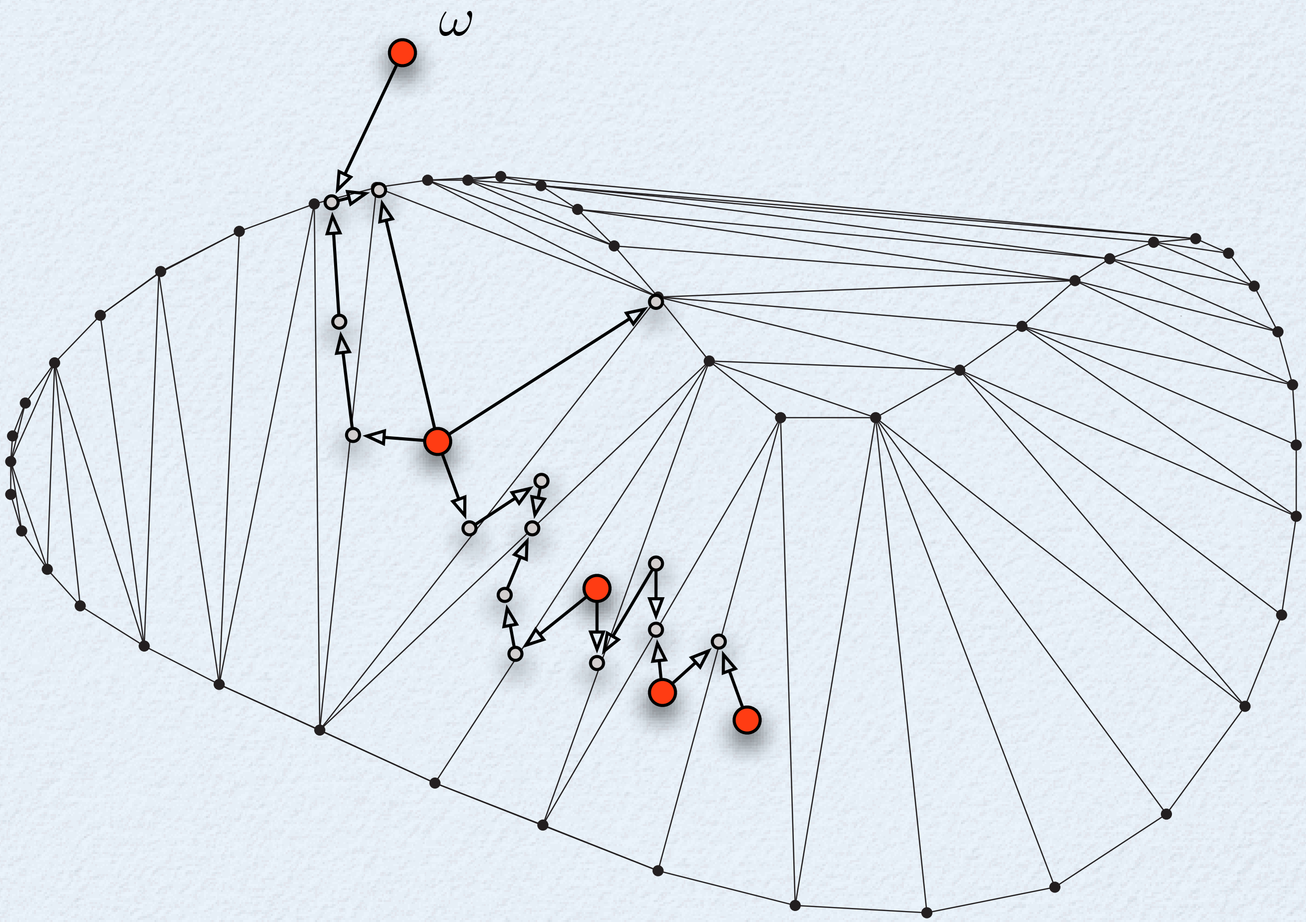

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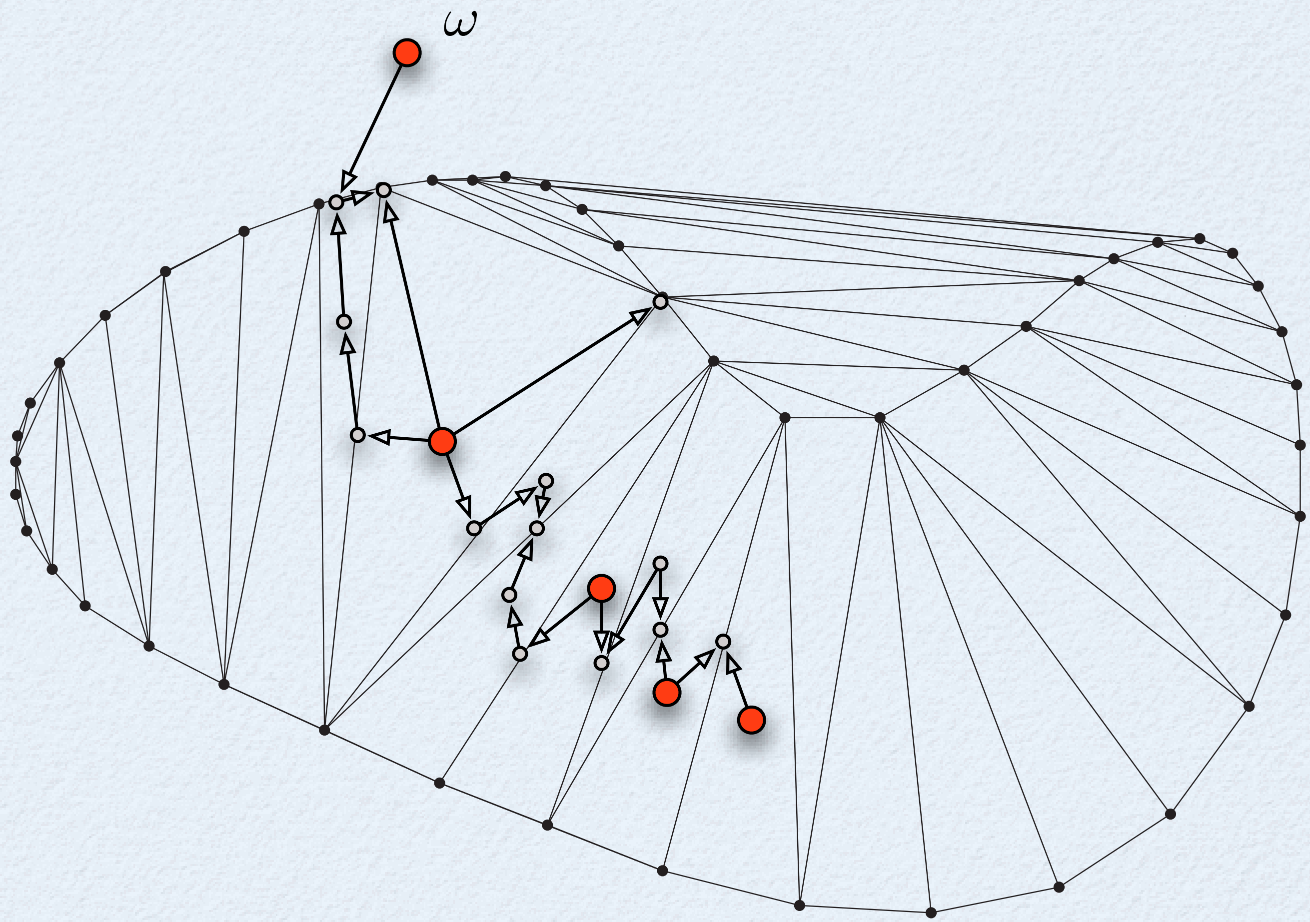
For every $\tau \in \text{Del } P$, if τ is reachable from no "centered" simplex other than ω, then remove τ.



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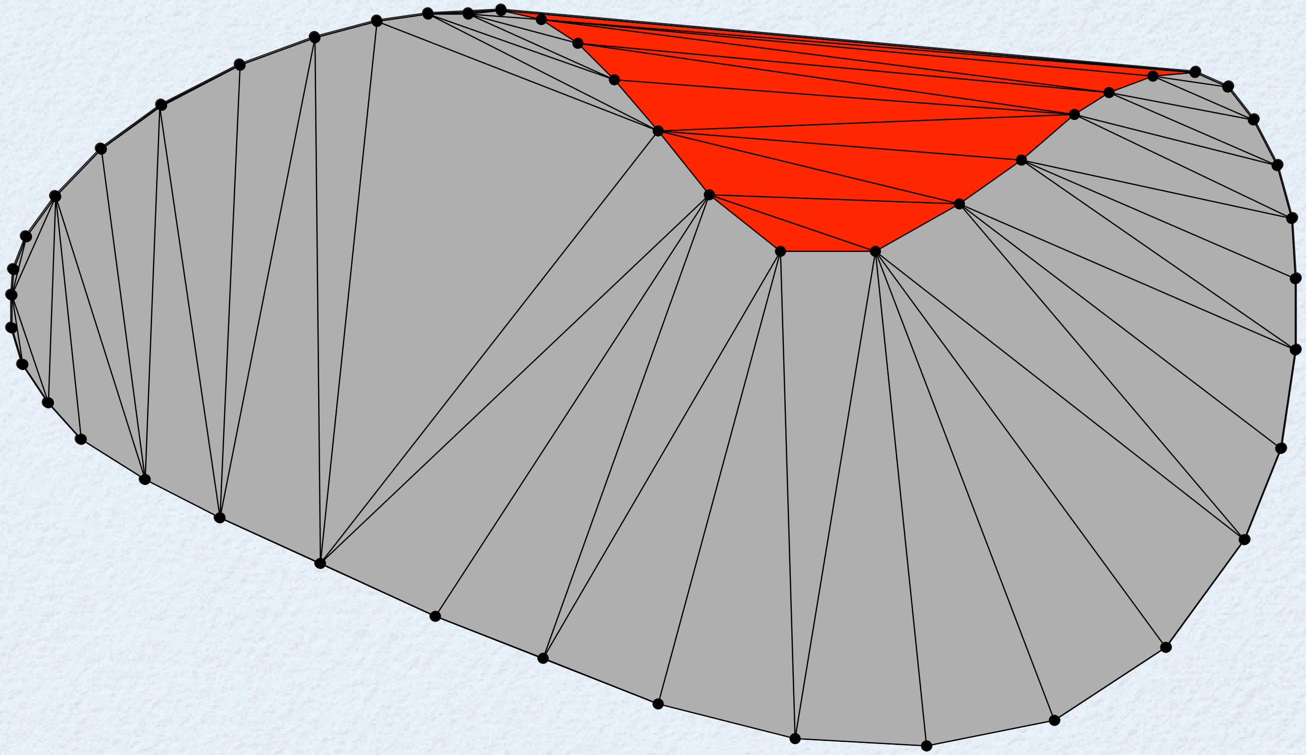

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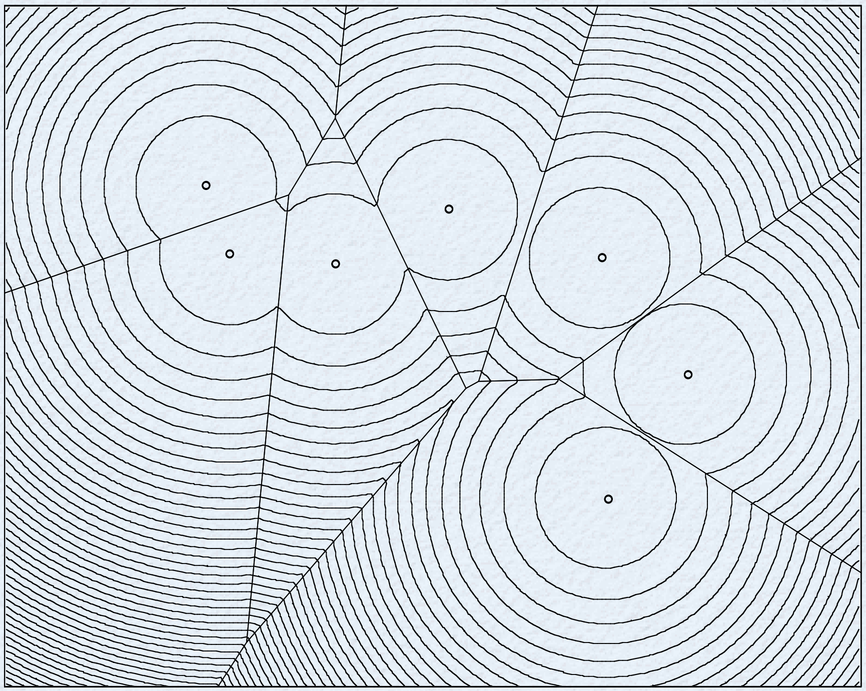
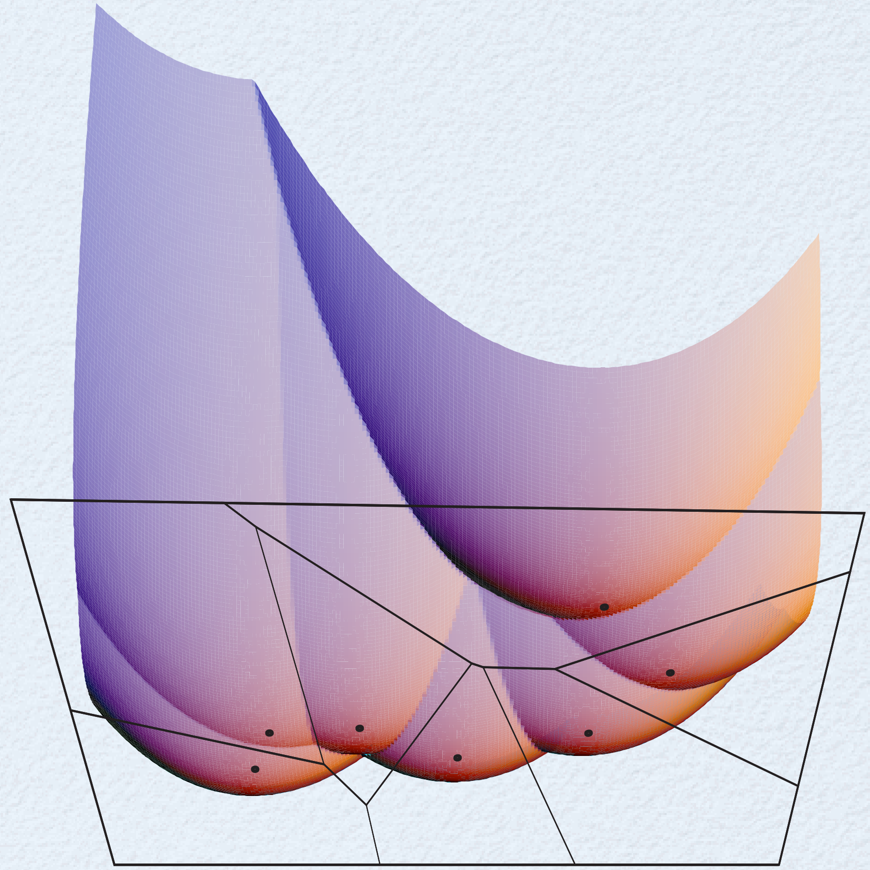
The Machinery

(Squared) Distance to Discrete Point Sets





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The squared distance function induced by P is

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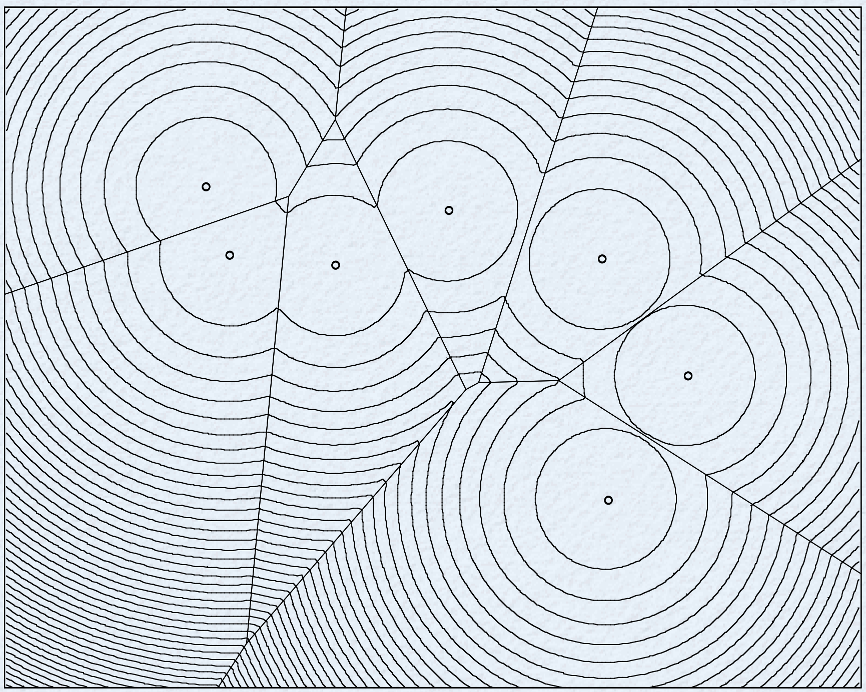

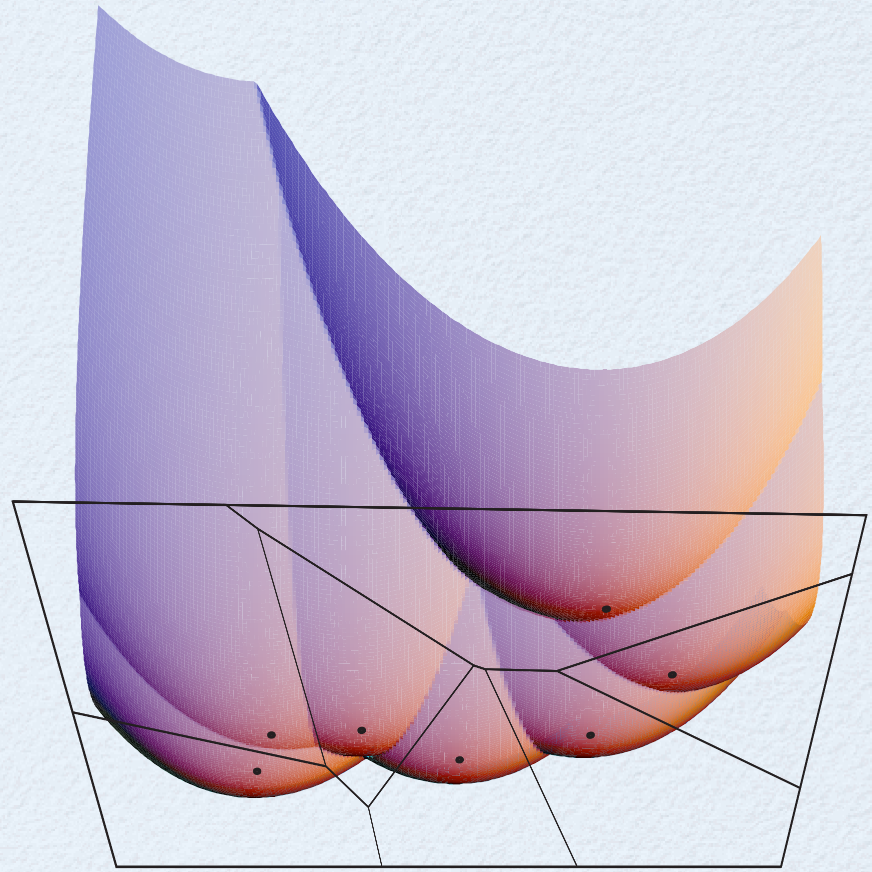

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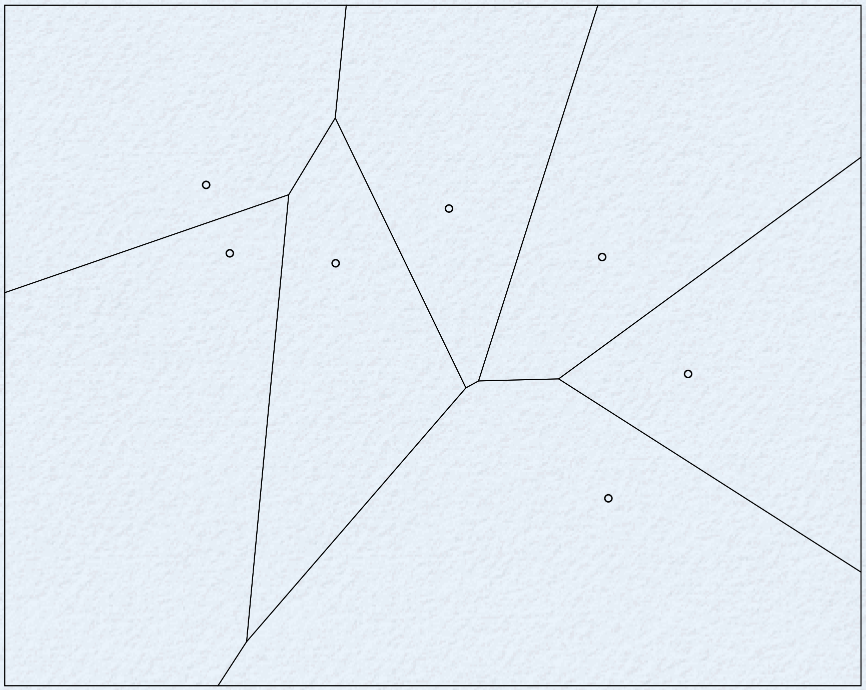
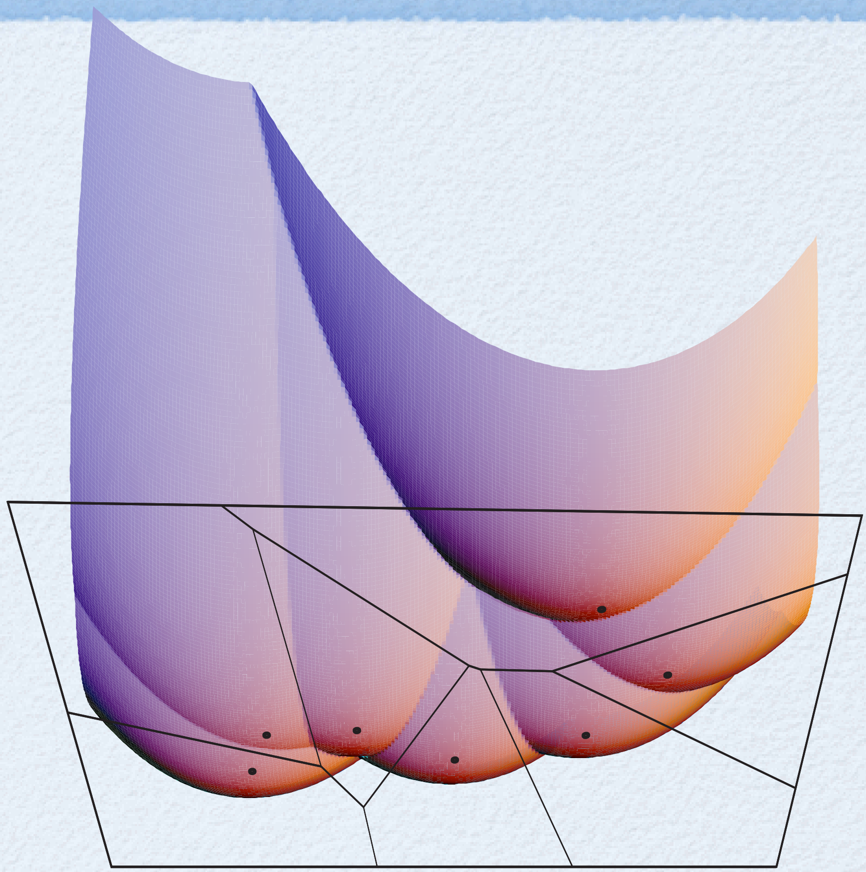
Observation. h is smooth at points with a unique closest point in P.

Generalized Gradient





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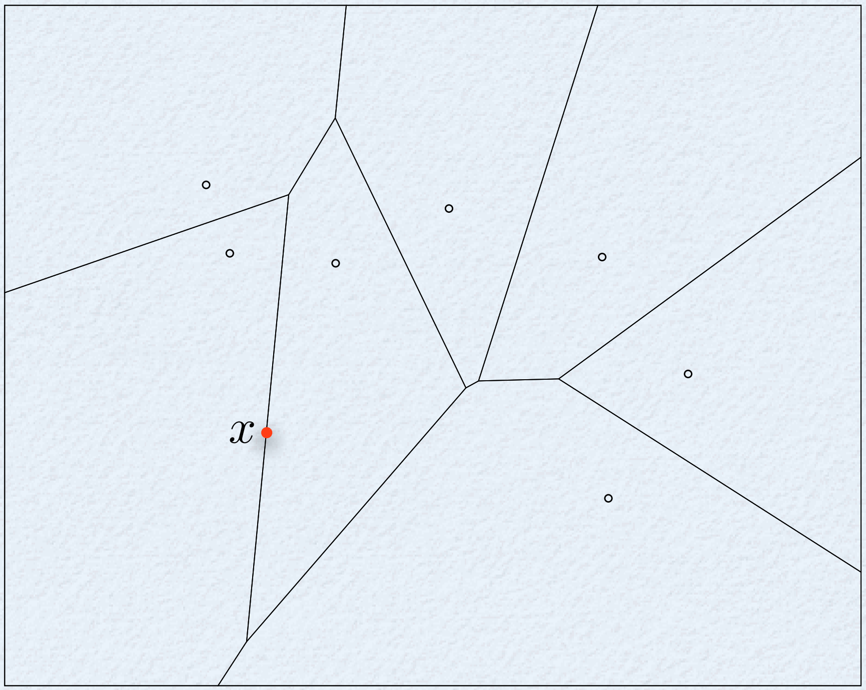

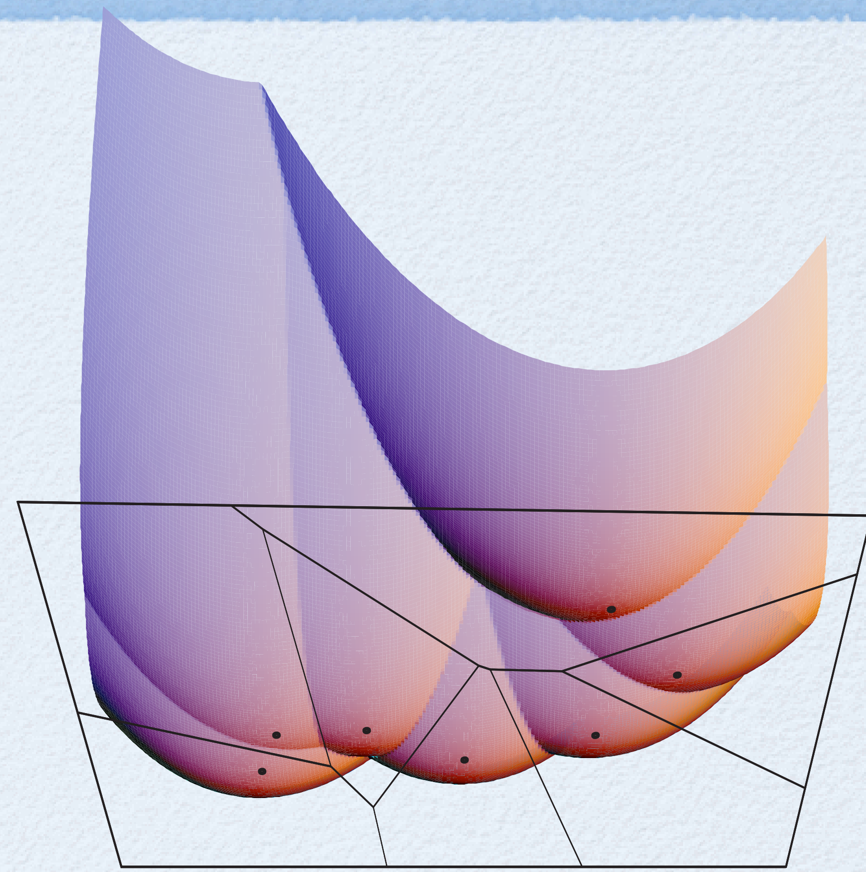

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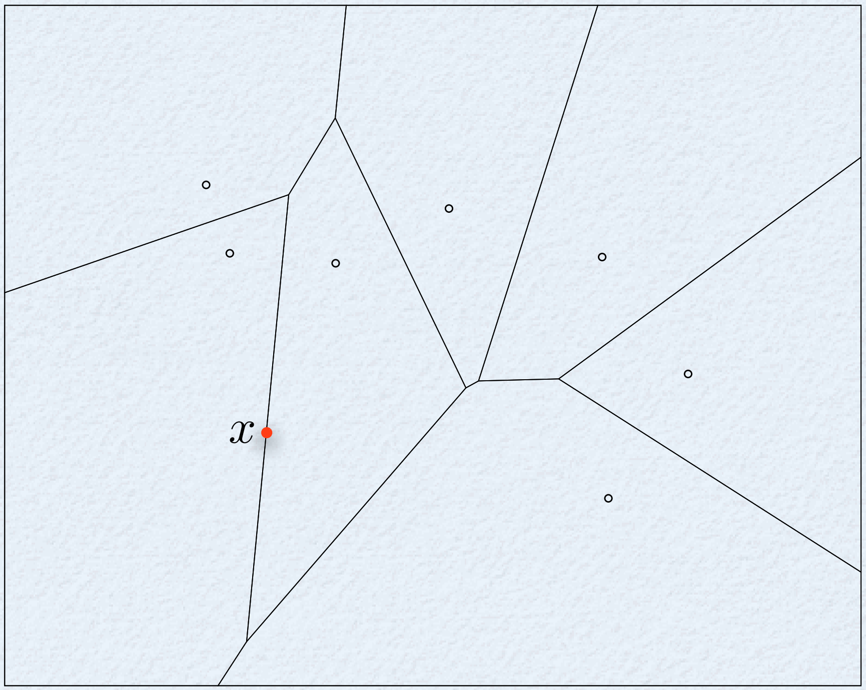

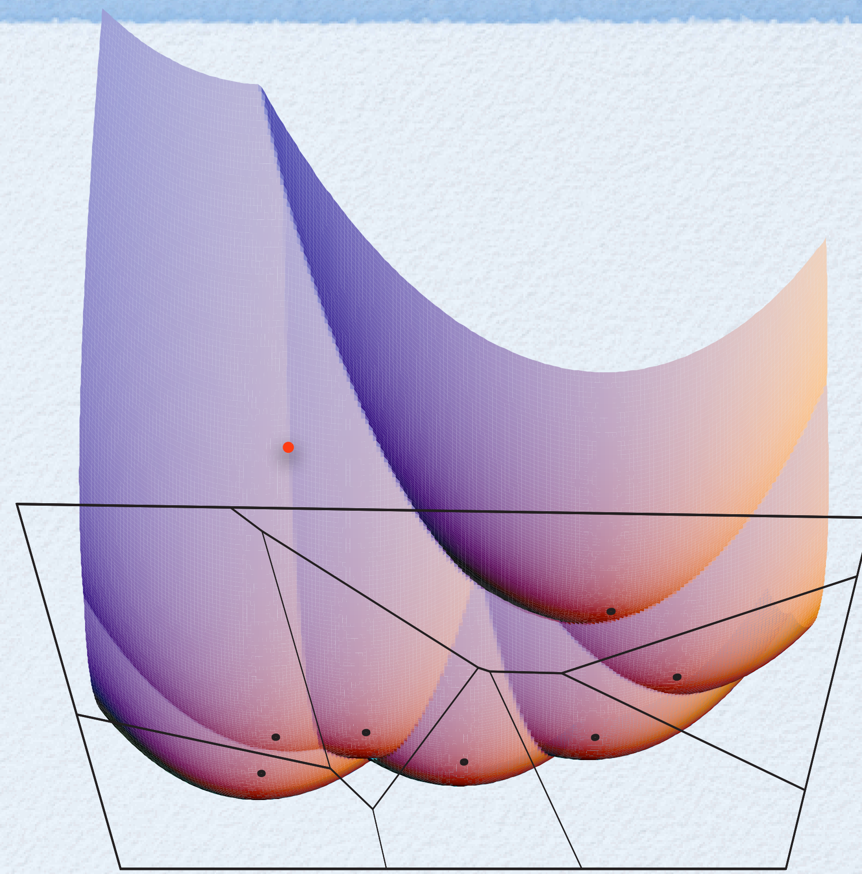

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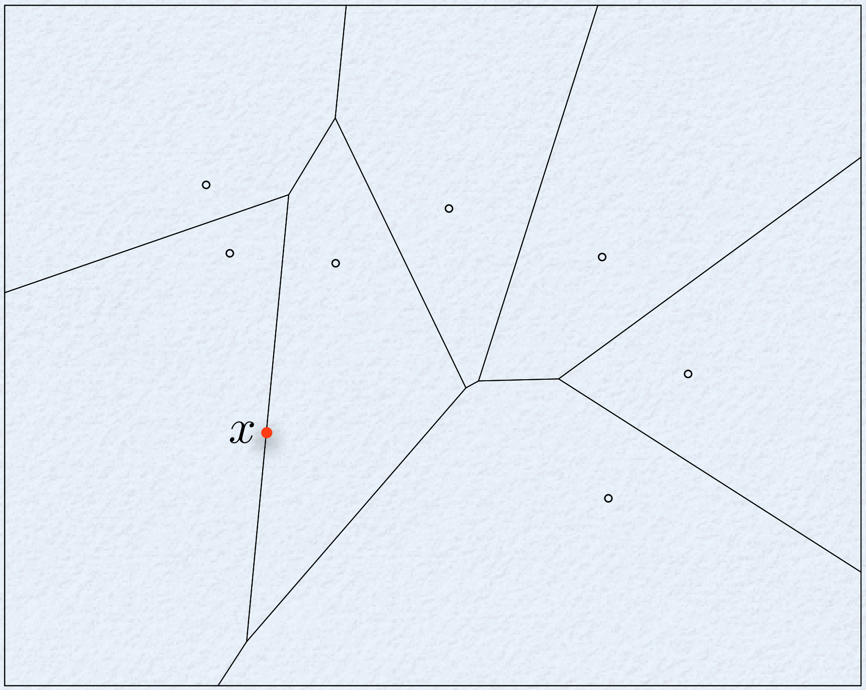

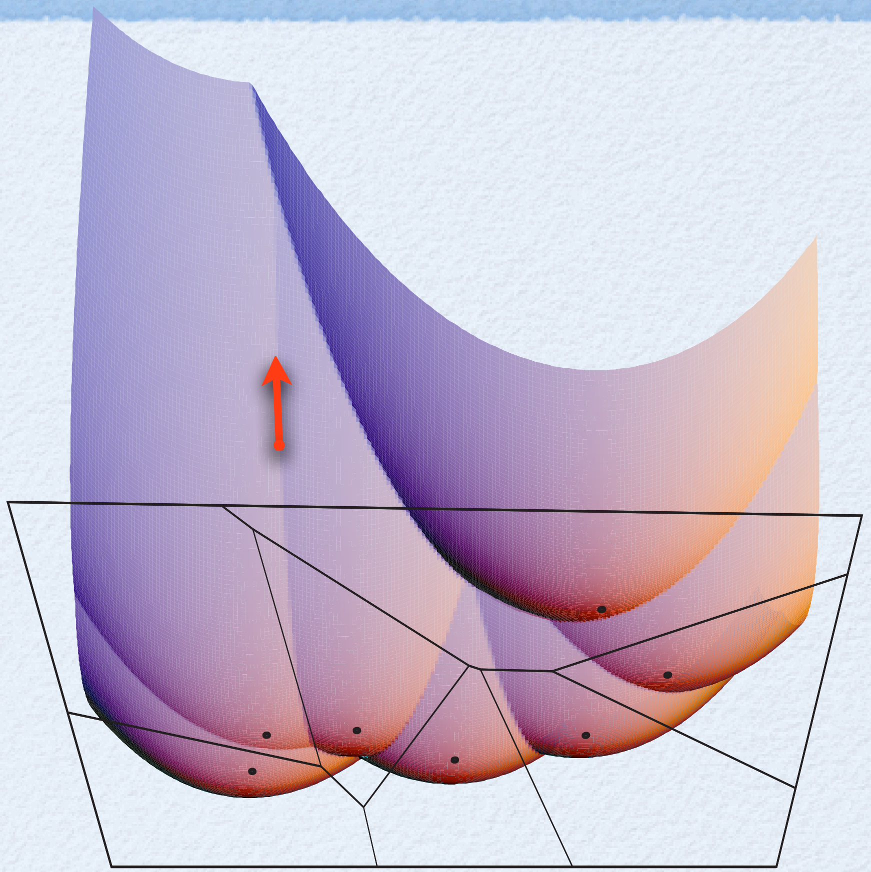

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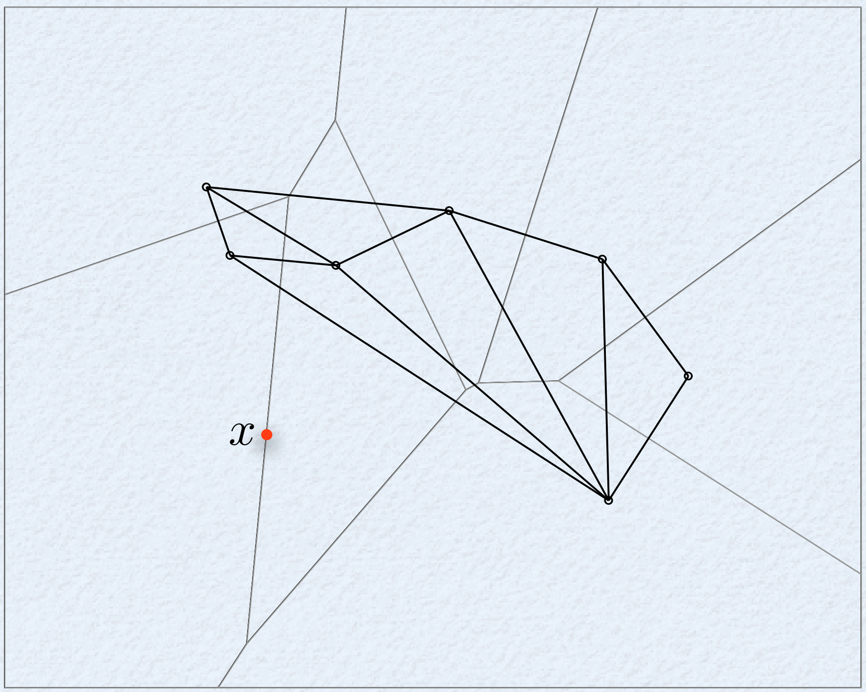

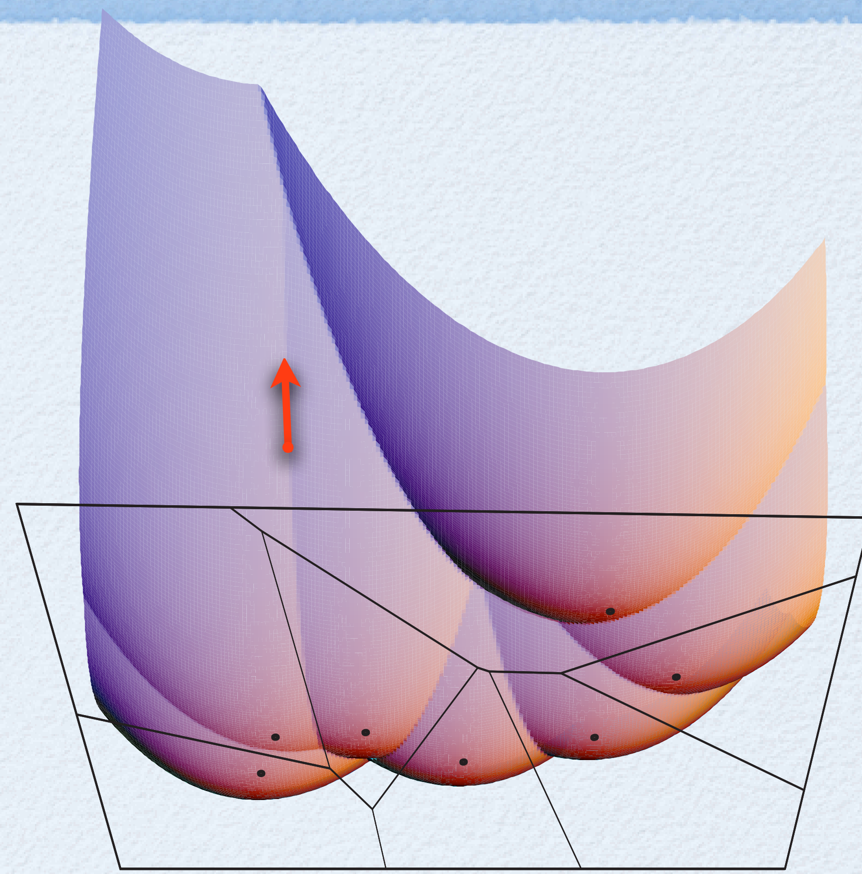

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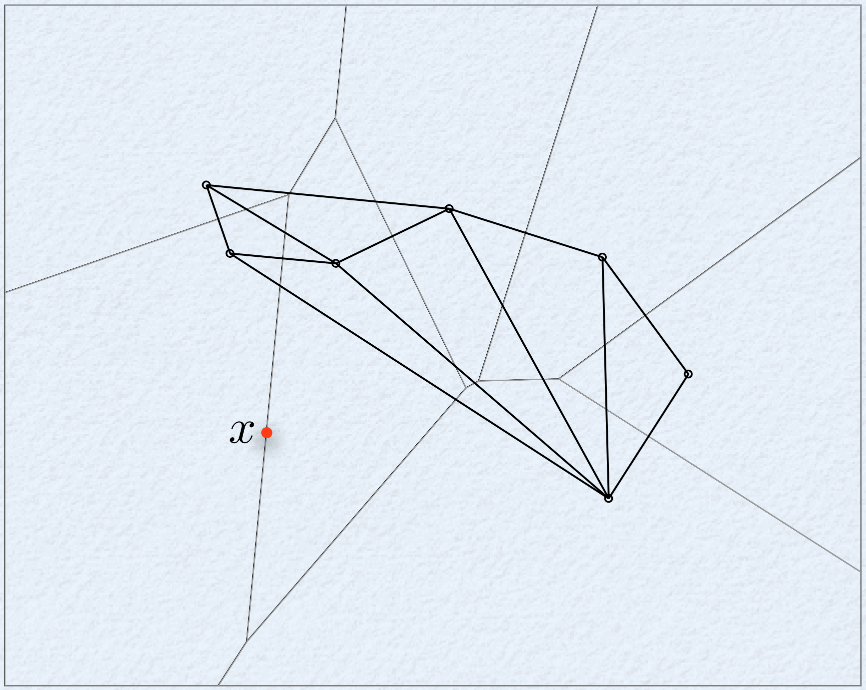

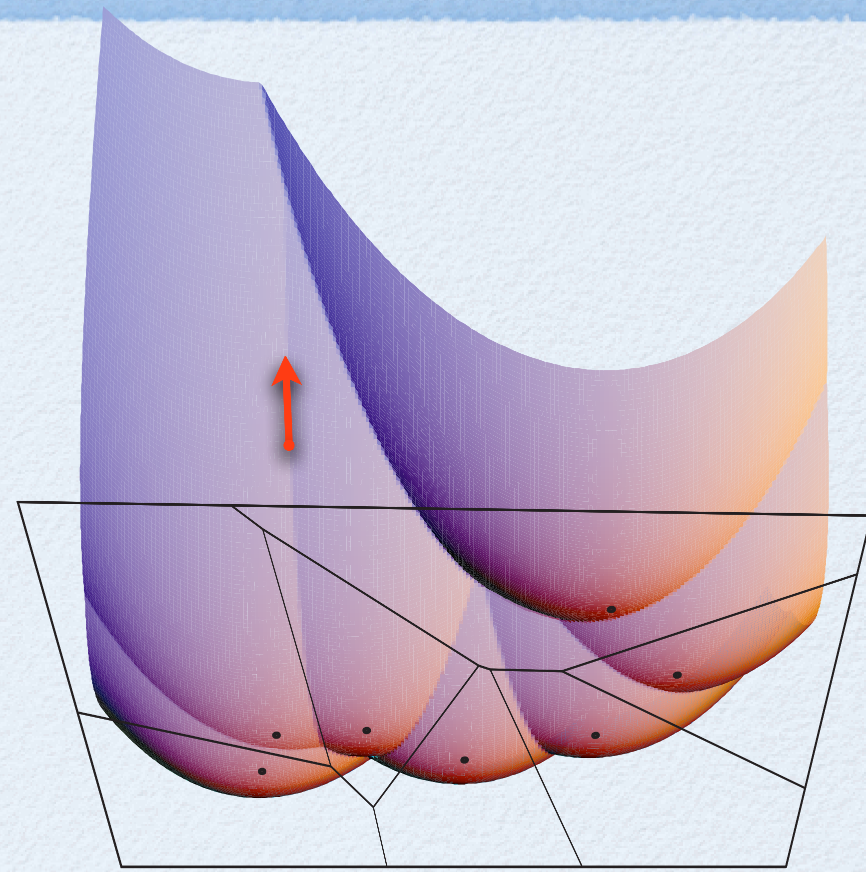

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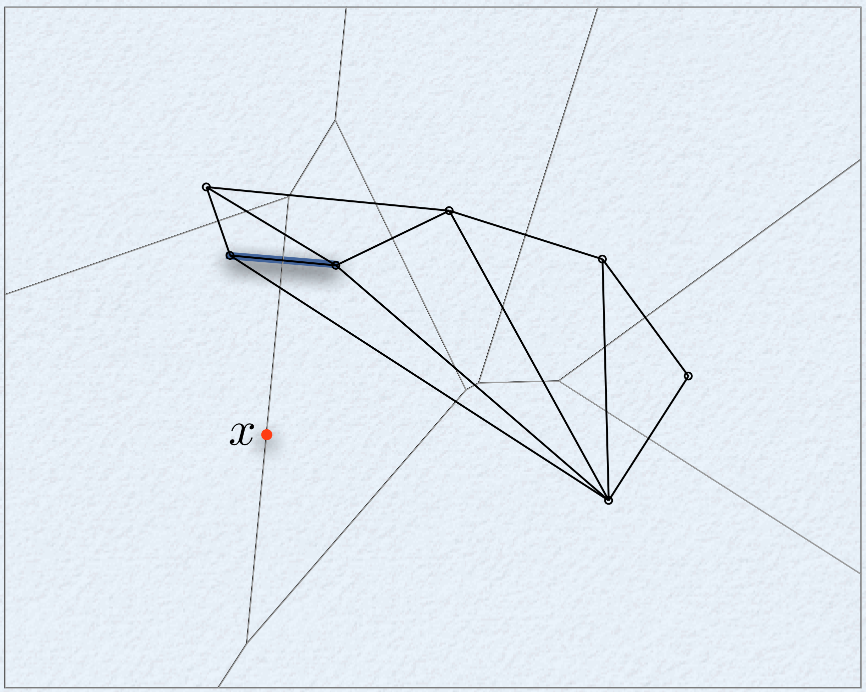

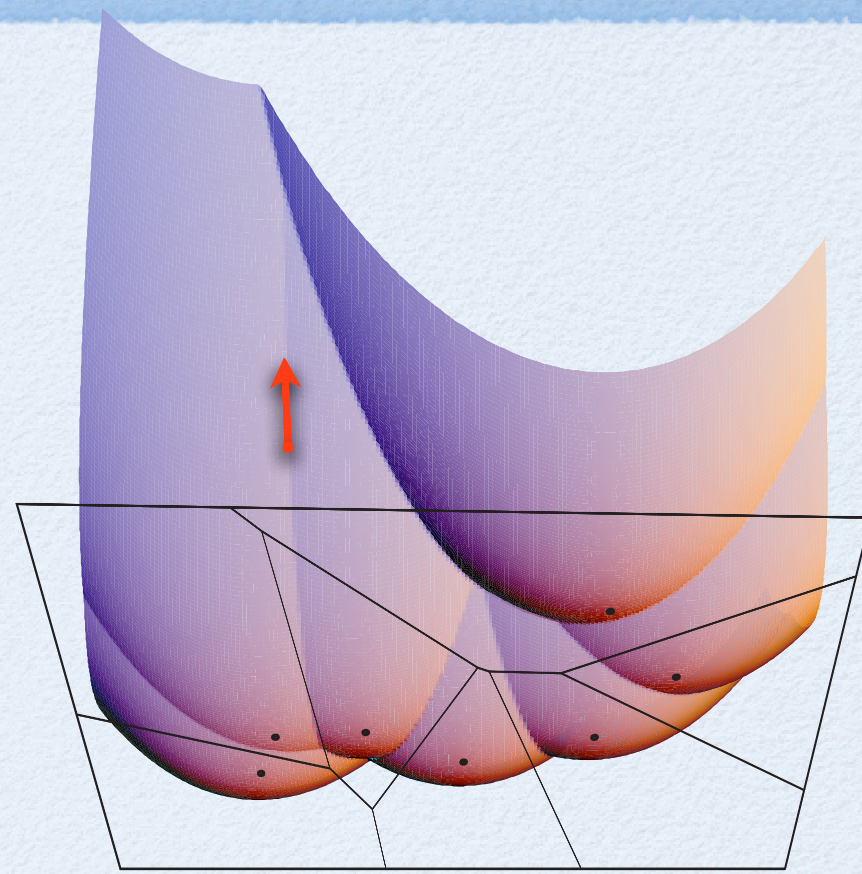

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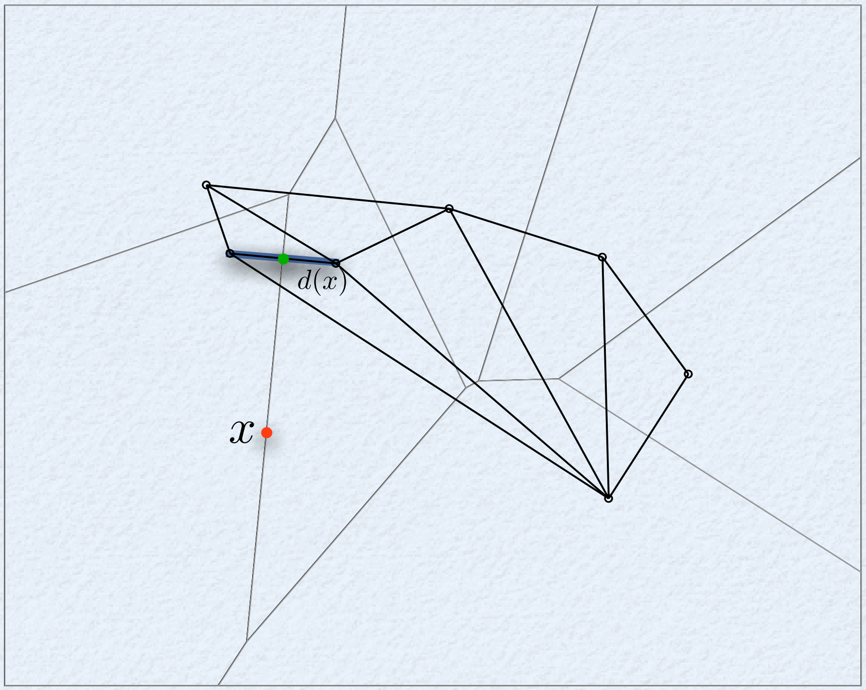

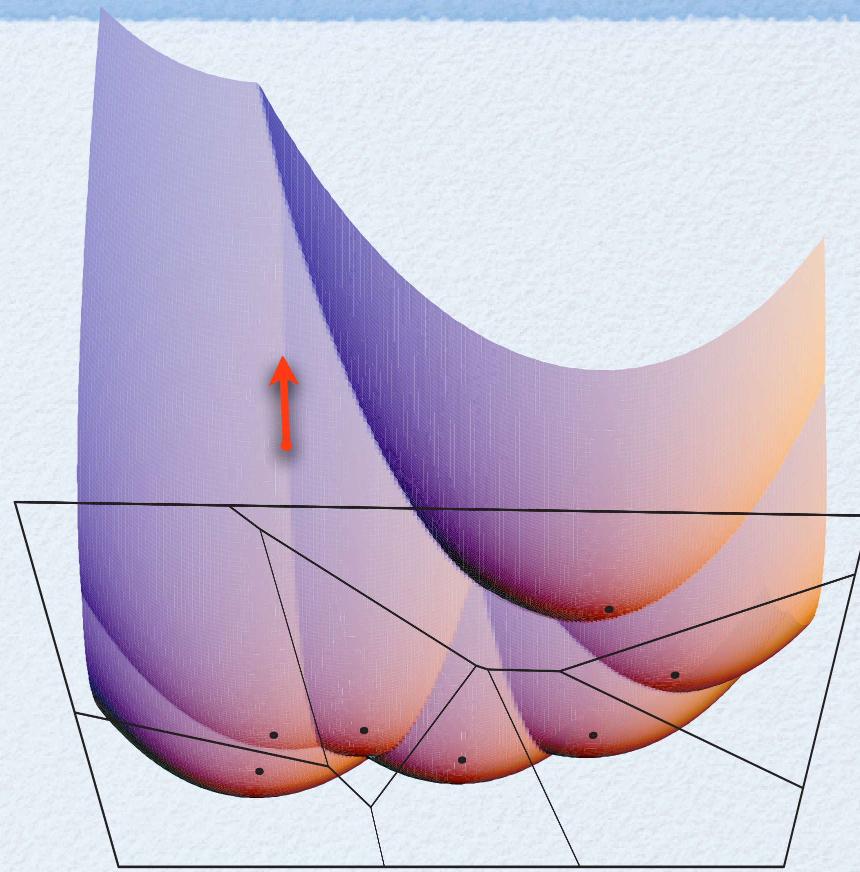

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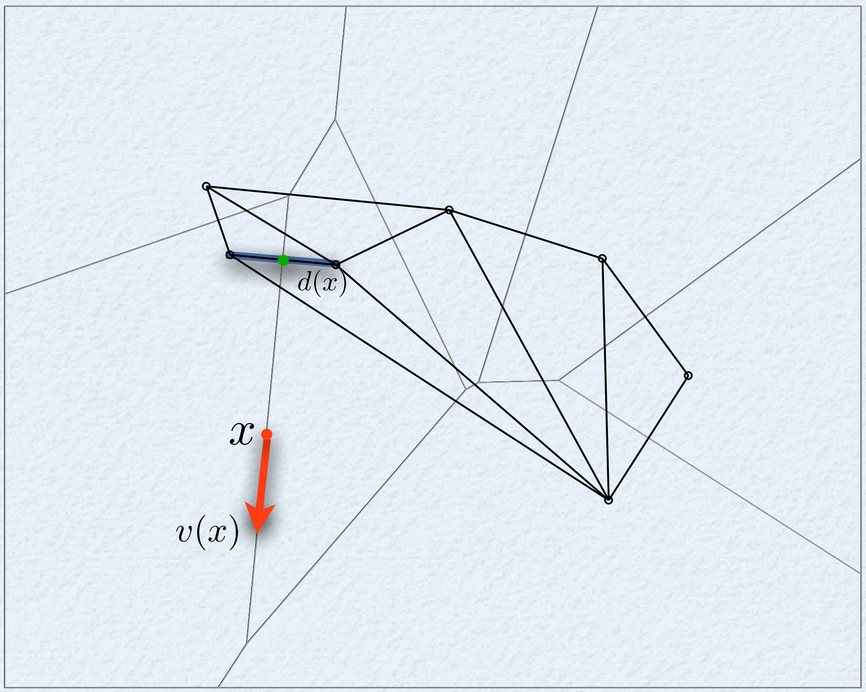

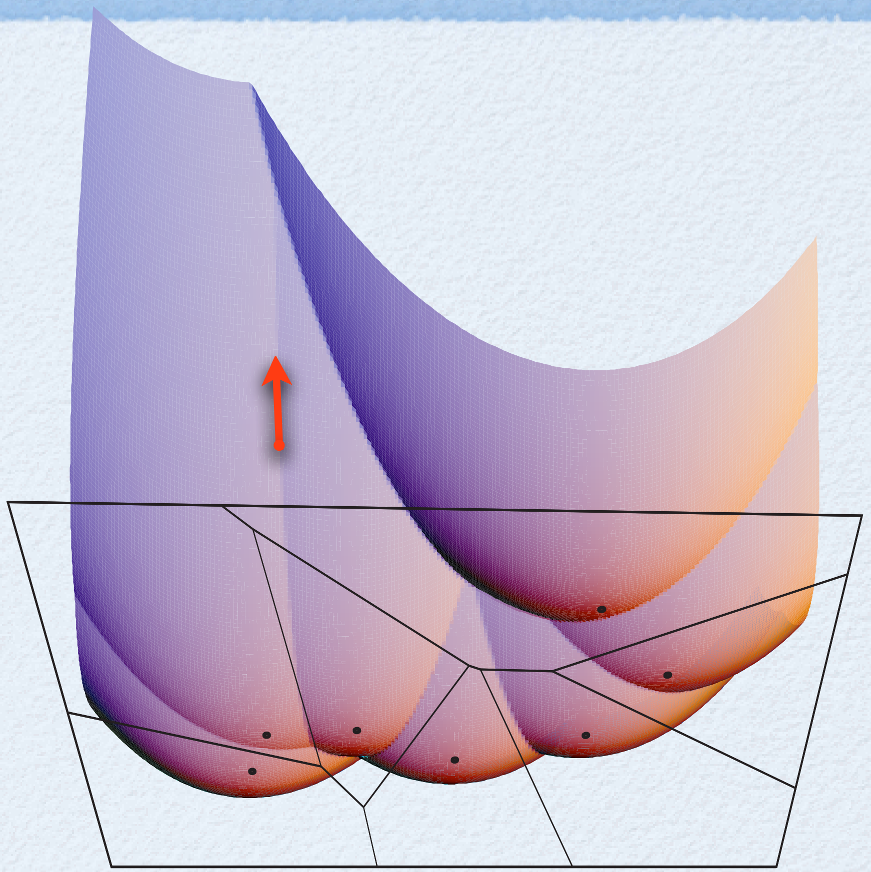

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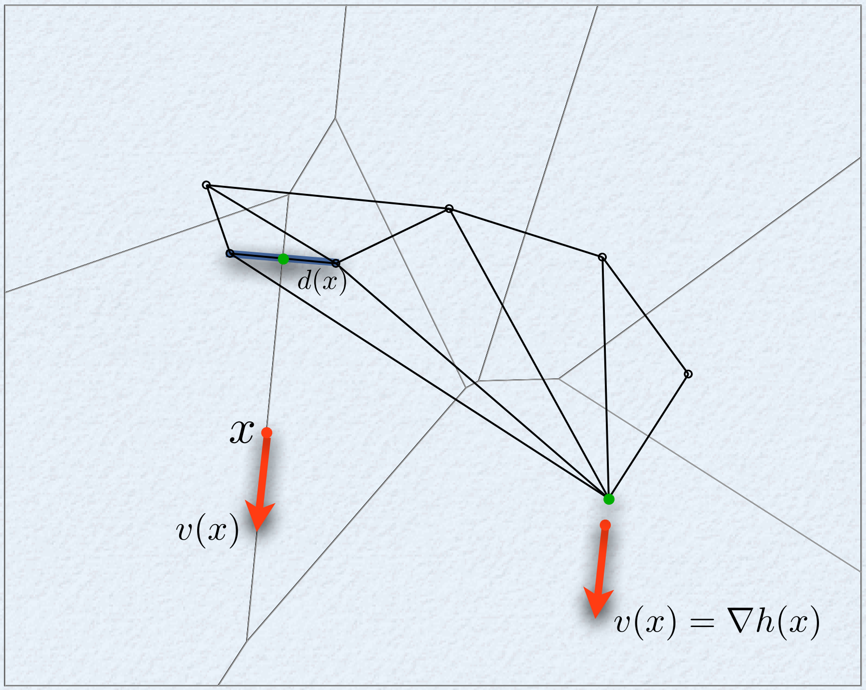

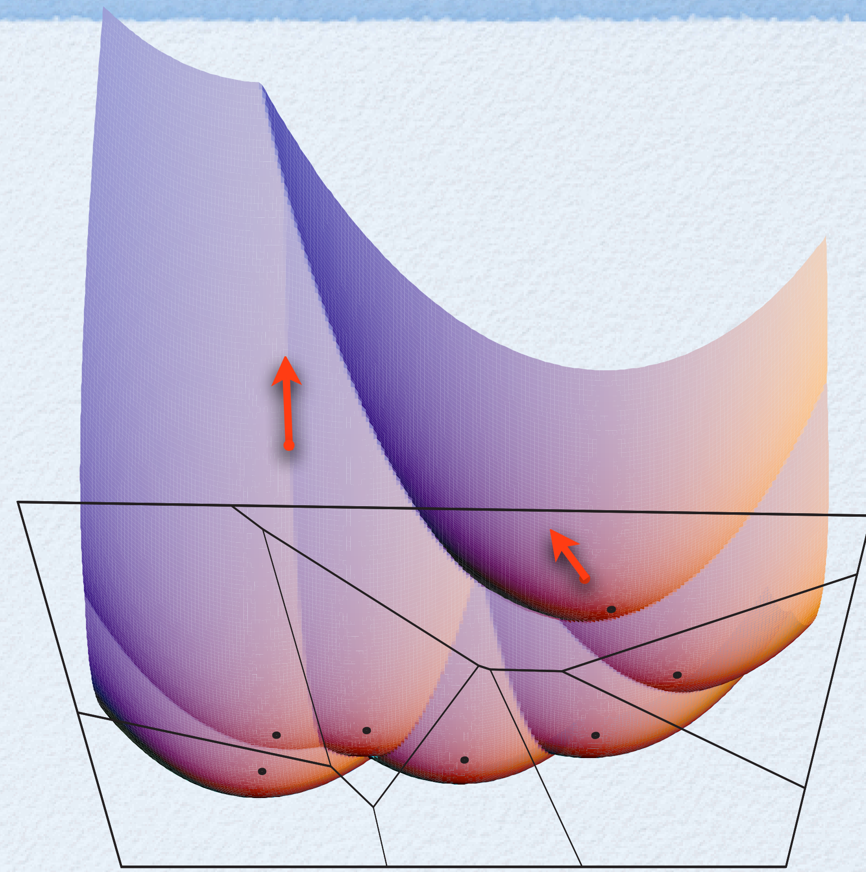

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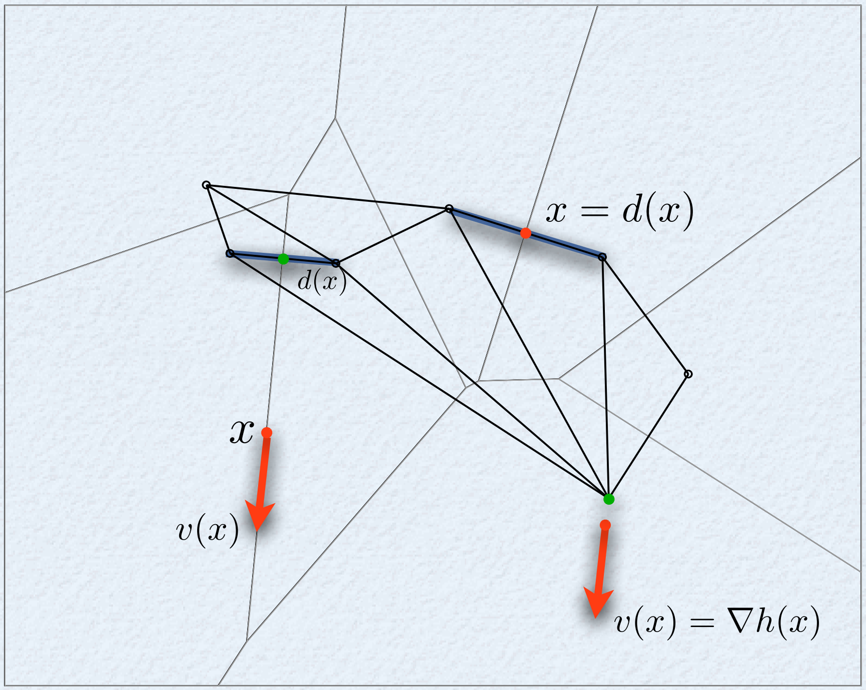

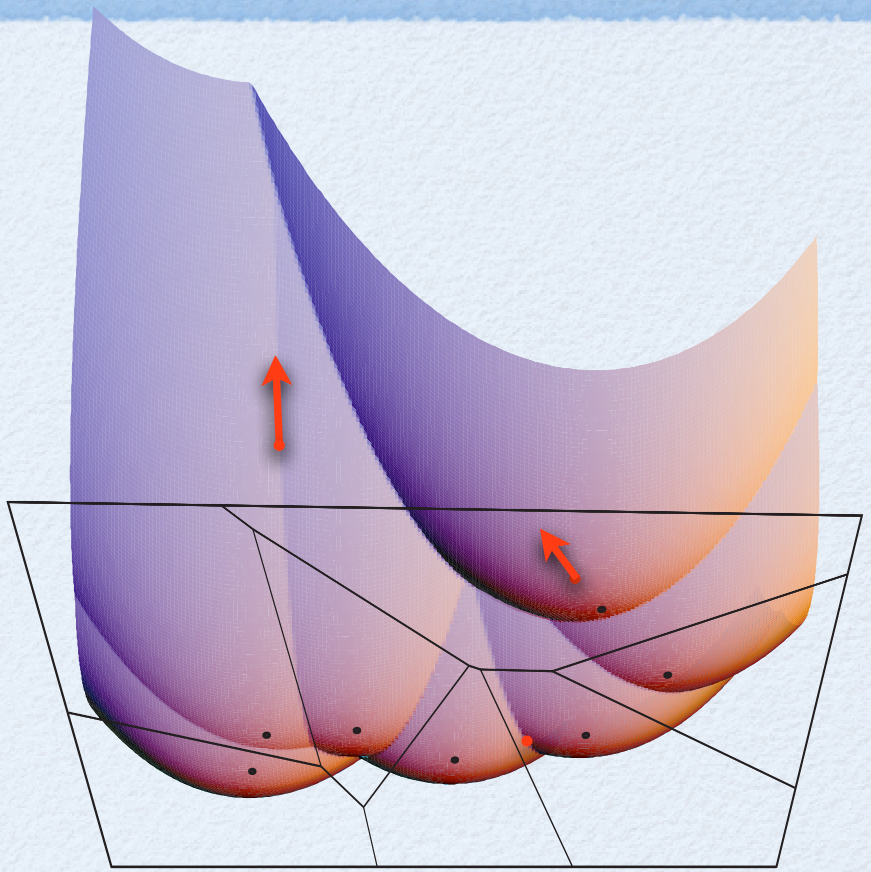

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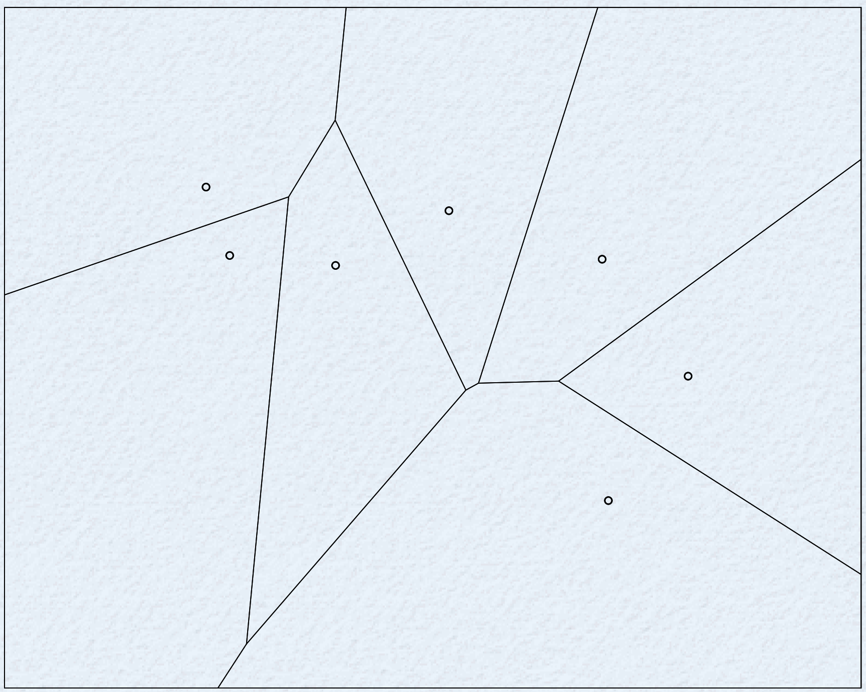
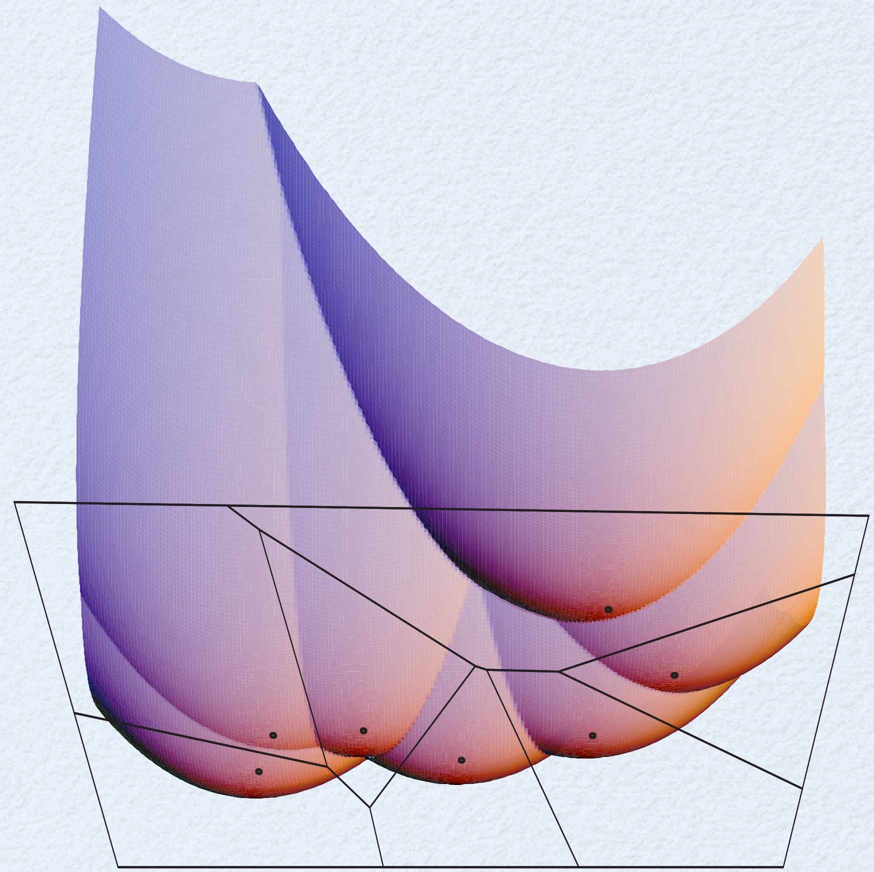

Moving at point x in with speed $v(x)$ results a flow map $\phi : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$.

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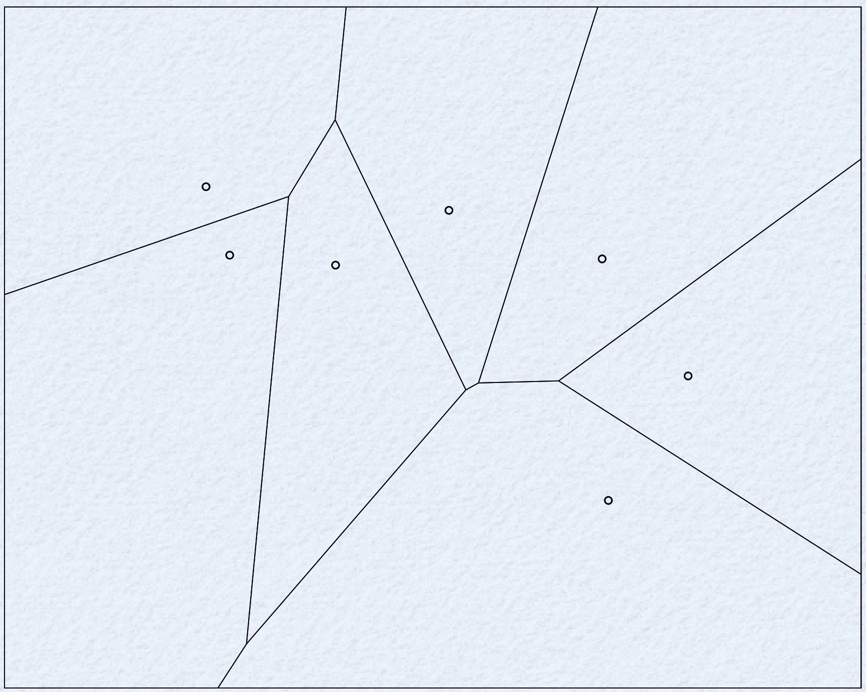

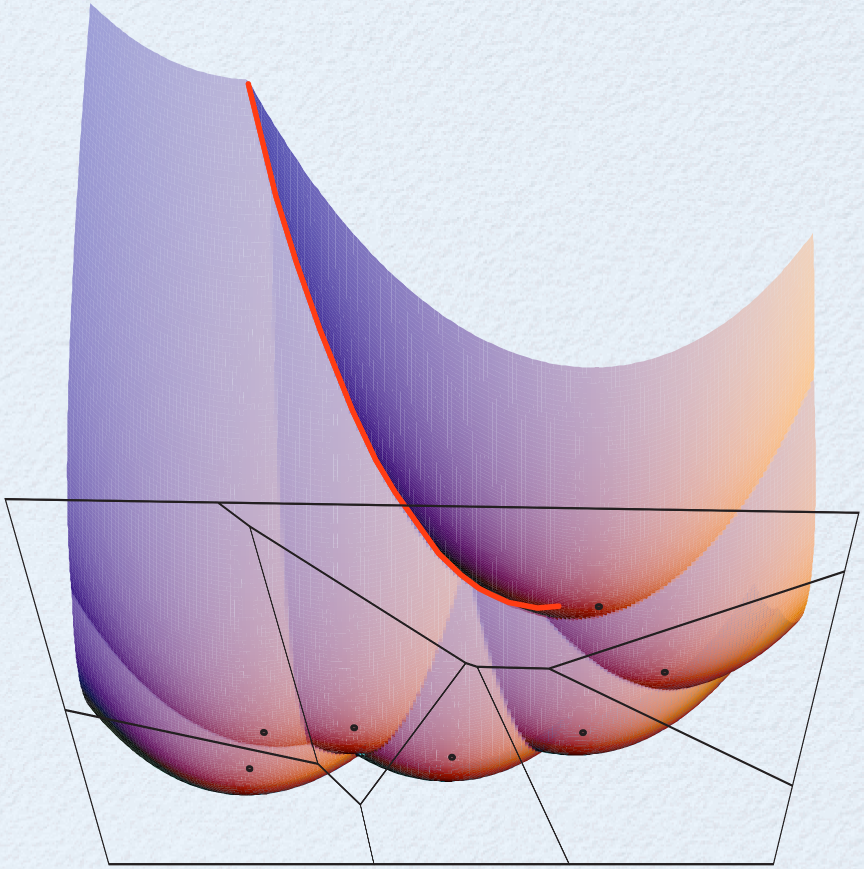

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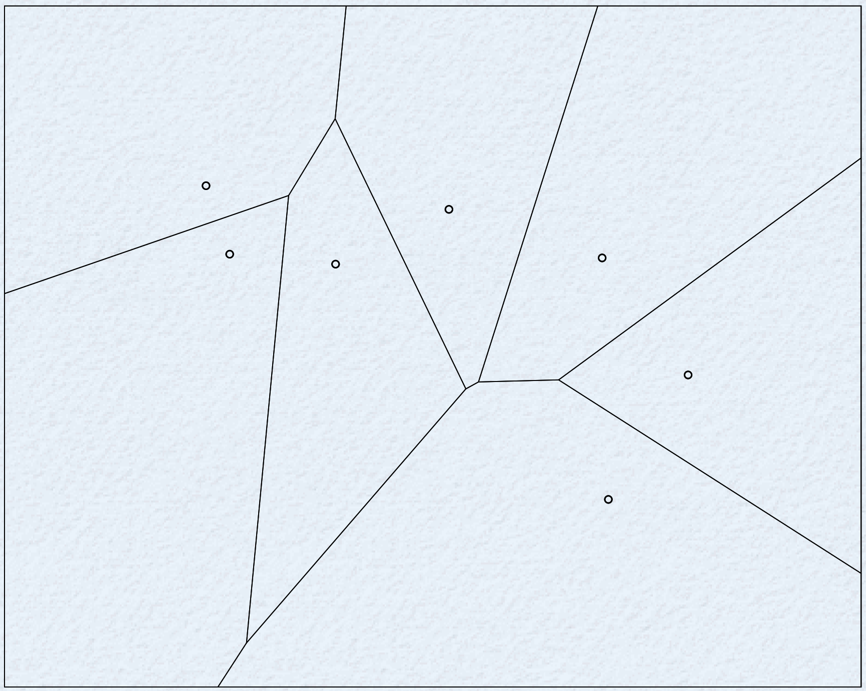

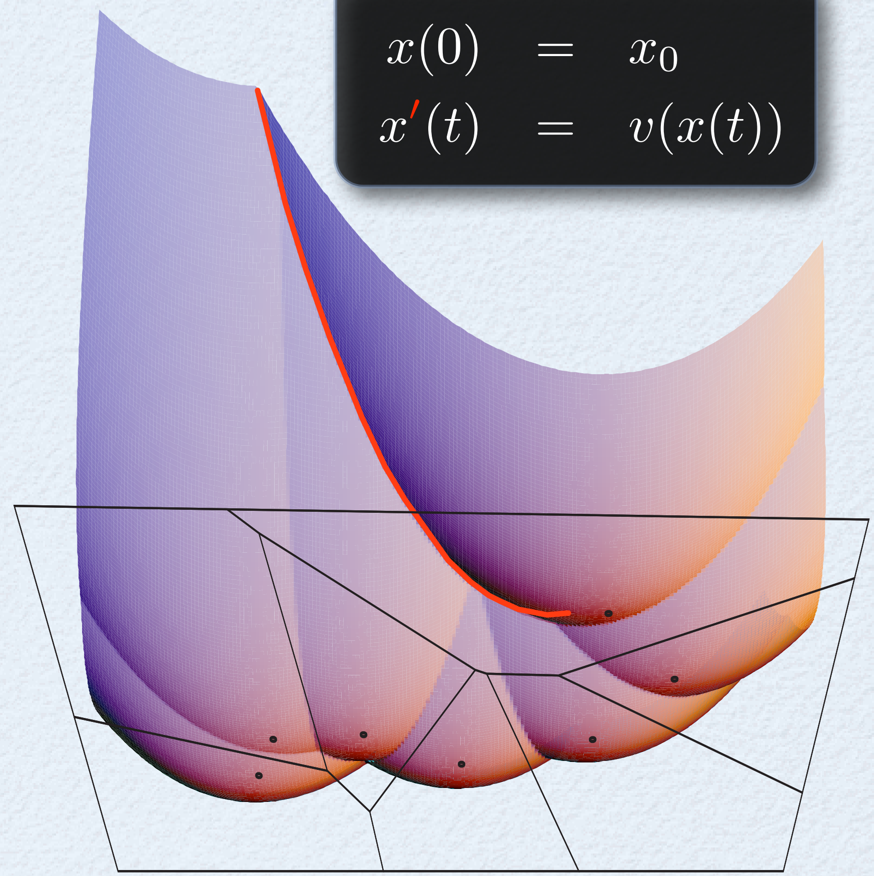

$$\begin{aligned} x(0) &= x_0 \\ x'(t) &= v(x(t)) \end{aligned}$$

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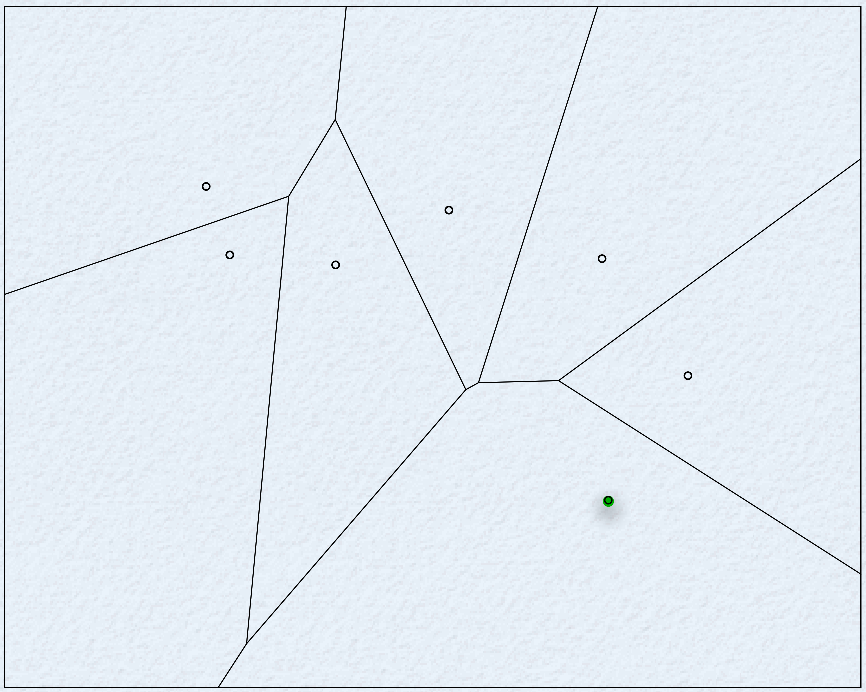

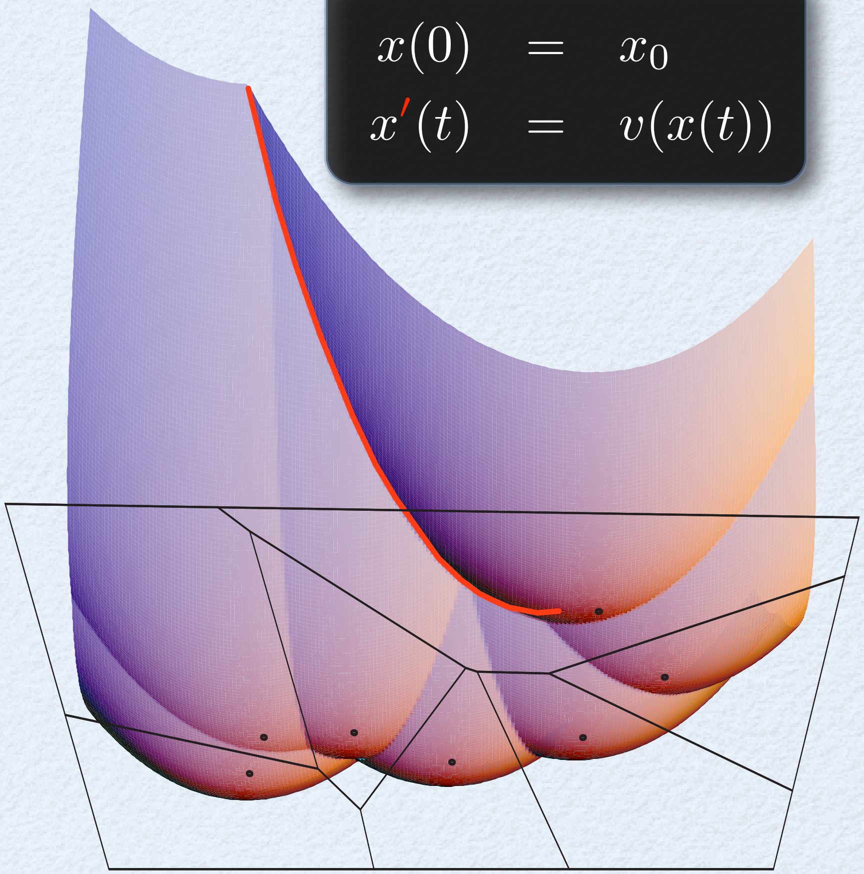

$$\begin{aligned} x(0) &= x_0 \\ x'(t) &= v(x(t)) \end{aligned}$$

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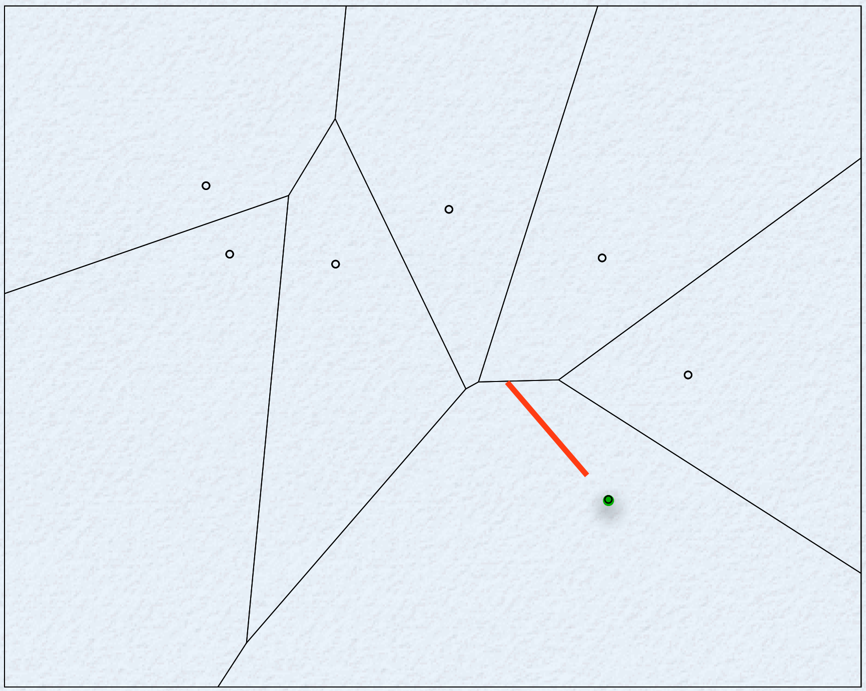

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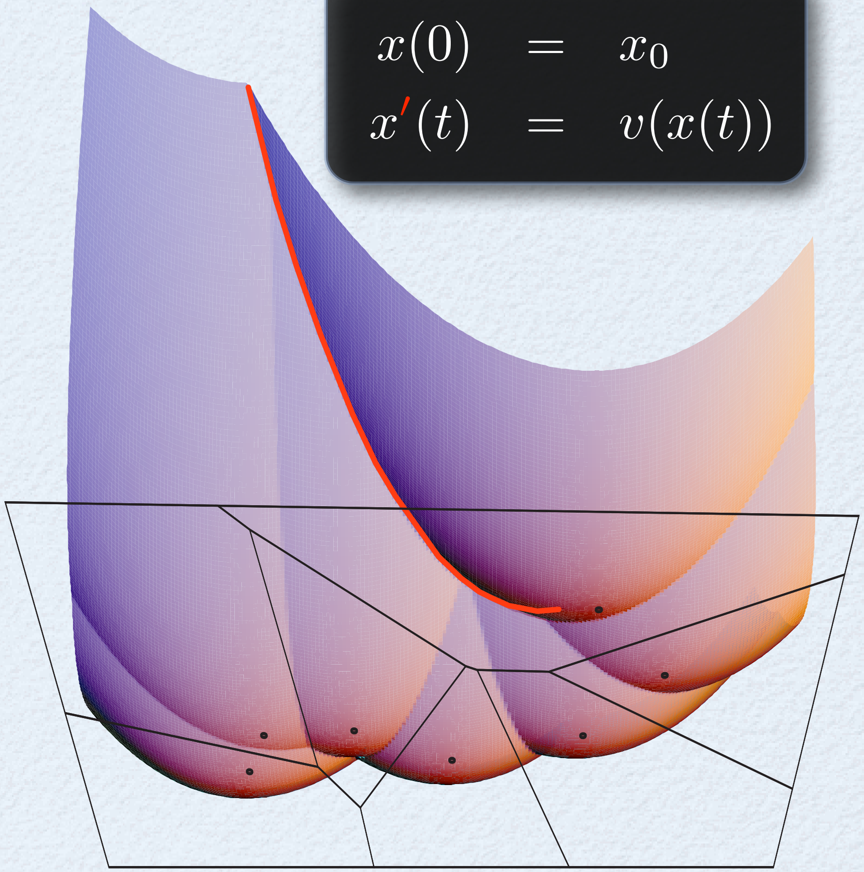

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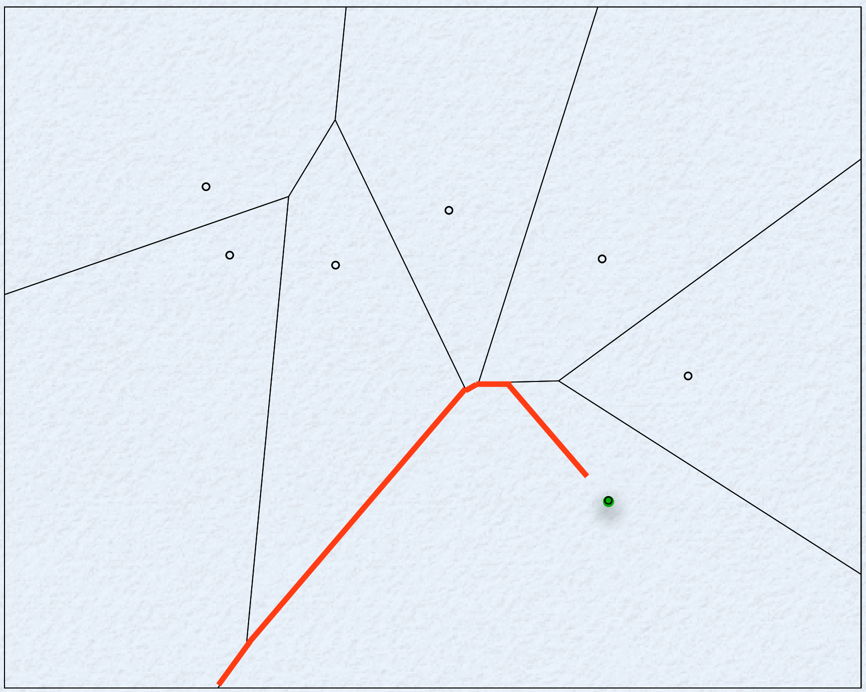

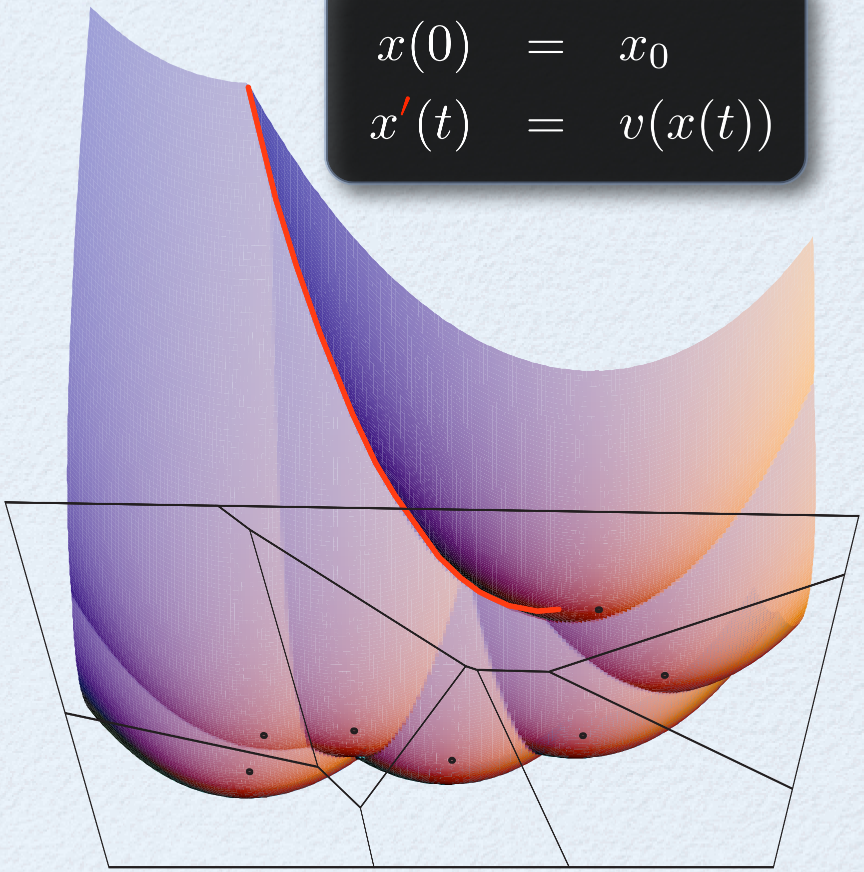

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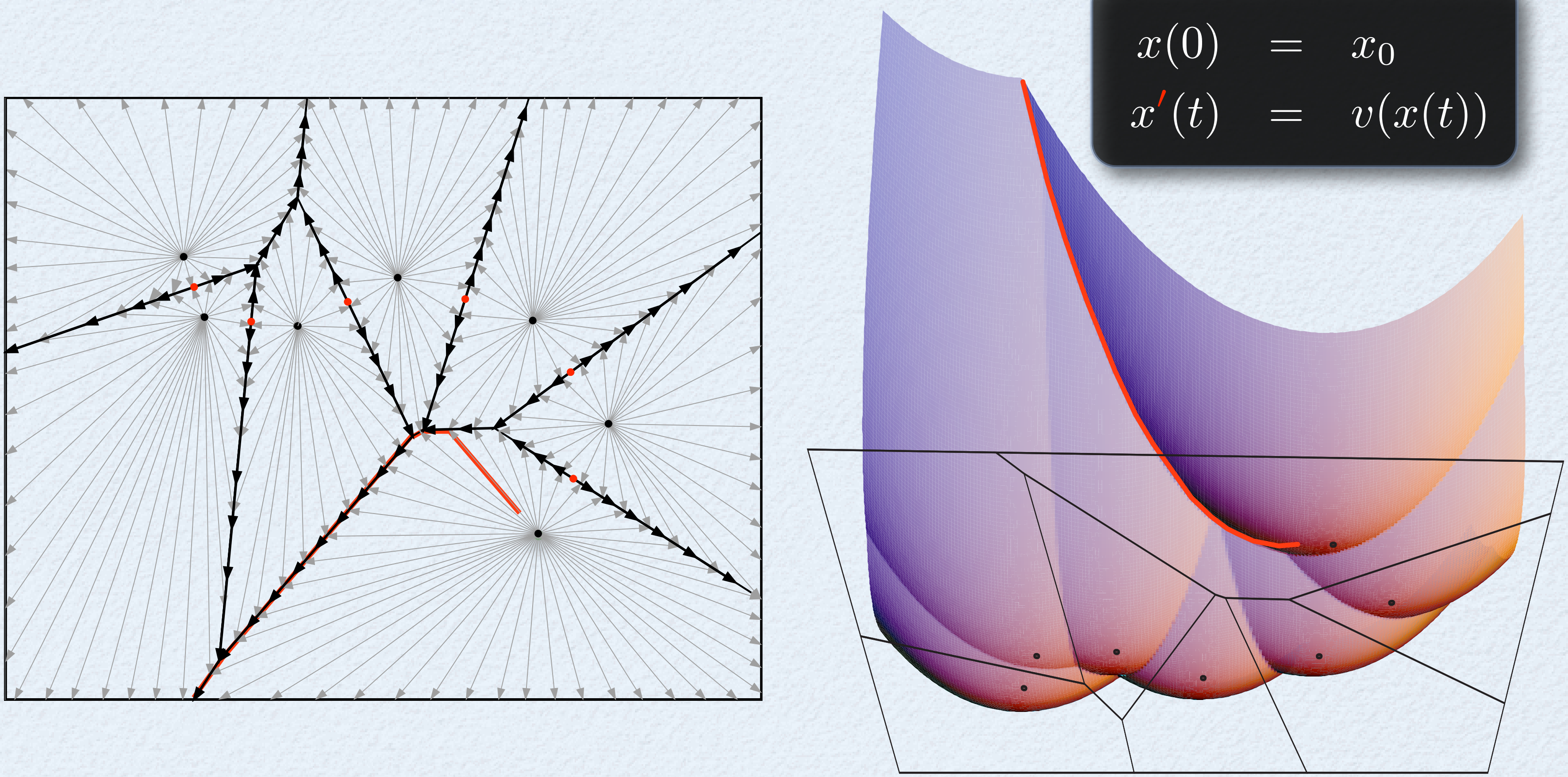

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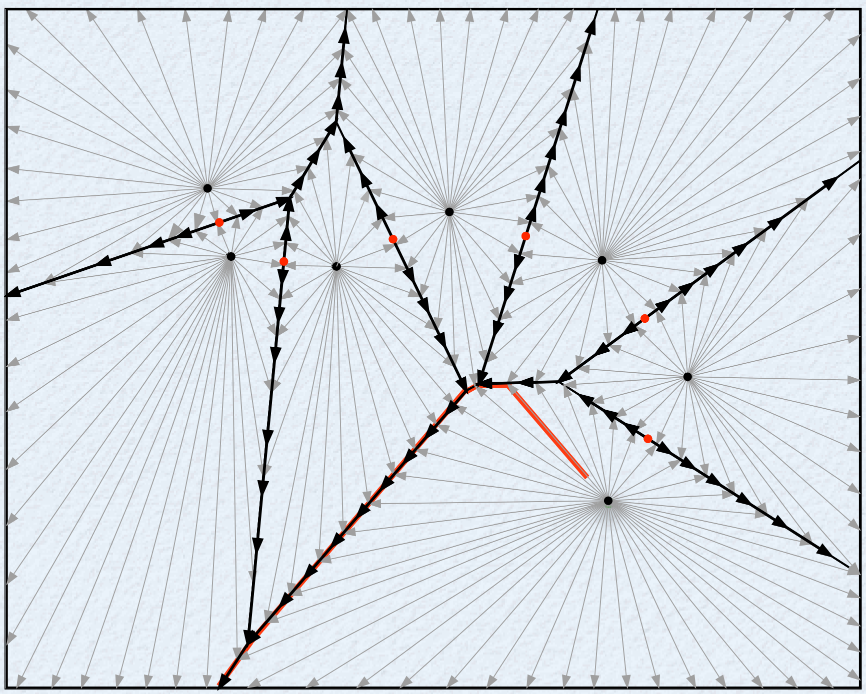

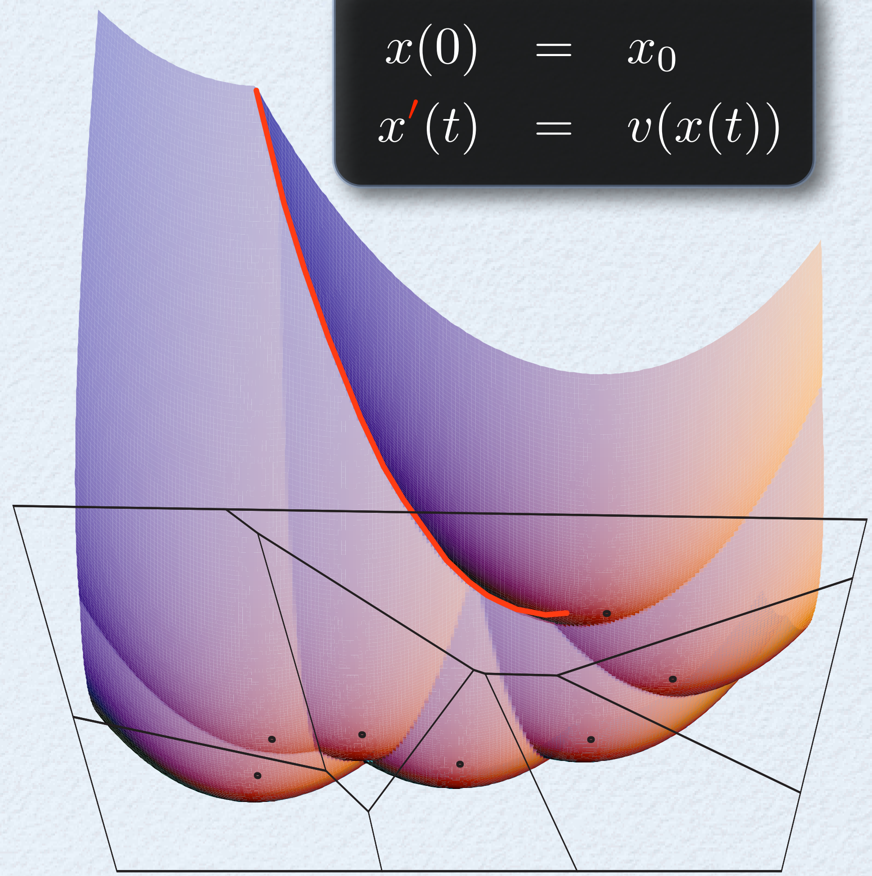

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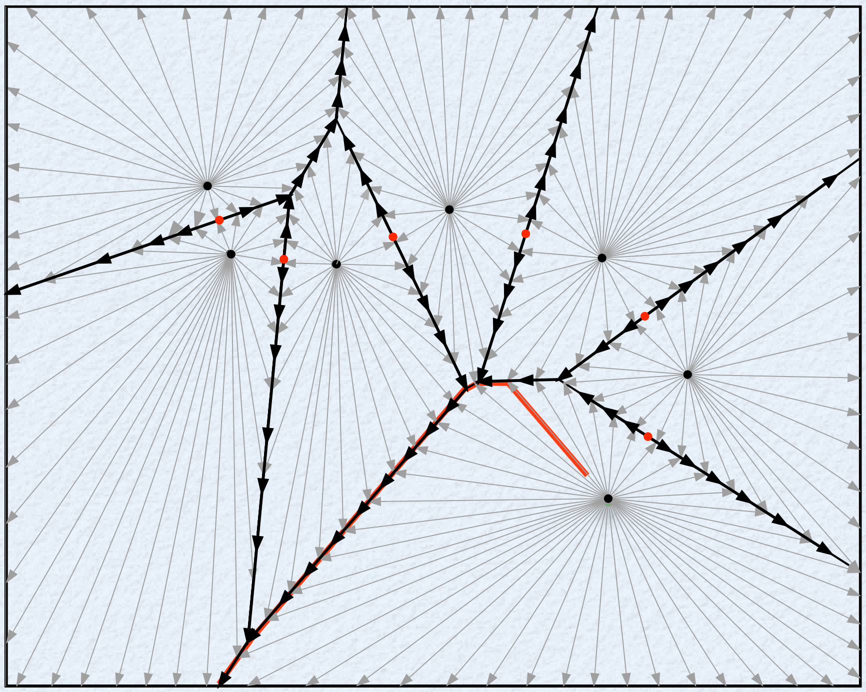

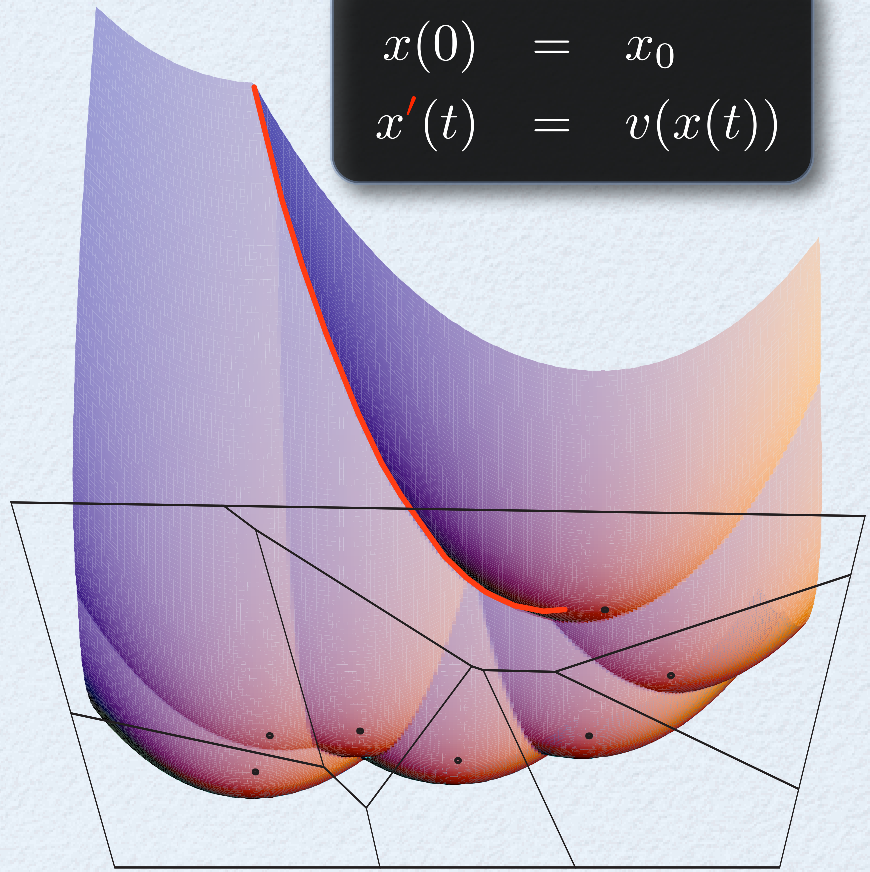

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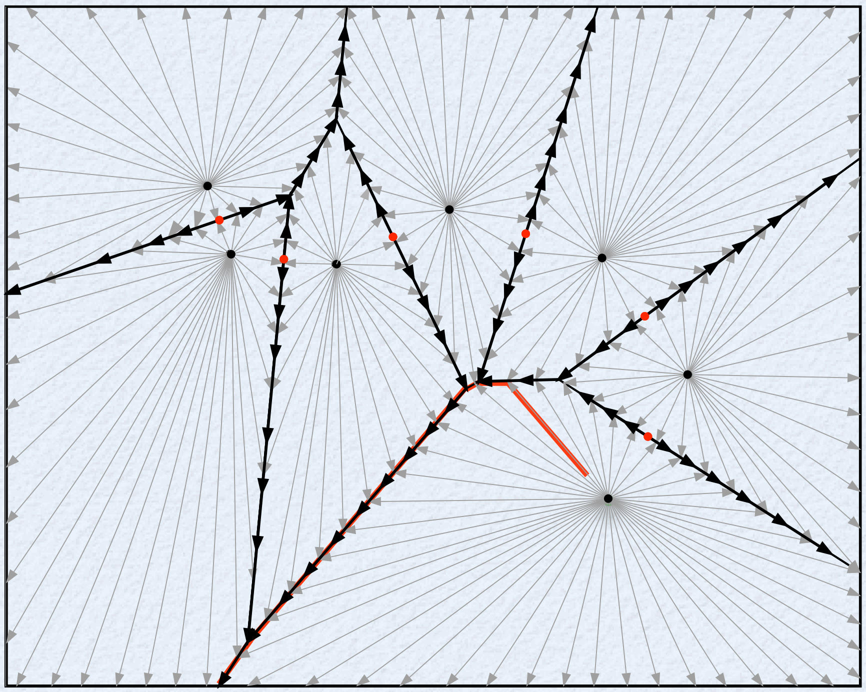

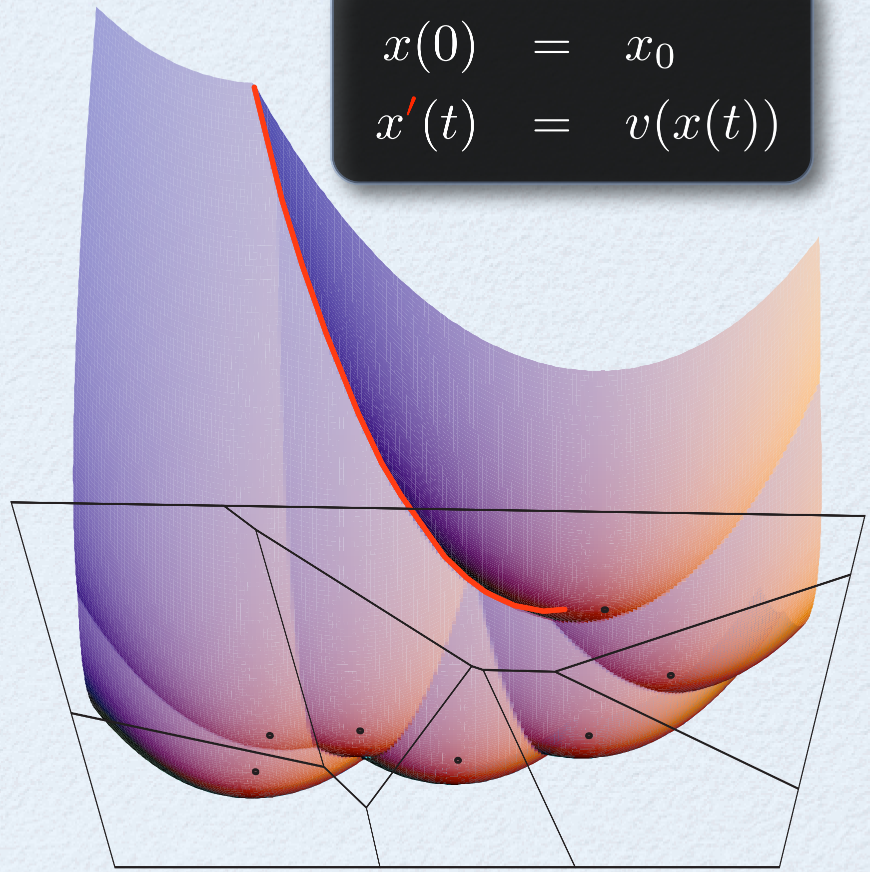

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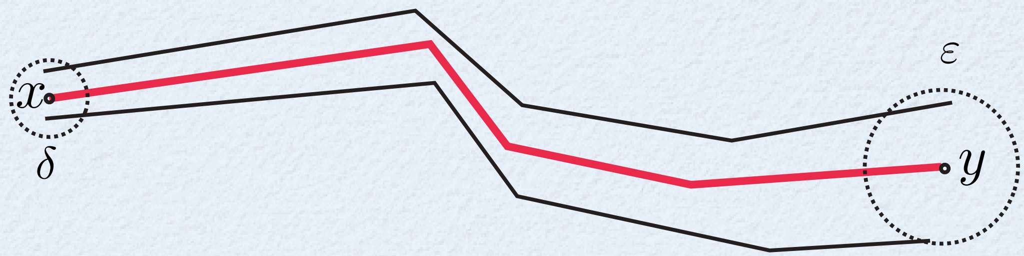
Continuity of the Induced Flow

Theorem. The flow map $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous on both variables.



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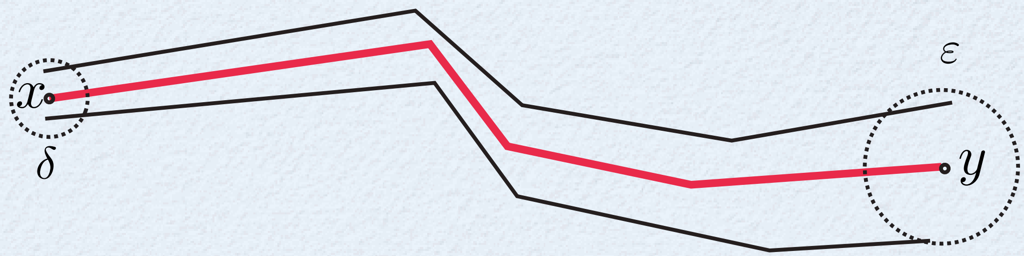

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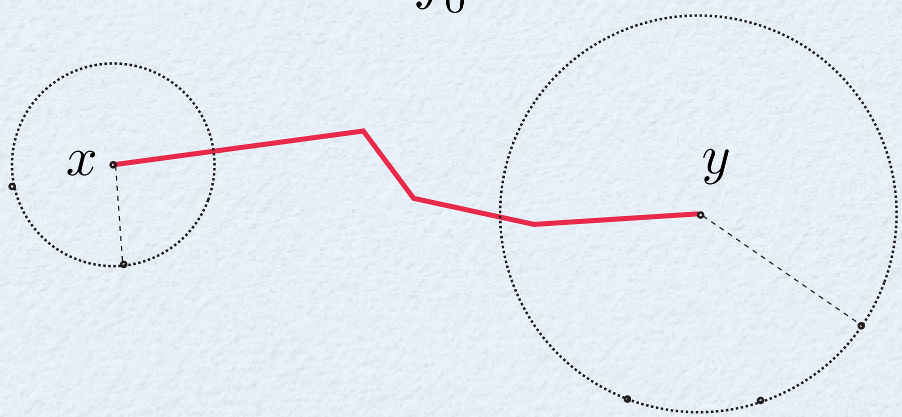
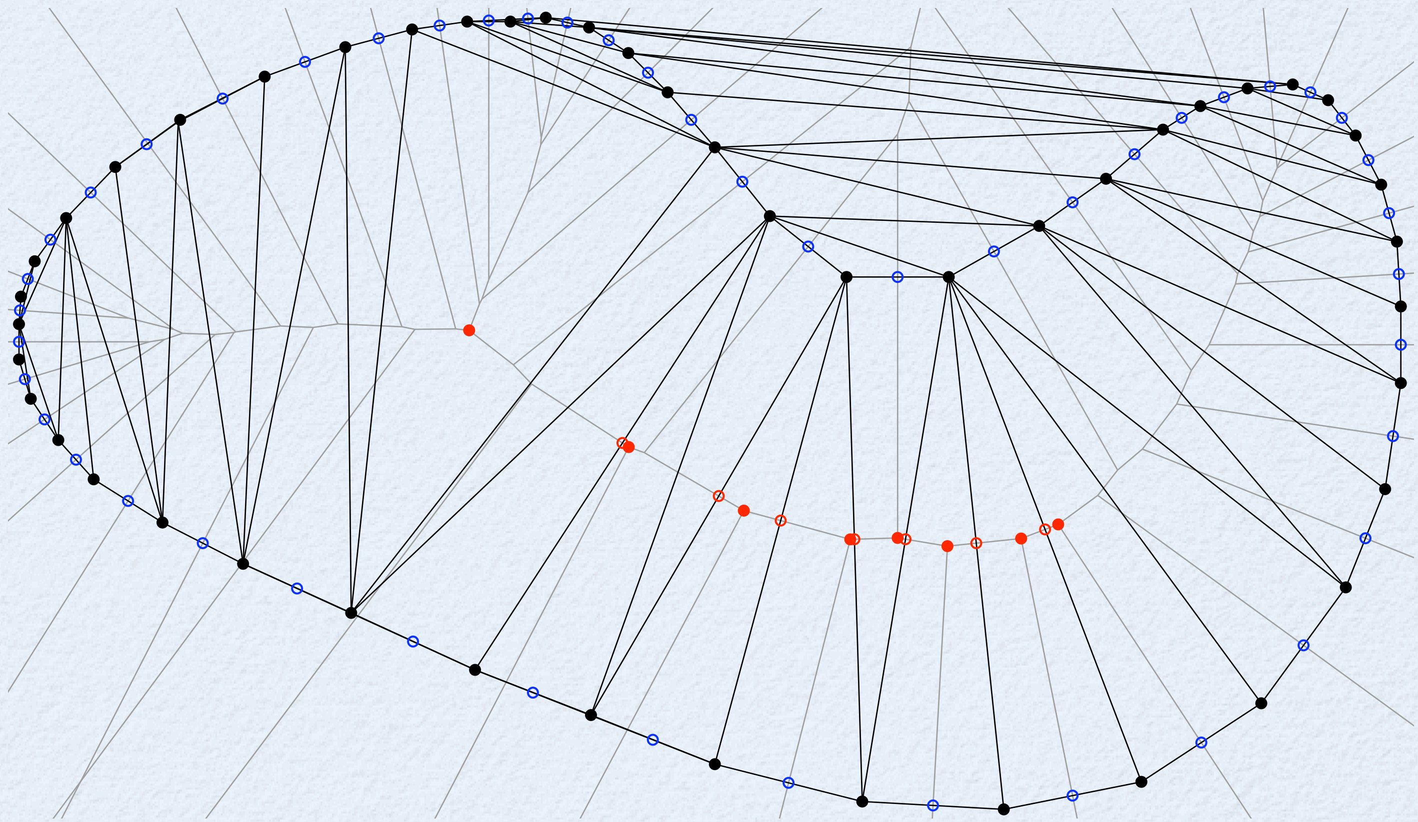
Theorem. For $y = \phi(t, x)$,

$$h(y) = h(x) + \int_0^t \|v(\phi(\tau, x))\|^2 d\tau.$$



Critical Points of Distance Function

A point c with $v(c) = 0$ is called critical.



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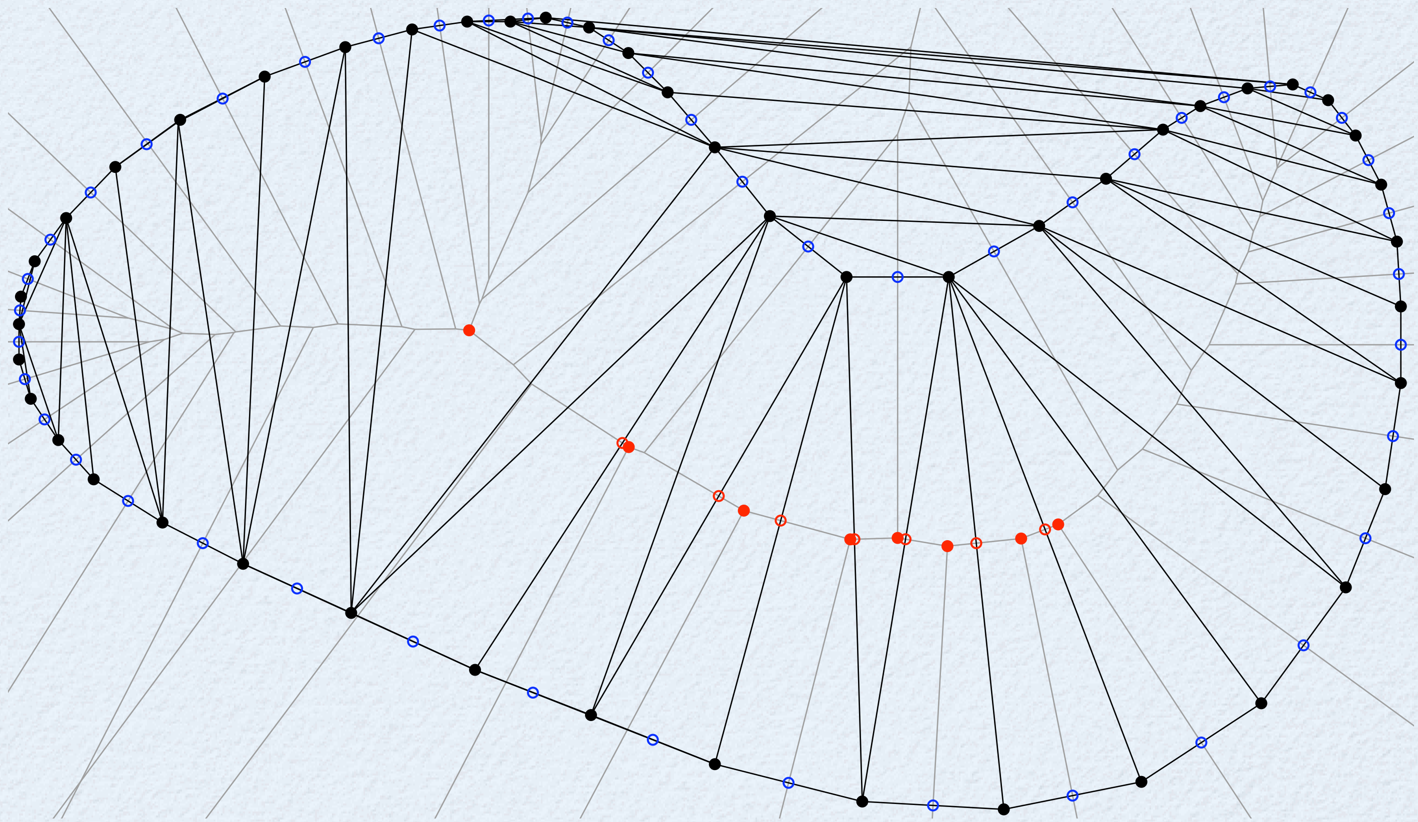
A point c is critical iff $\{c\} = V(c) \cap D(c)$.



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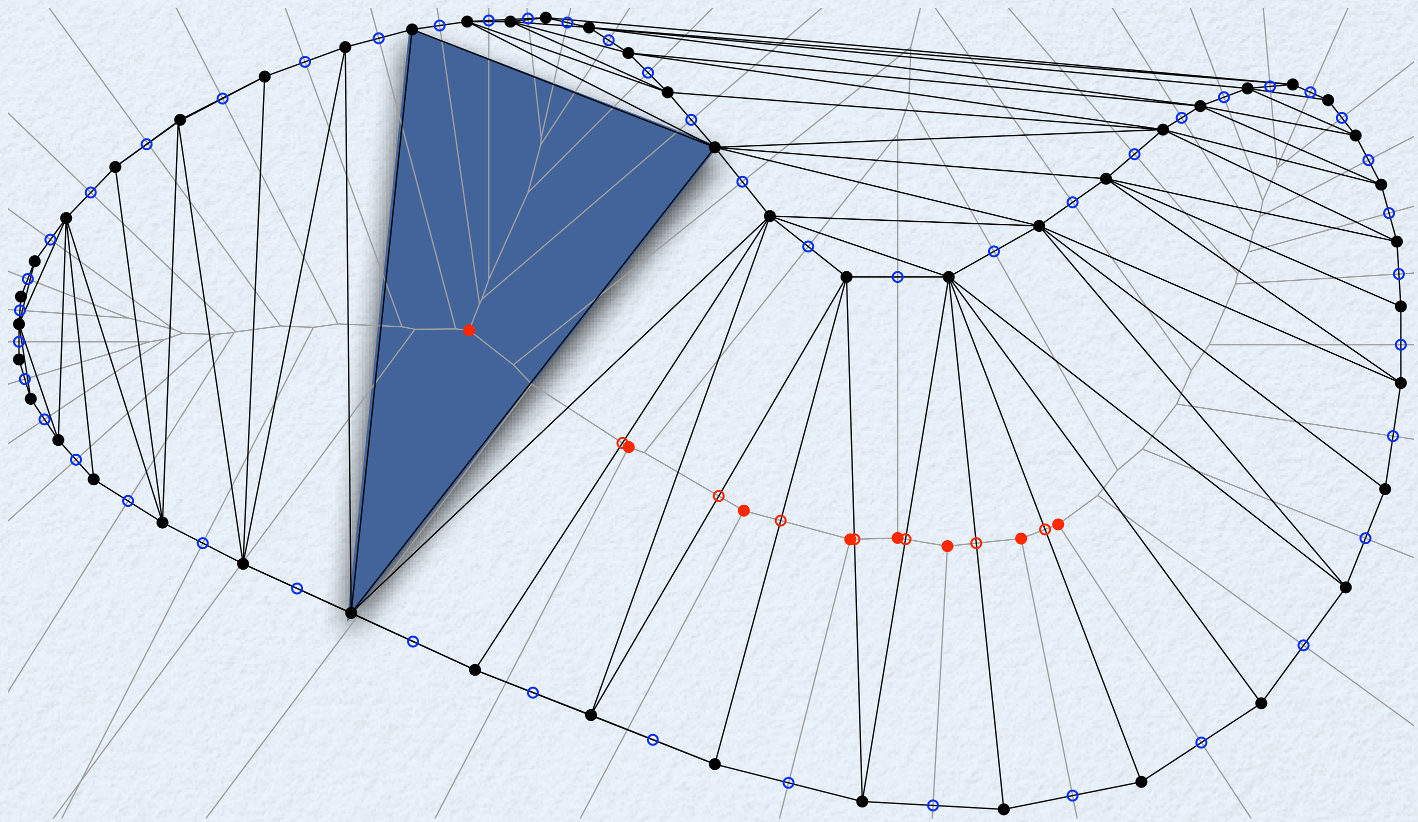

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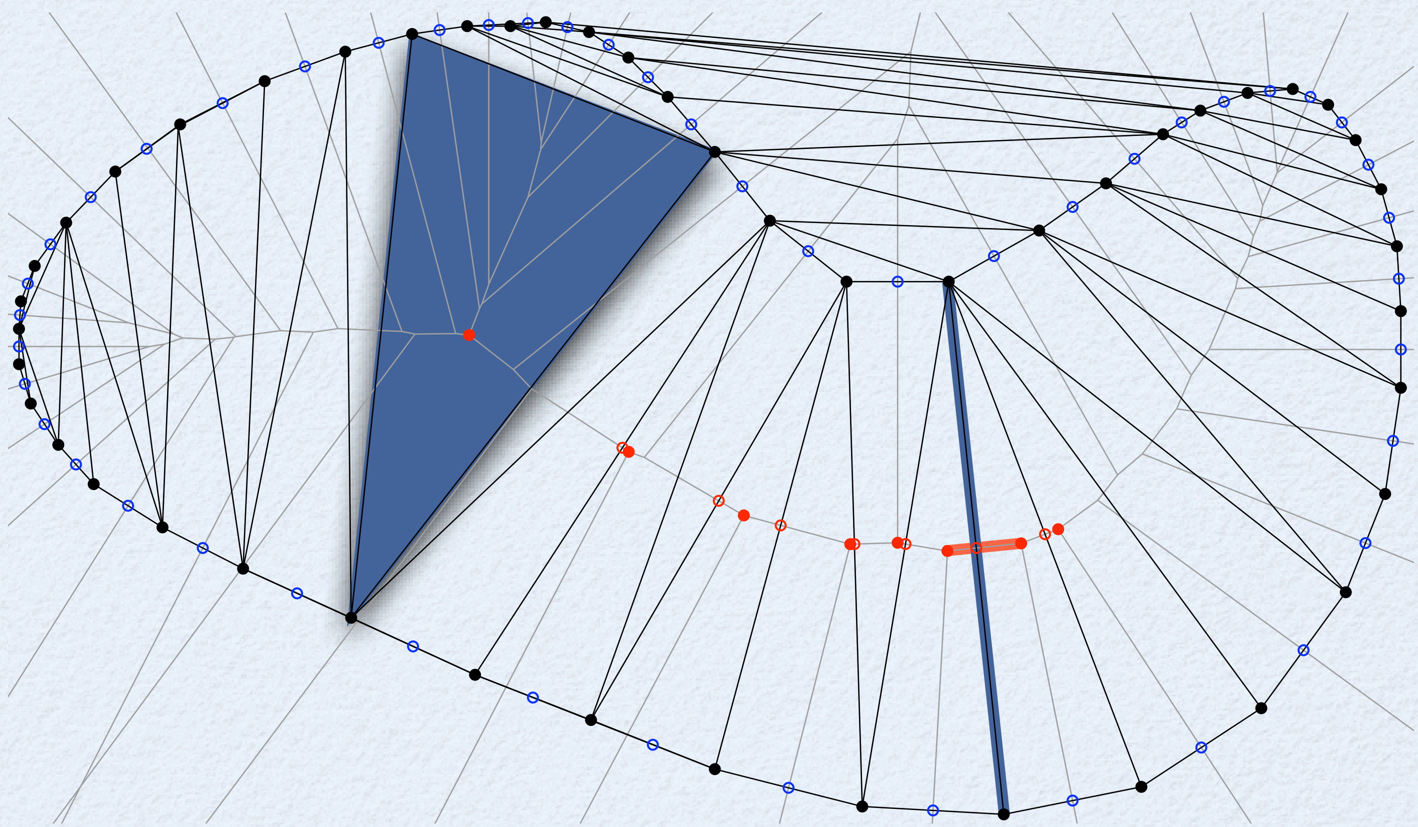
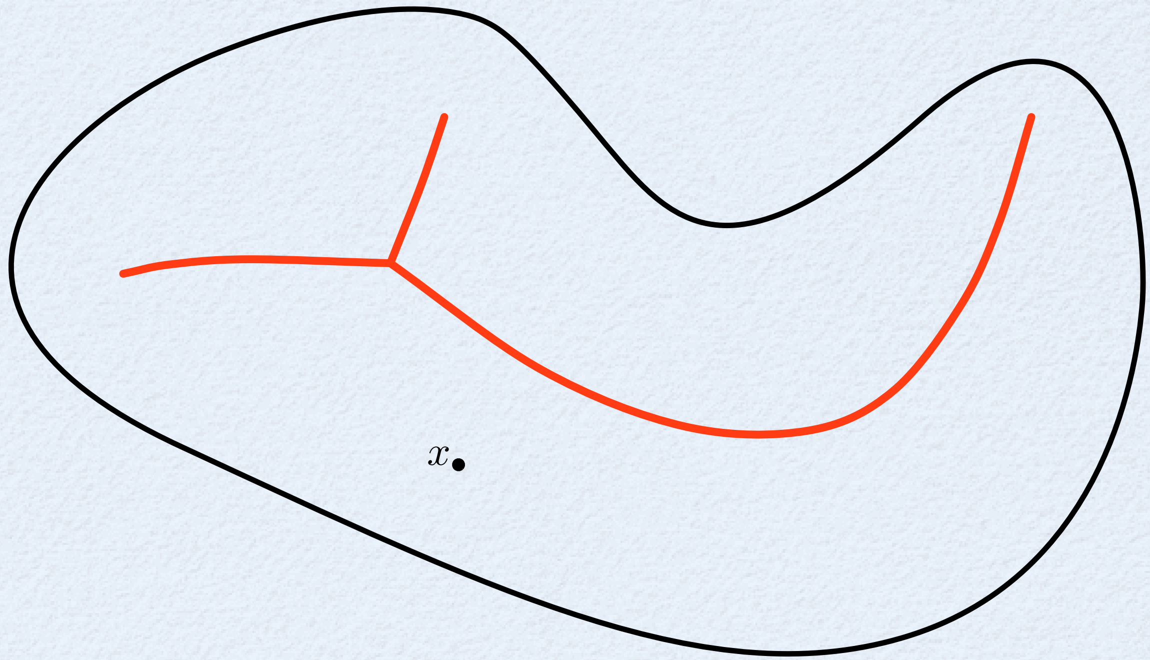


Separation of Critical Points



The δ-tubular neighborhoods of Σ and M:

$$\begin{aligned} \Sigma_\delta &= \left\{ x \in \mathbb{R}^n \setminus M : \|x - \hat{x}\| < \delta f(\hat{x}) \right\} \\ M_\delta &= \left\{ x \in \mathbb{R}^n \setminus \Sigma : \|x - \check{x}\| < \delta f(\hat{x}) \right\} \end{aligned}$$

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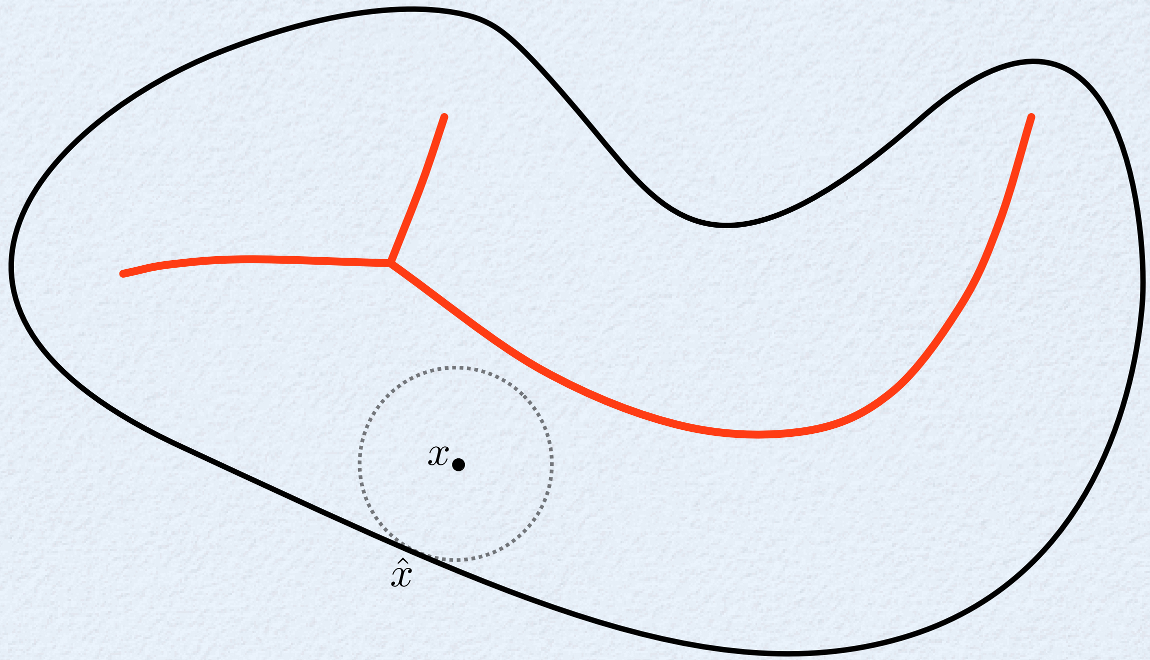

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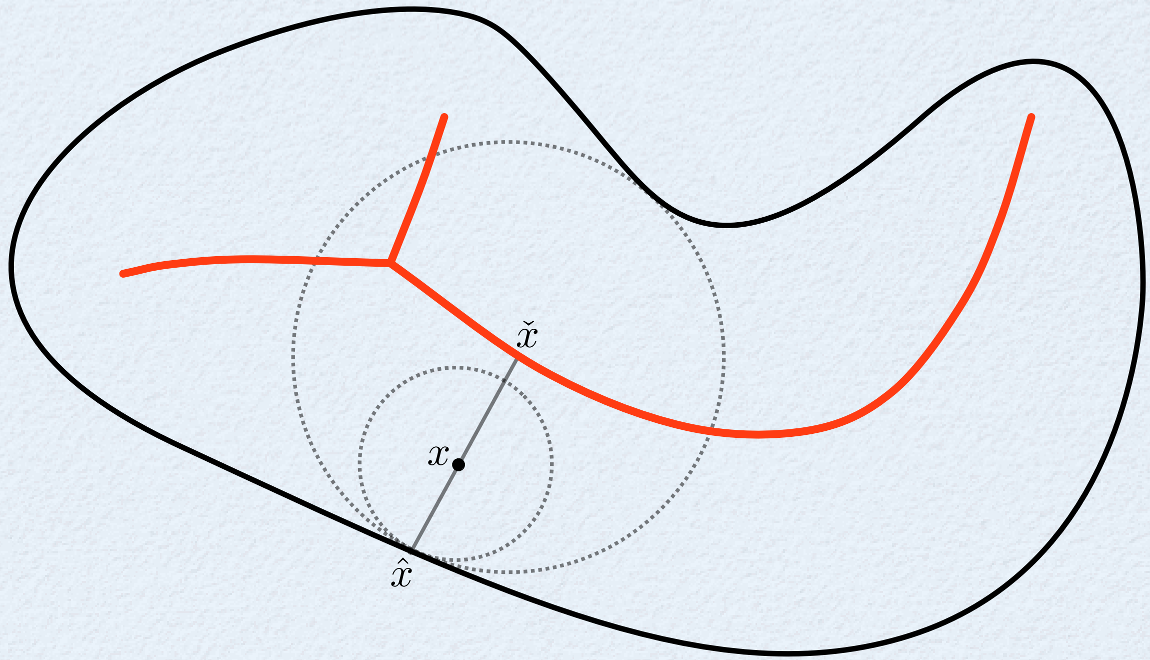

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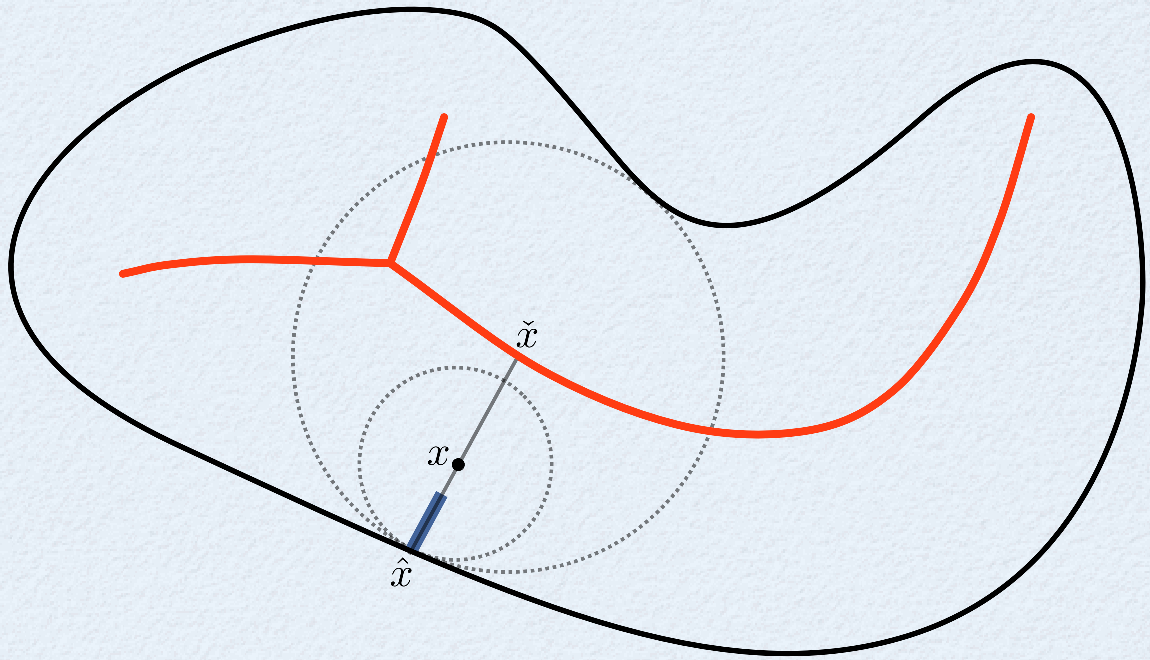

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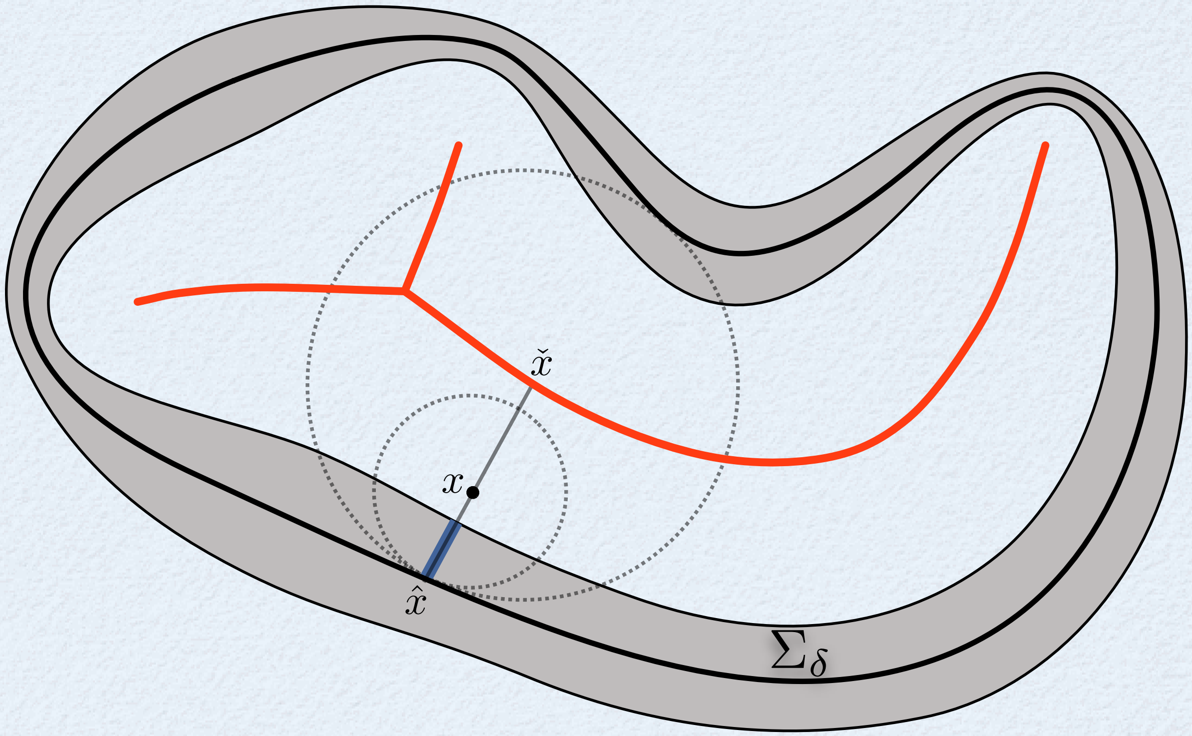

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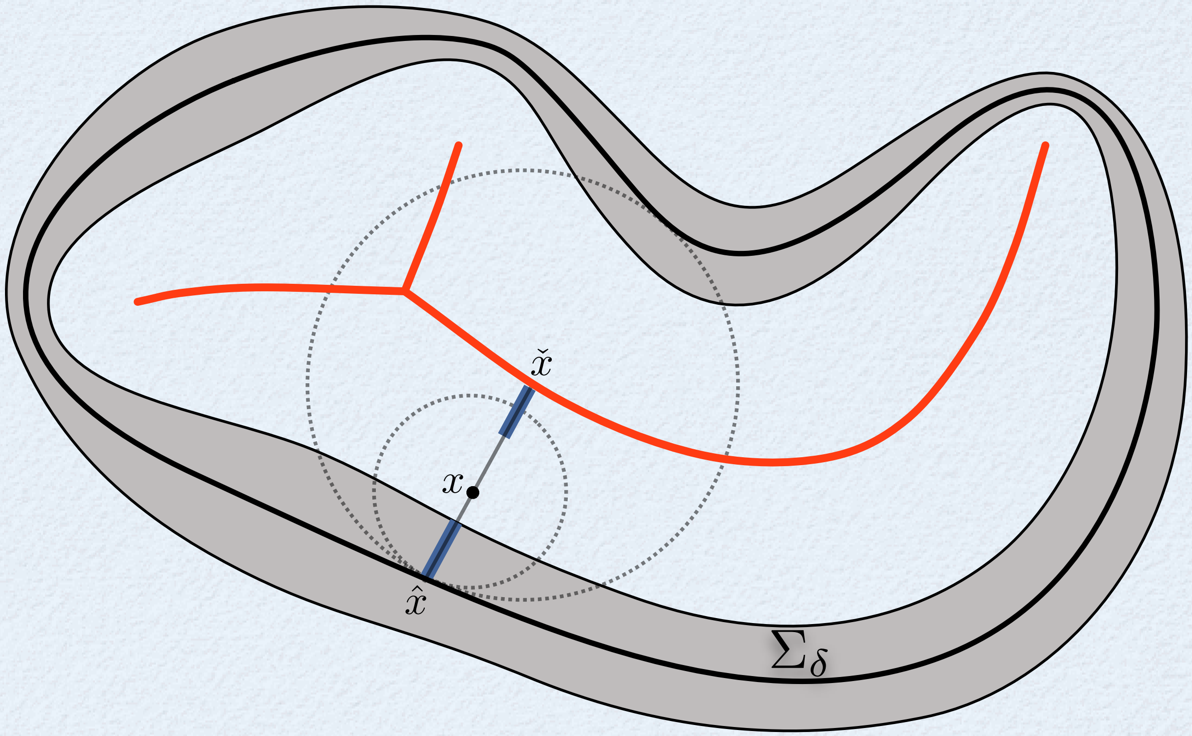

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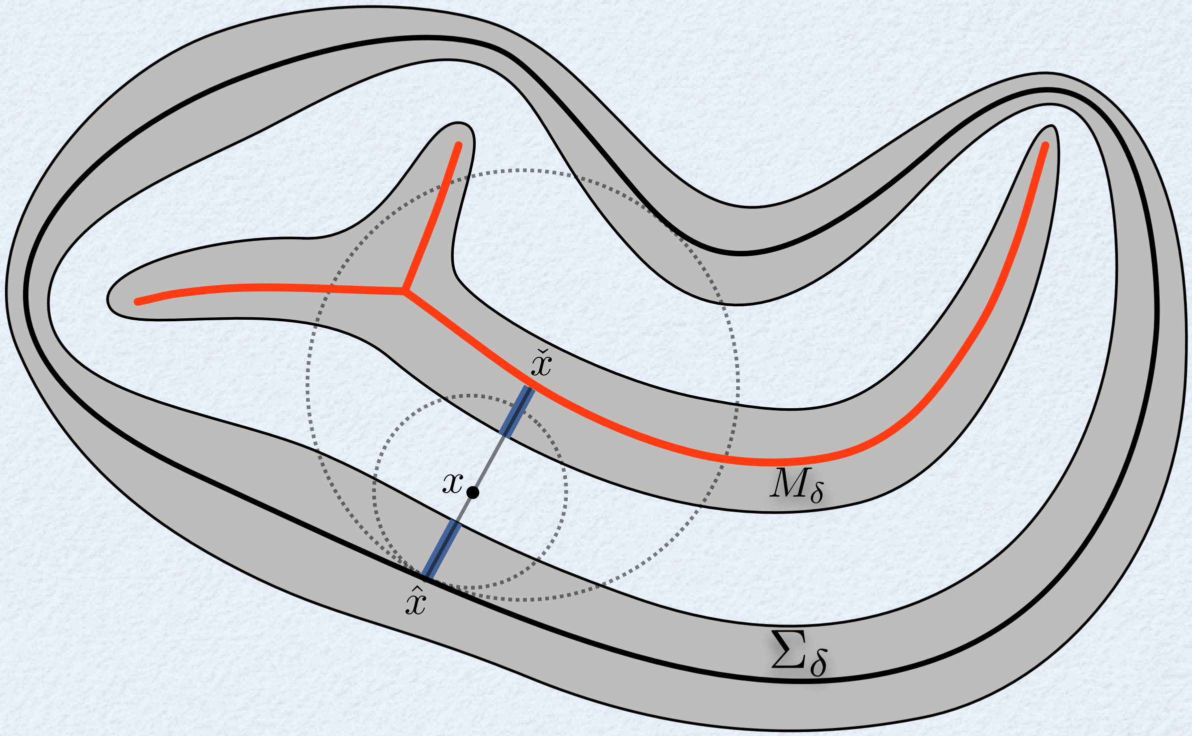

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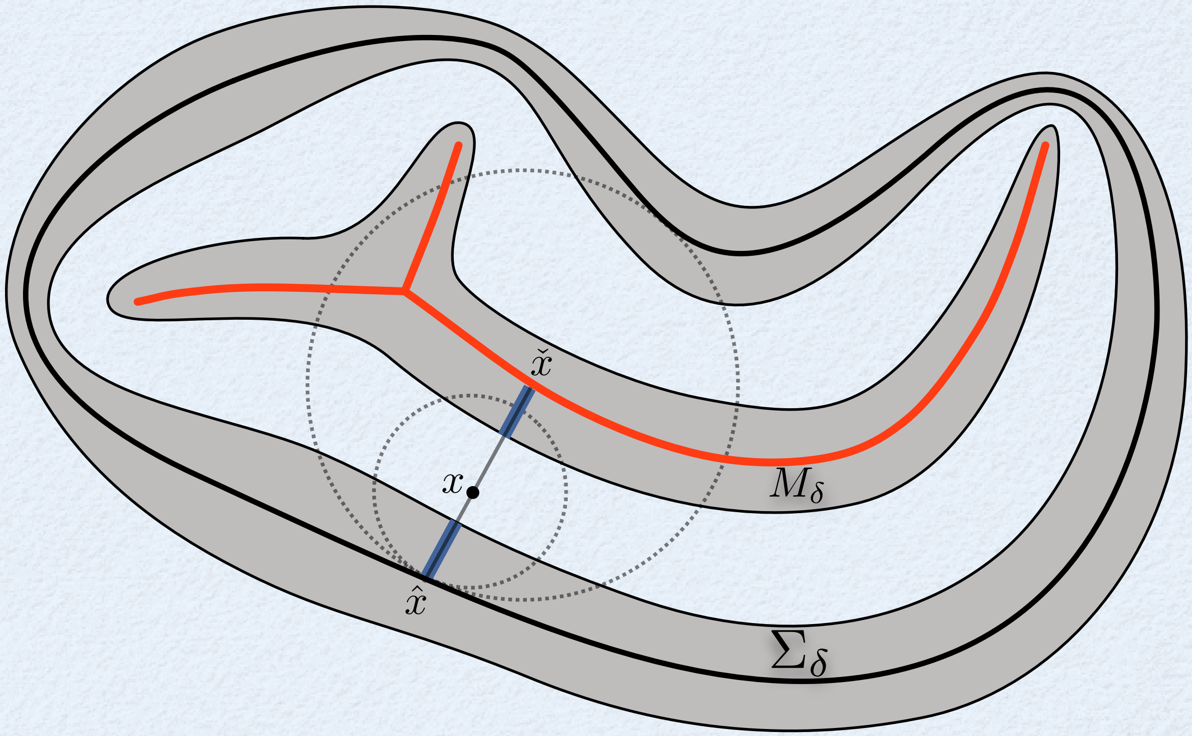
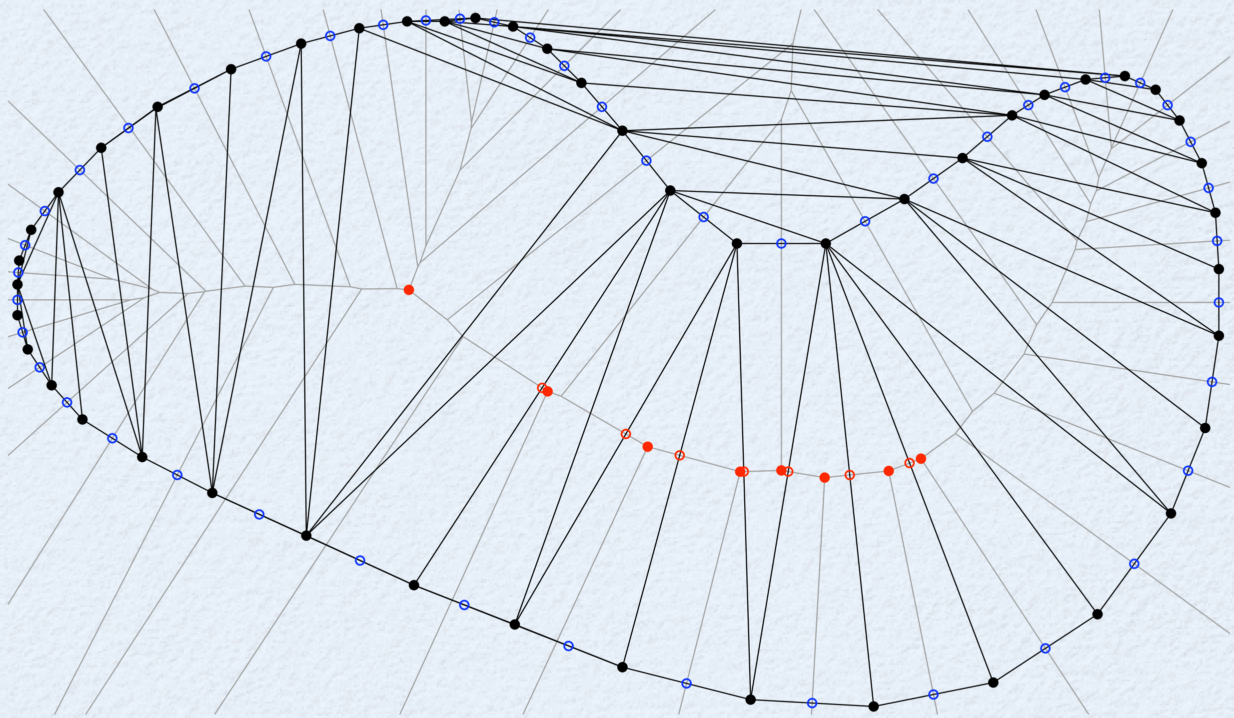


Theorem [DGRS'05]
If h is induced by an ε-sample of Σ with $\varepsilon < 1/\sqrt{3}$, the all critical points of h are contained in either Σ_{ε^2} or $M_{2\varepsilon^2}$.

Stable Manifold of a Critical Point

Stable manifold of a critical point c is everything that flows into c.

$$\mathsf{Sm}(c) = \{x : \phi(\infty, x) = c\}.$$



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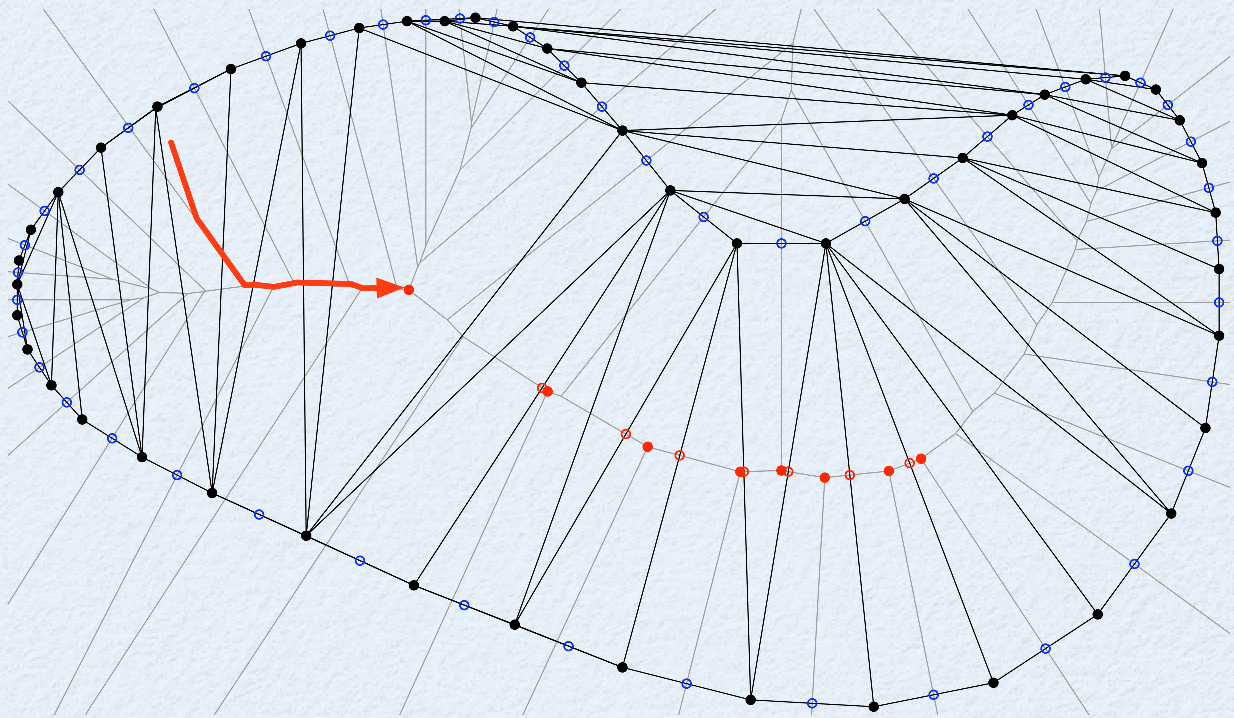

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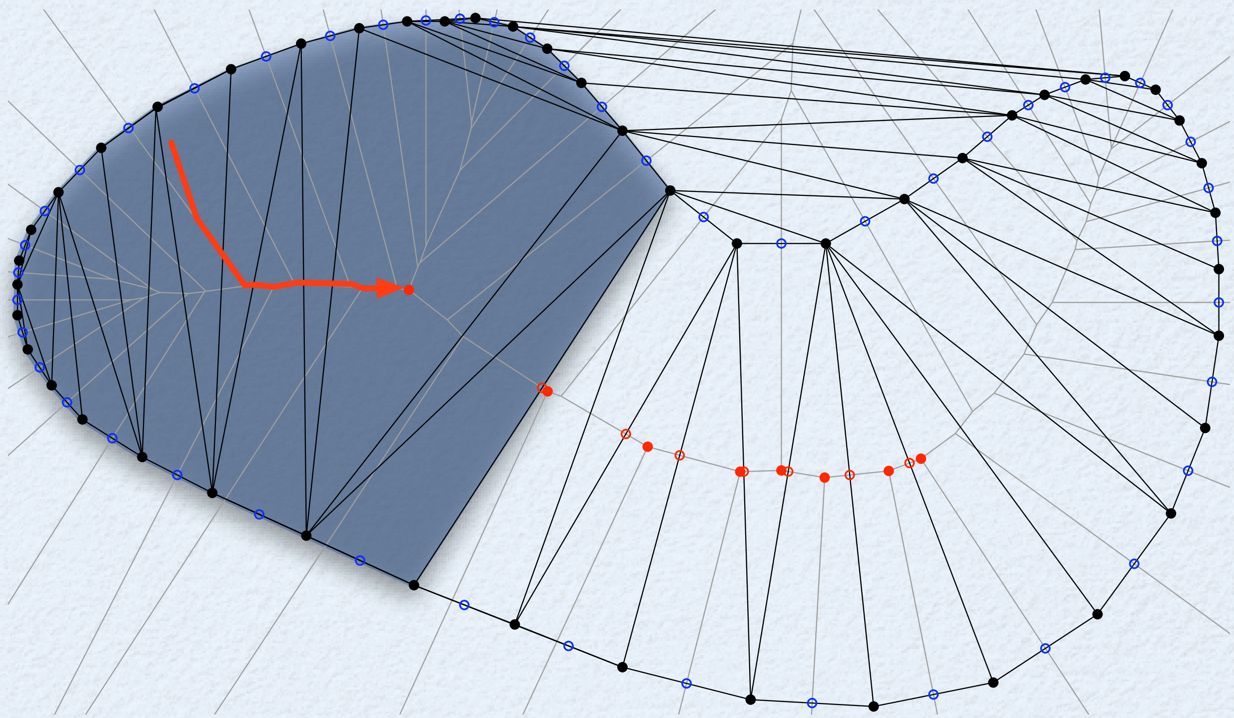

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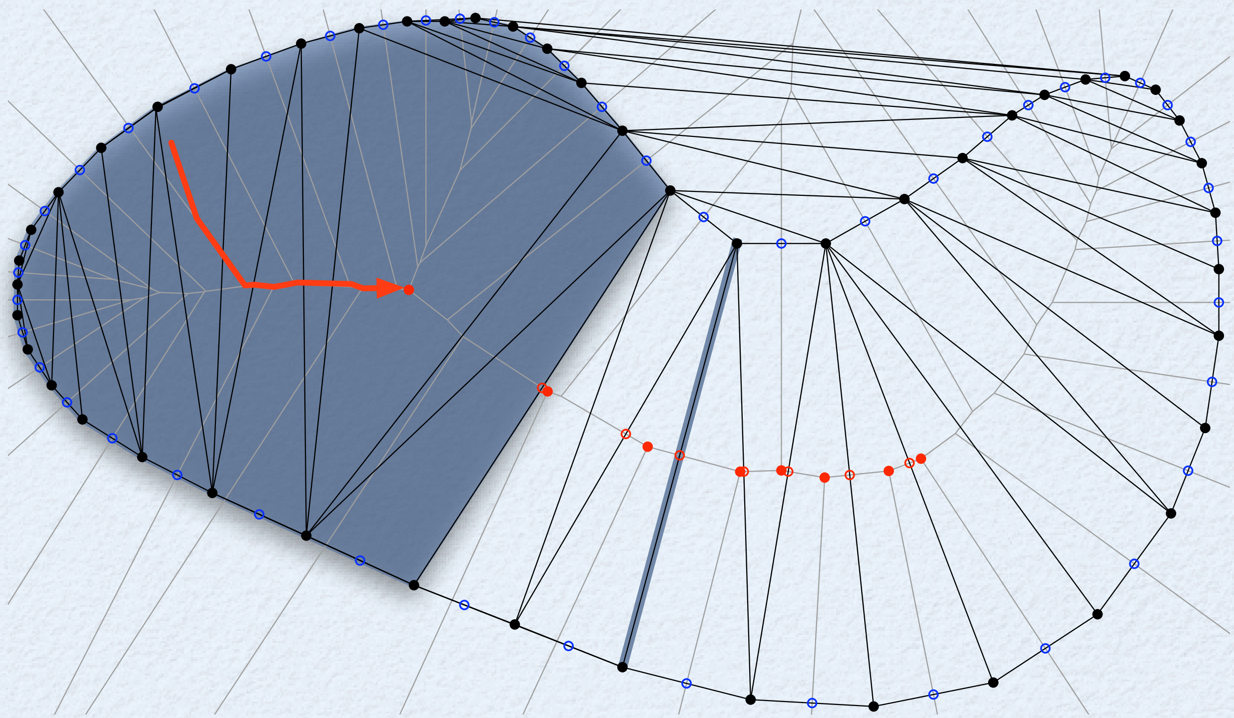

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Unstable manifold of a critical point c is everything that flows "out of" c.

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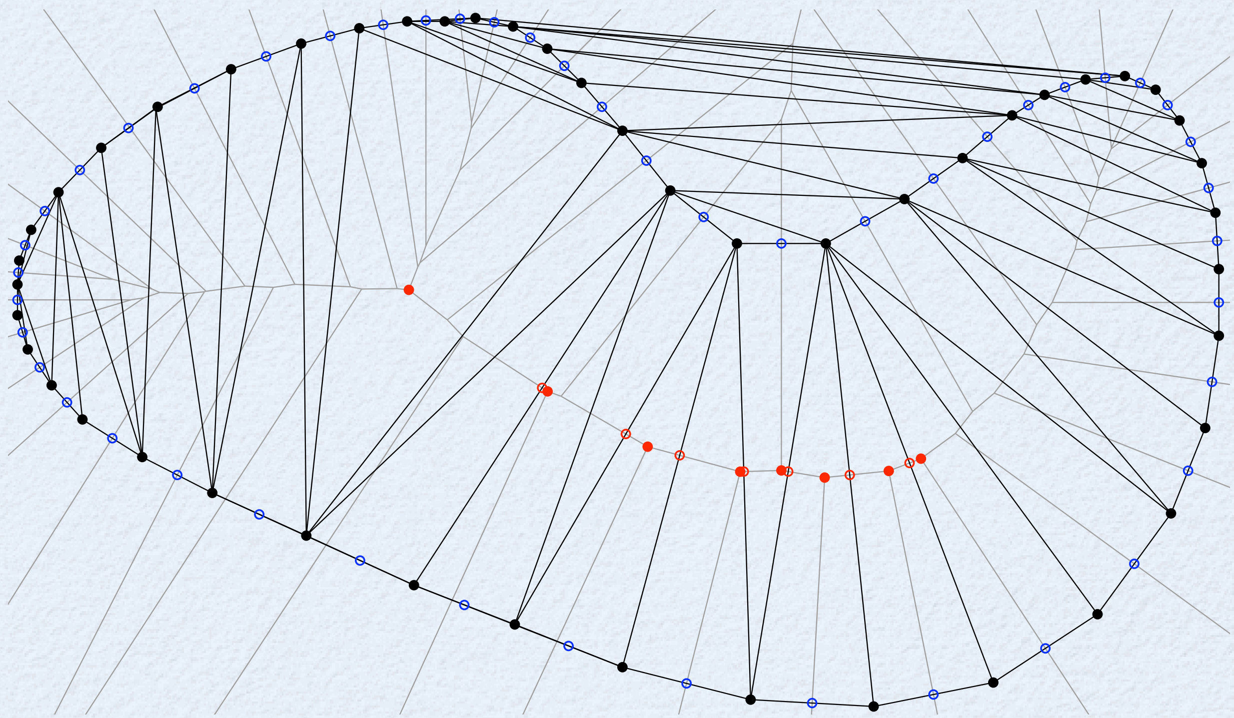

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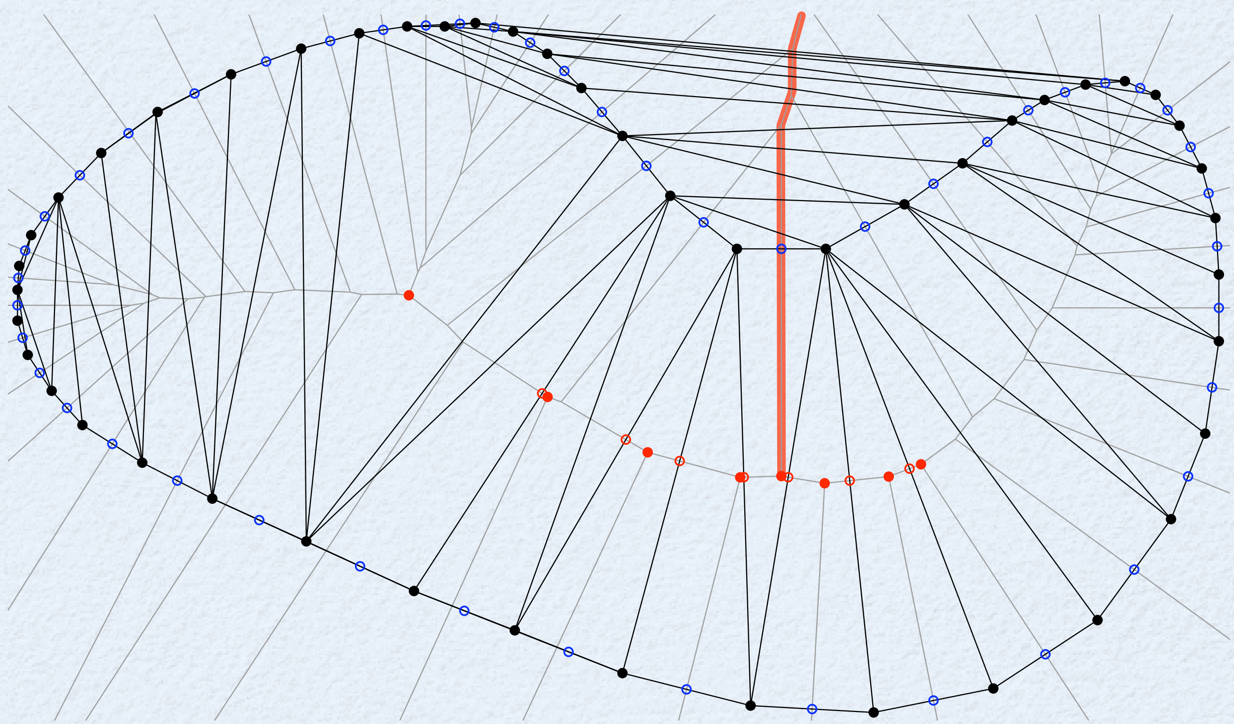

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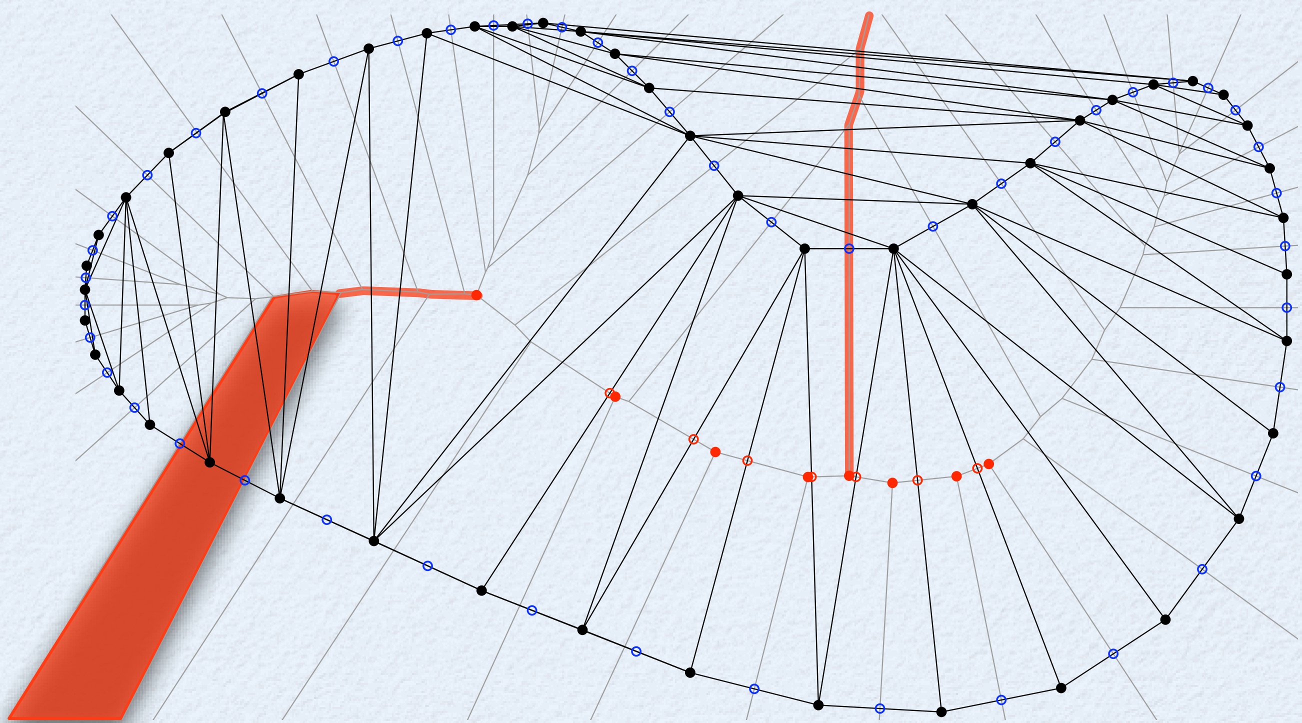


A Criterion for Homotopy Equivalence

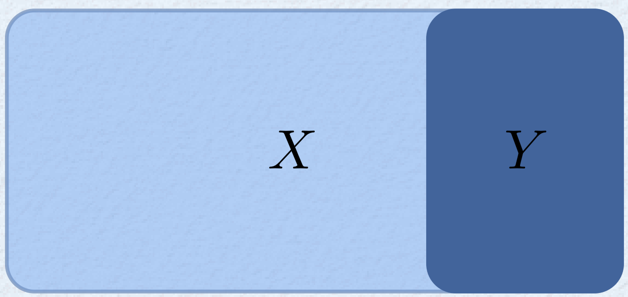
Proposition. Let X and $Y \subseteq X$ be arbitrary sets and

$$H : [0,1] \times X \to X$$

be a continuous function (on both variables) satisfying

1. $\forall x \in X : H(0,x) = x$
2. $\forall y \in Y, \forall t \in [0,1] : H(t,y) \in Y$
3. $\forall x \in X : H(1,x) \in Y$

Then X and Y have the same homotopy type.



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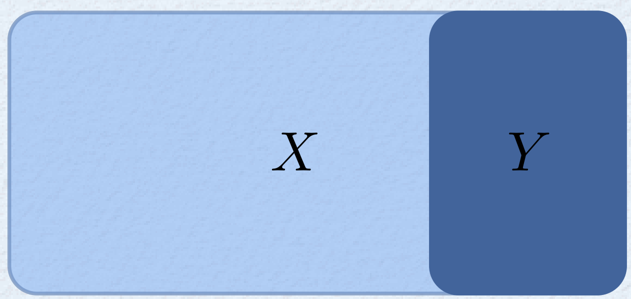

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1. $\forall x \in X : H(0,x) = x$ Identity at time 0
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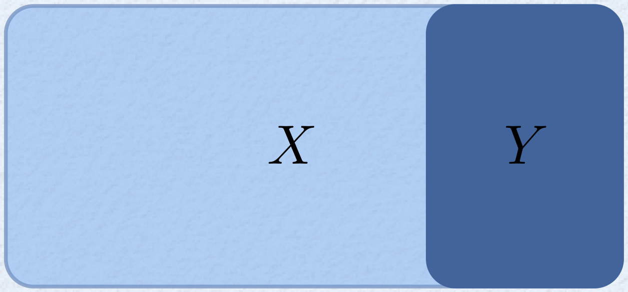

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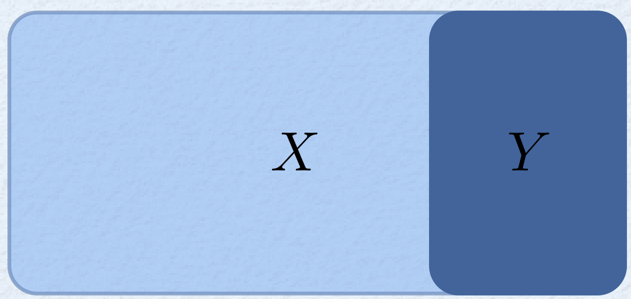

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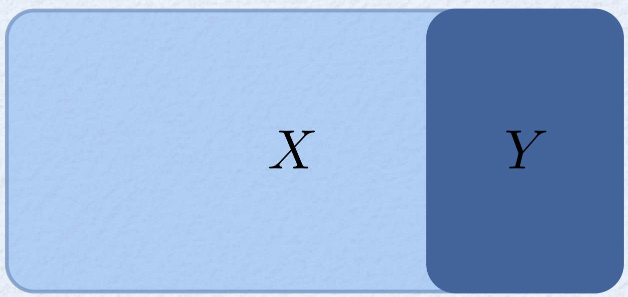

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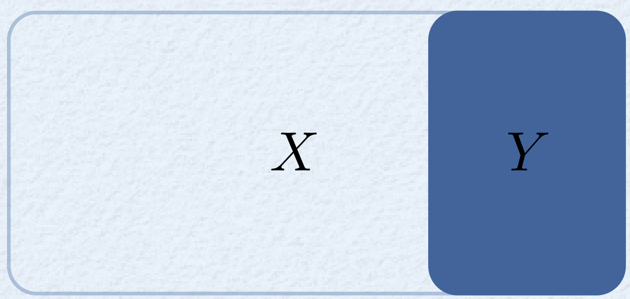

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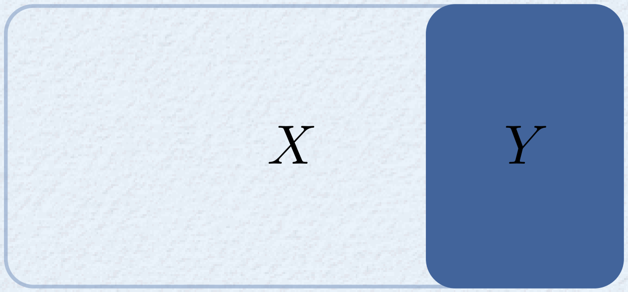

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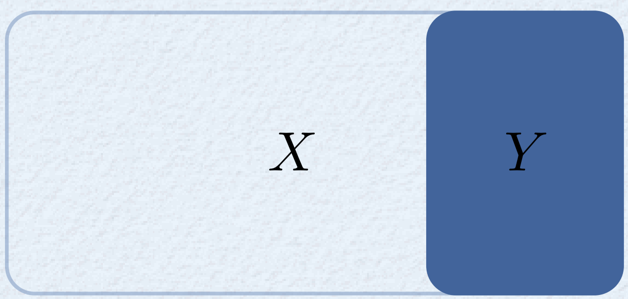

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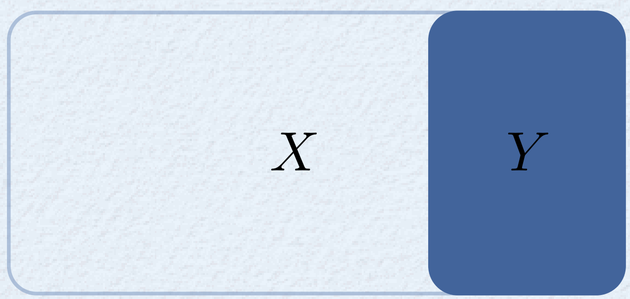
2. $\forall y \in Y, \forall t \in [0, T] : \phi(t, y) \in Y$ — Nothing leaves Y
3. $\forall x \in X : \phi(T, x) \in Y$ — Everything in Y by time 1

Then X and Y have the same homotopy type.



A Criterion for Homotopy Equivalence

Proposition. Let X and $Y \subseteq X$ be arbitrary sets and

$$\phi : \underbrace{[0, T]}_{\text{time}} \times X \to X$$

be a continuous function (on both variables) satisfying

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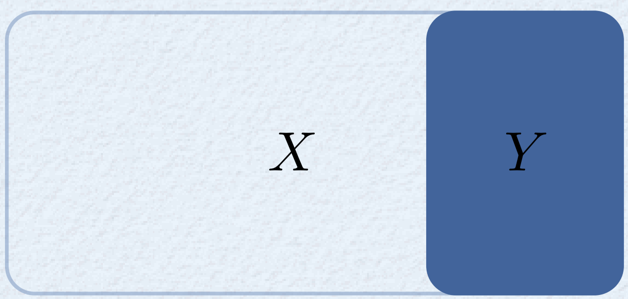

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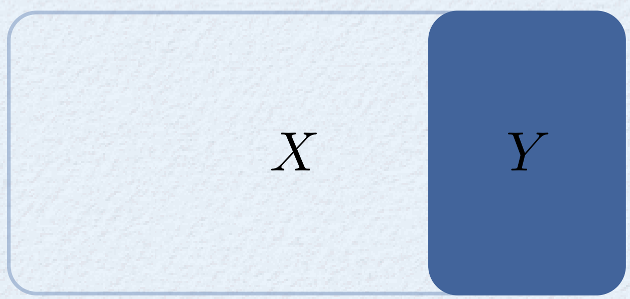

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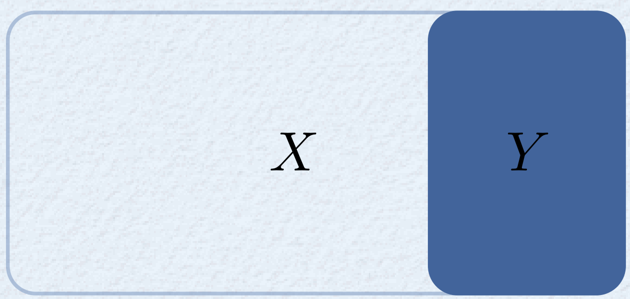

This is the idea Lieutier used in [Lieuteir'04] to show $M(S) \simeq S$.

In Other Words ...

Key Theorem. If $Y \subset X$ are bounded and

1. $\phi(X) = X$ and $\phi(Y) = Y$, and
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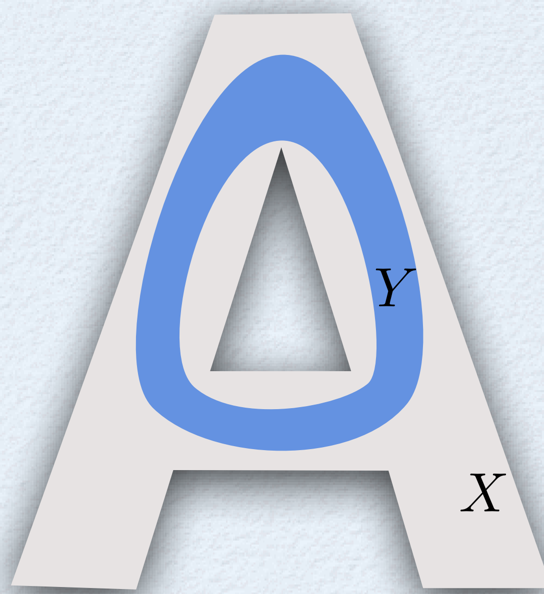

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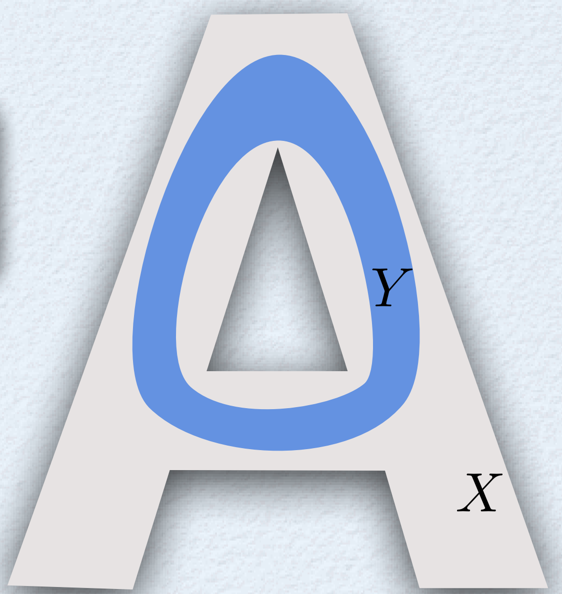

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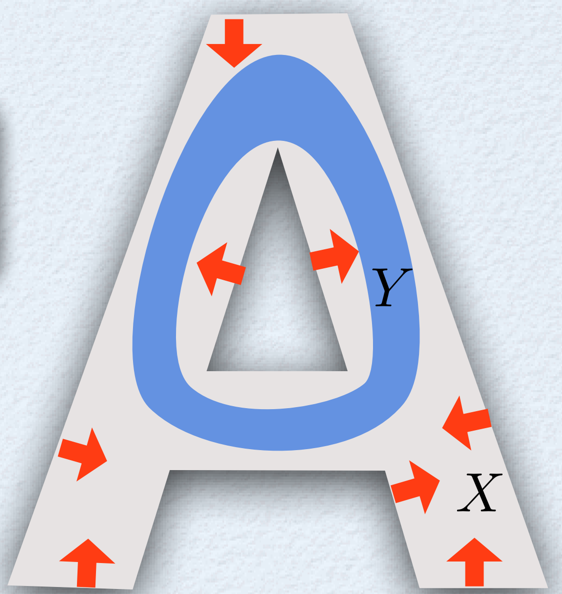

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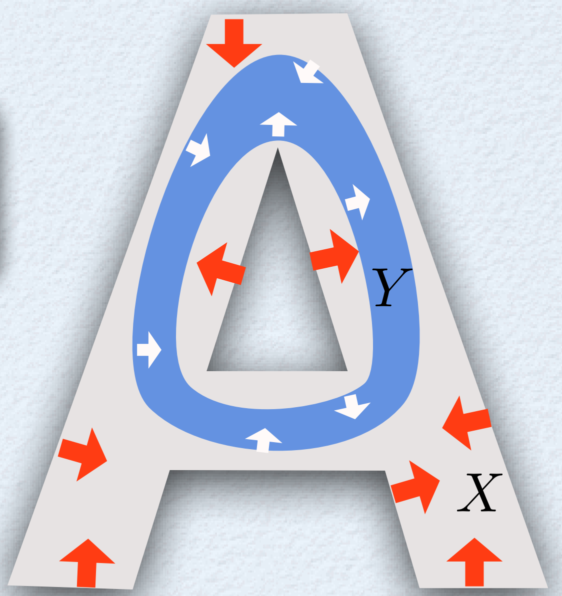

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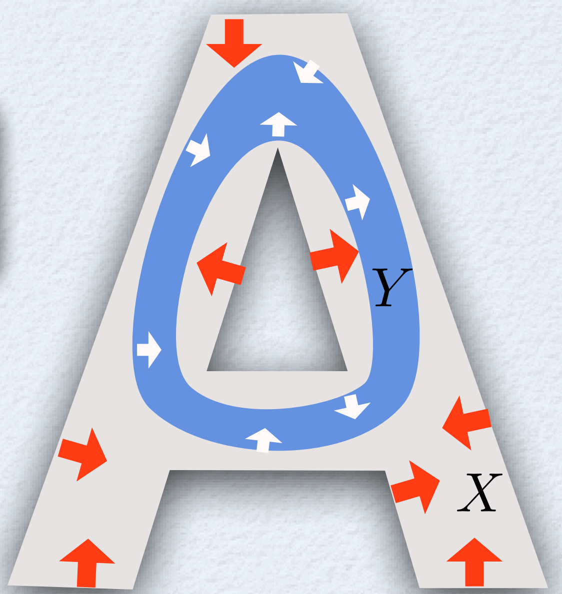


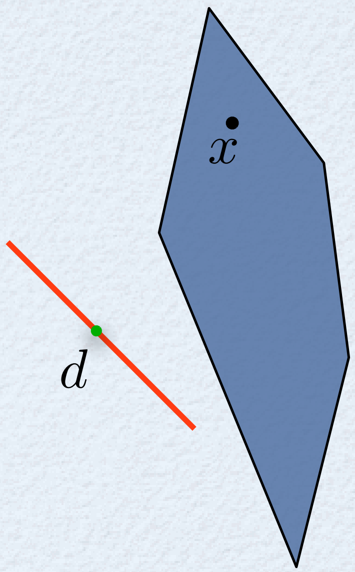
Proof. If $\phi(t,x) \notin Y$, then

$$\begin{aligned} h(\phi(t,x)) &= h(x) + \int_0^t \|v(\phi(\tau,x))\|^2 d\tau \\ &\geq h(x) + \int_0^t c^2 d\tau \\ &= h(x) + tc^2 \\ &< d_H(X,P)^2. \end{aligned}$$

A Handy Lower Bound for Speed

If $V(x) \cap D(x) = \emptyset$ then

$$\begin{aligned} \|v(x)\| &= 2 \cdot \|x - d(x)\| \\ &\geq 2 \cdot \text{dist}(V(x), D(x)). \end{aligned}$$



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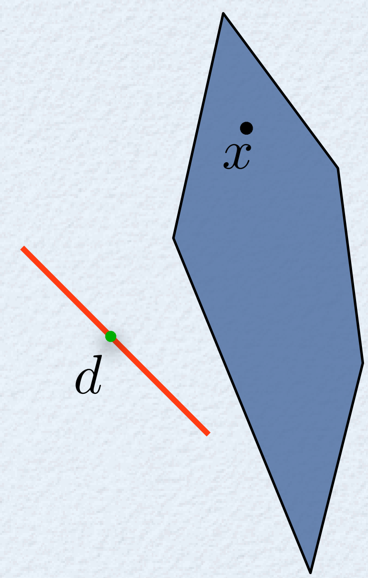
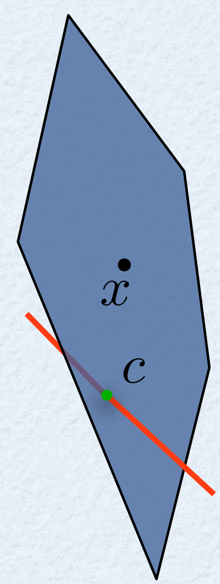


If $V(x) \cap D(x) = \{c\}$ then $x \in \mathsf{Um}(c)$.



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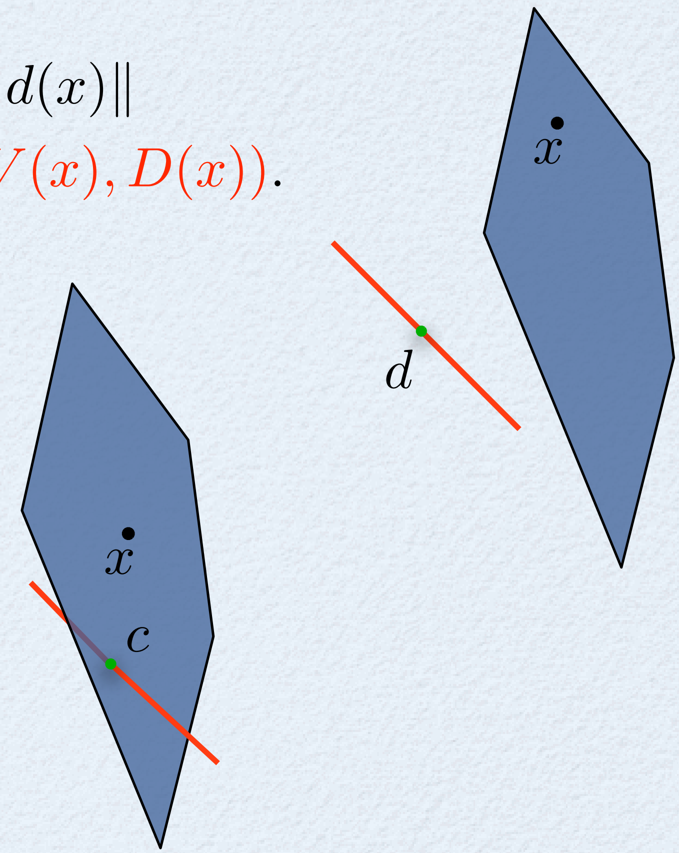
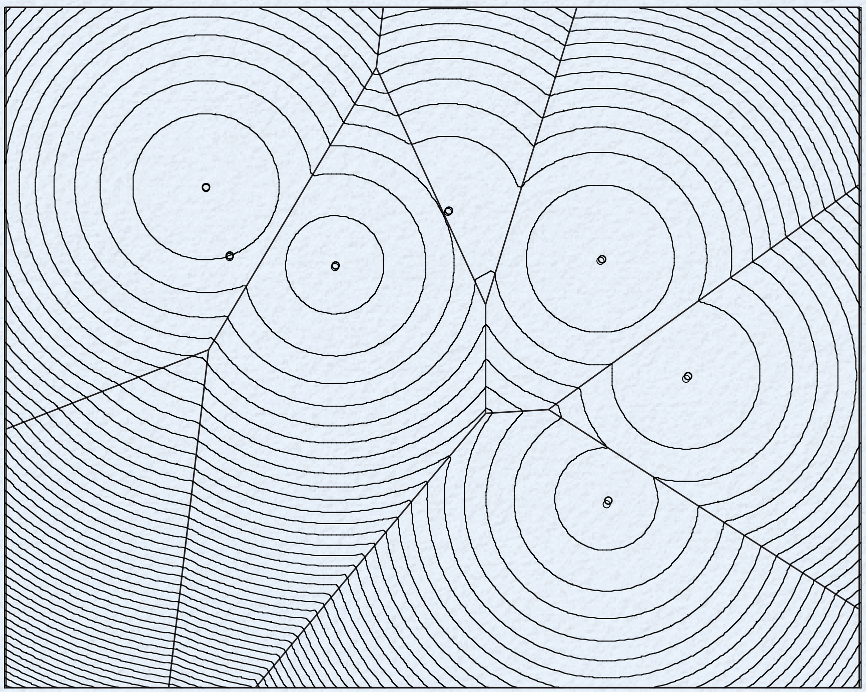
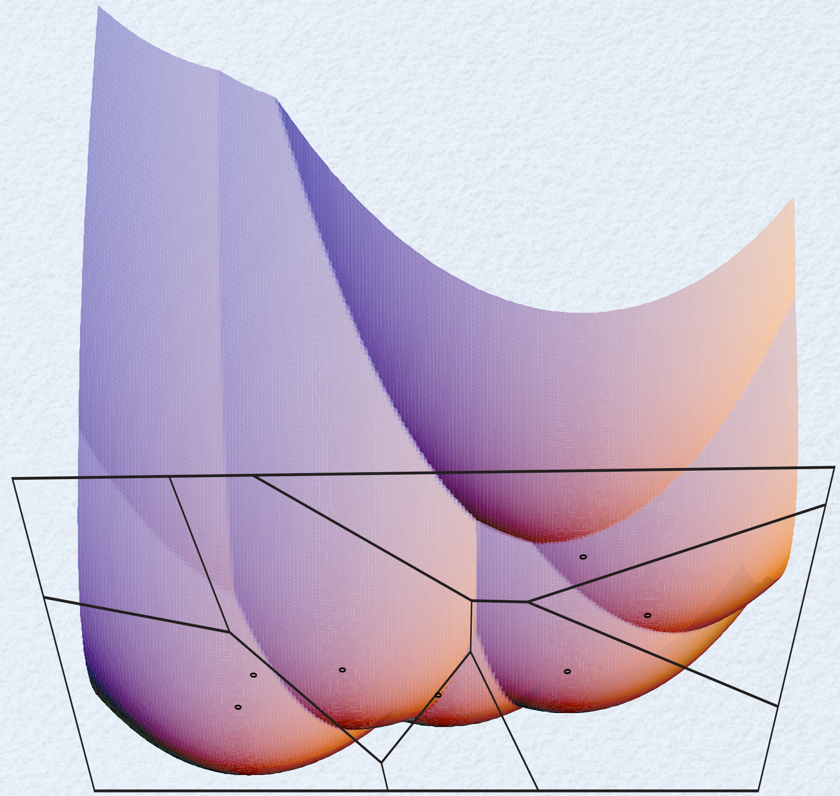


So, if $\mathsf{Um}(c) \subset Y$ we are fine!

Flow Induced by Weighted Points





Squared distance to p with weight w_p is $\|x - p\|^2 - w_p$.

The squared distance to a set P of weighted points is

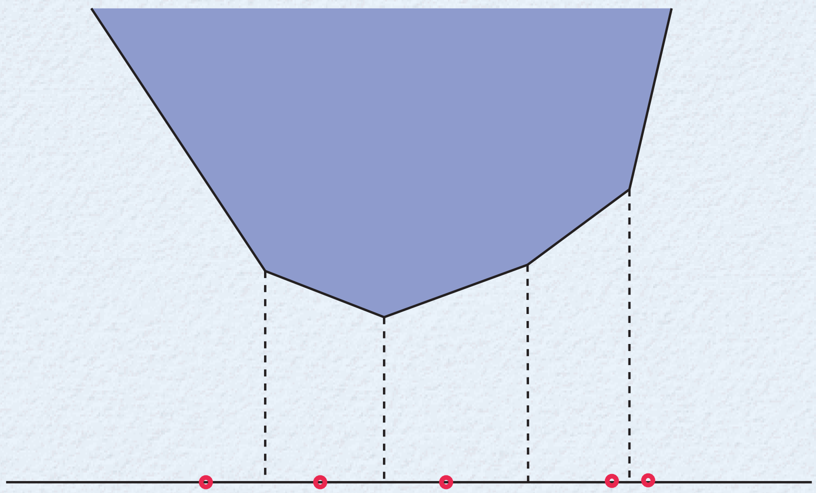
$$h(x) = \min_{p \in P} \|x - p\|^2 - w_p.$$

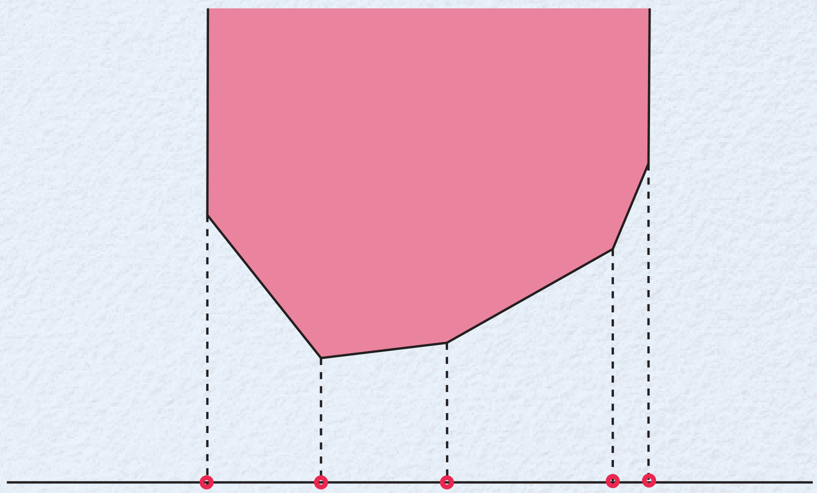
The
WRAP Algorithm

Polarity

For every set P of weighted points there is a set Q of weighted points such that

$$\text{Vor } P = \text{Del } Q \qquad \text{and} \qquad \text{Del } P = \text{Vor } Q$$





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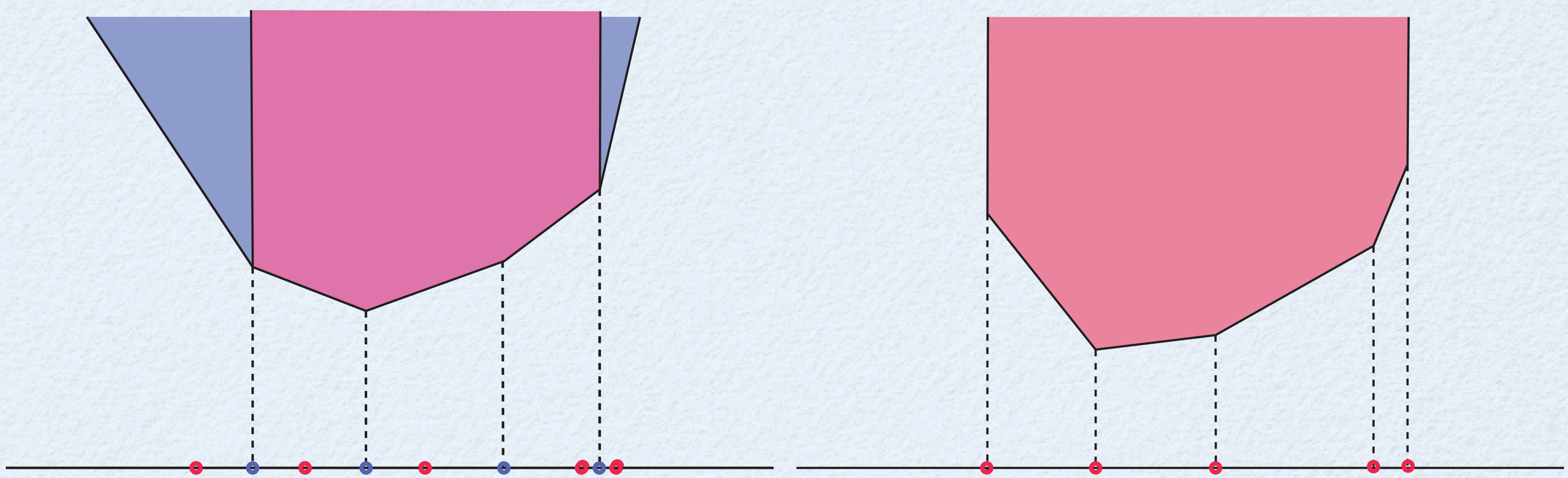
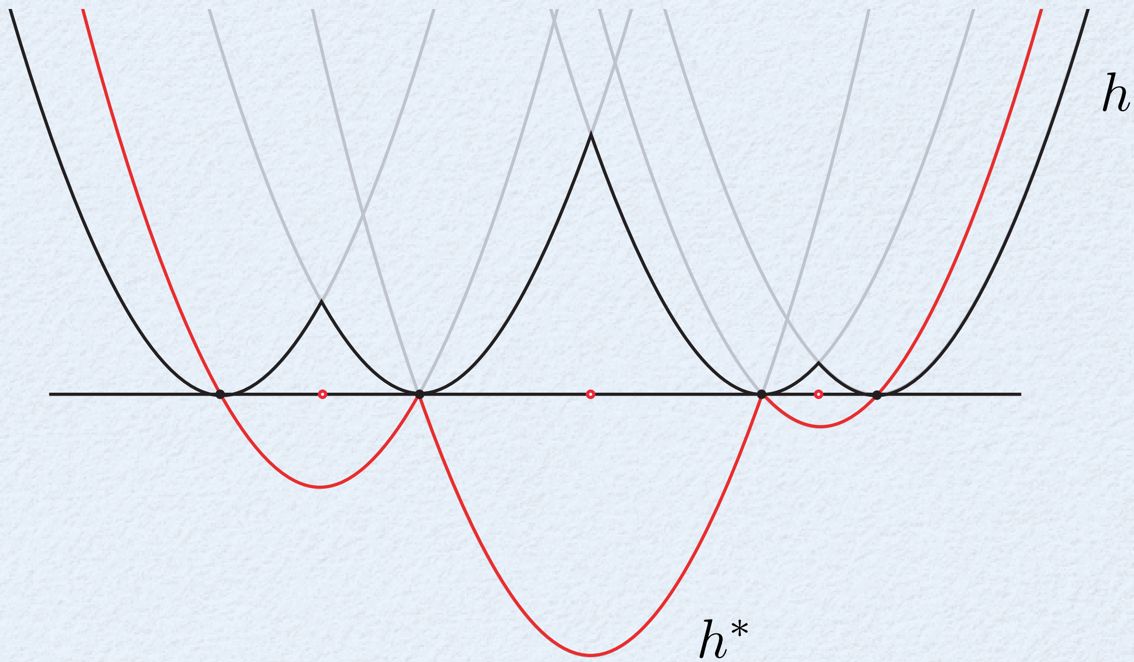


Voronoi Vertices as Weighted Points



For unweighted P, Q is the Voronoi vertices of P and for $q \in Q$:

$$w_q = \mathsf{dist}(q, P)^2.$$

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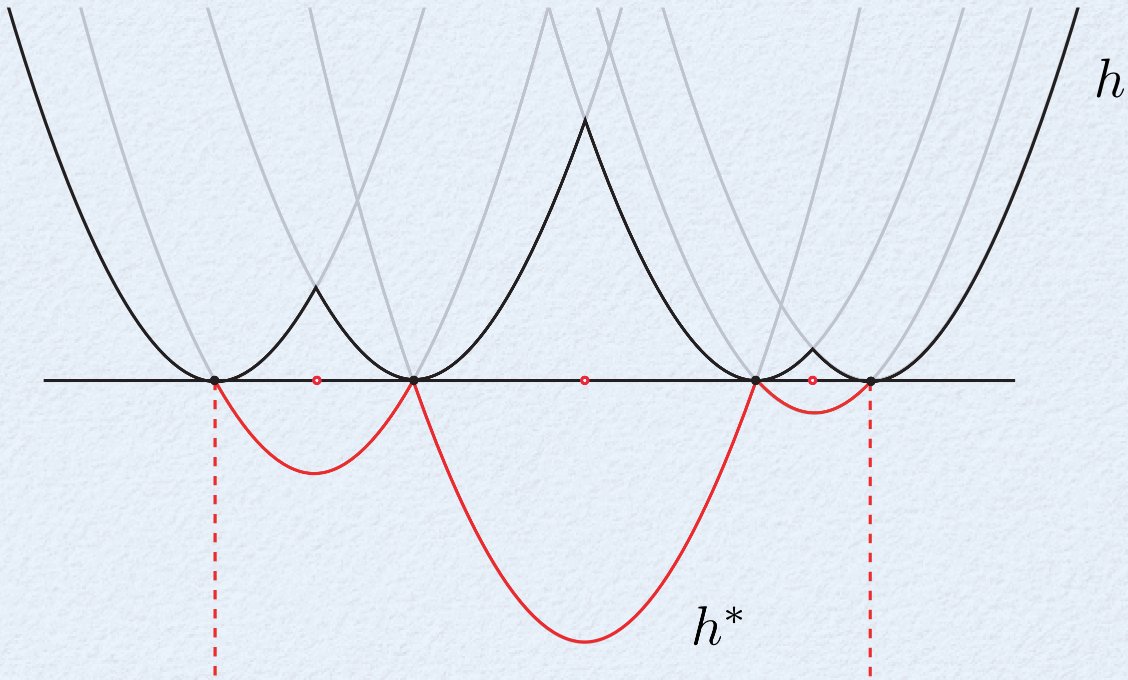

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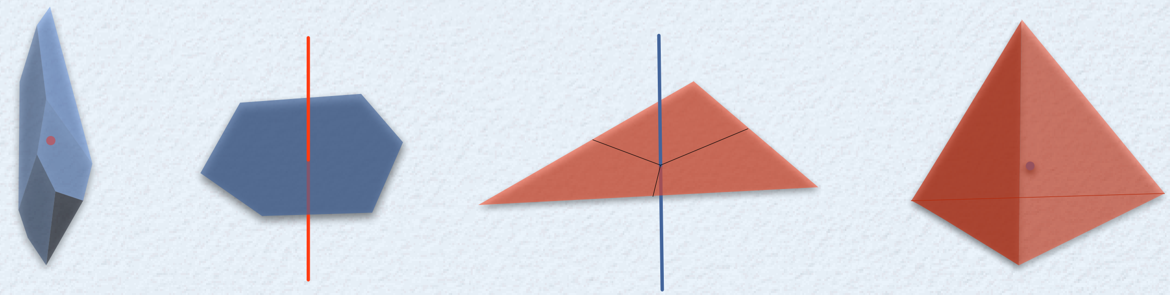
Critical Points of h^*

Observation. critical points of h^* and h are the same.



A simplex $\tau \in \mathrm{Del}\ P$ that contains a critical point is called a centered simplex.

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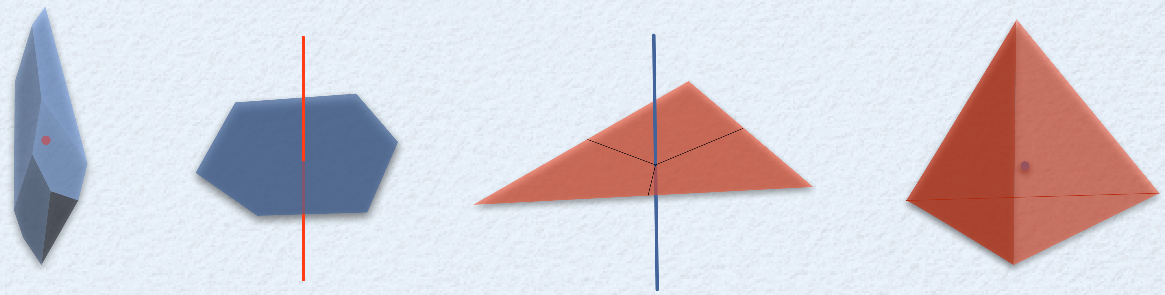

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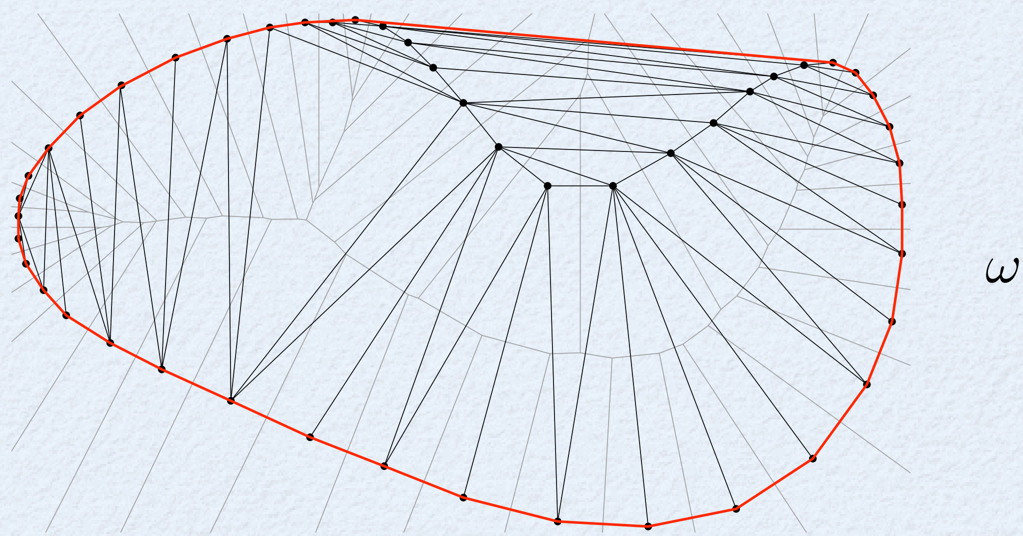
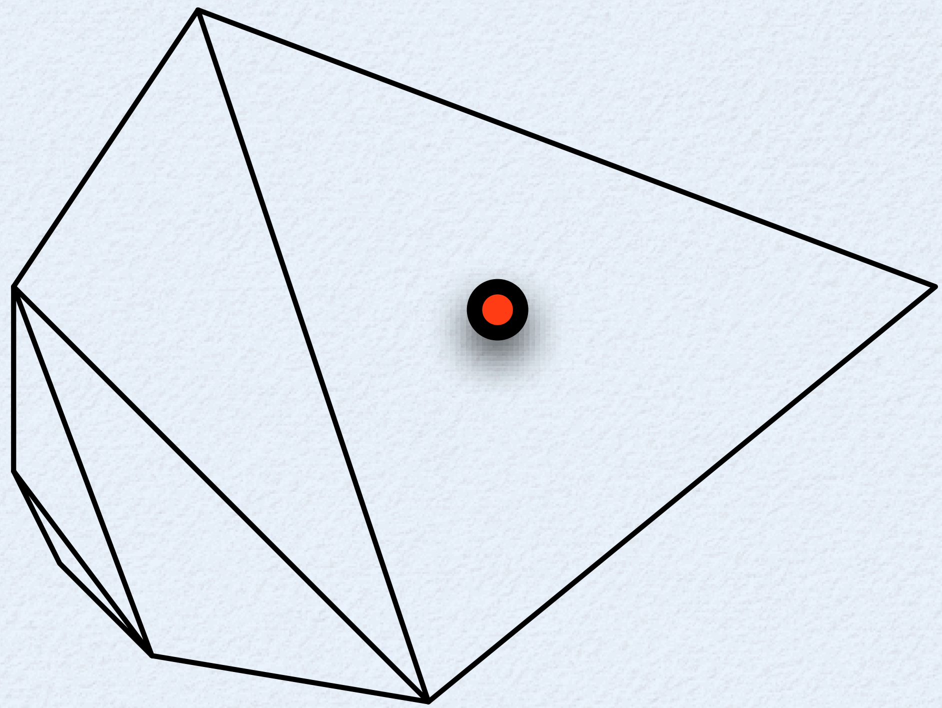
We treat $\mathbb{R}^n \setminus \mathrm{conv}\ P$ as an abstract critical simplex ω.



A Partial Order on Delaunay Simplices

$\tau \prec \sigma$: some flow line of ϕ^* visits relative interiors of σ and τ consecutively.



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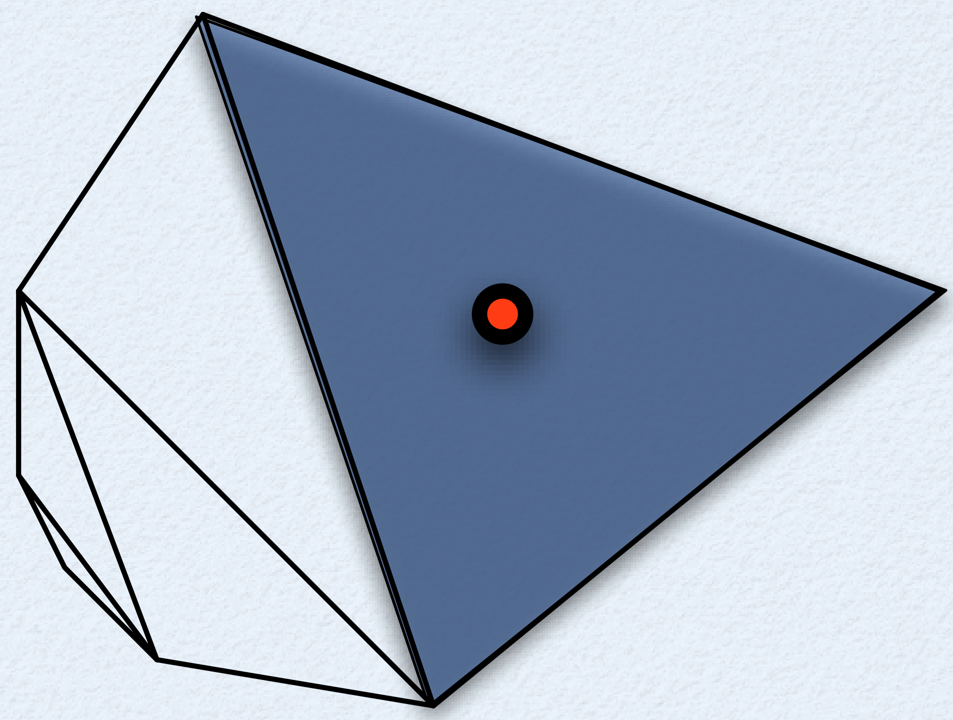

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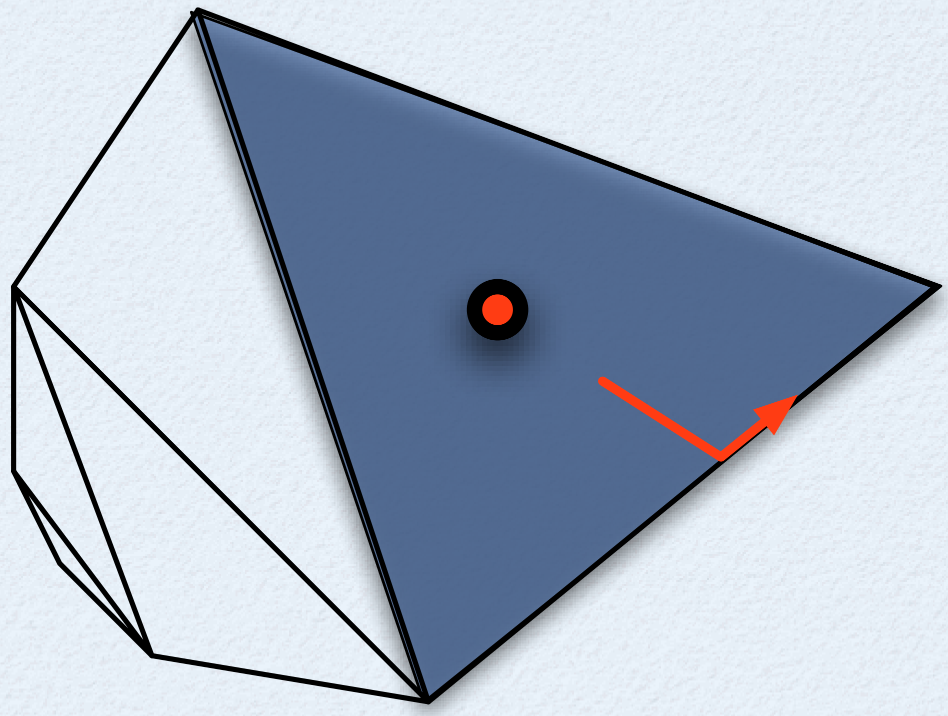

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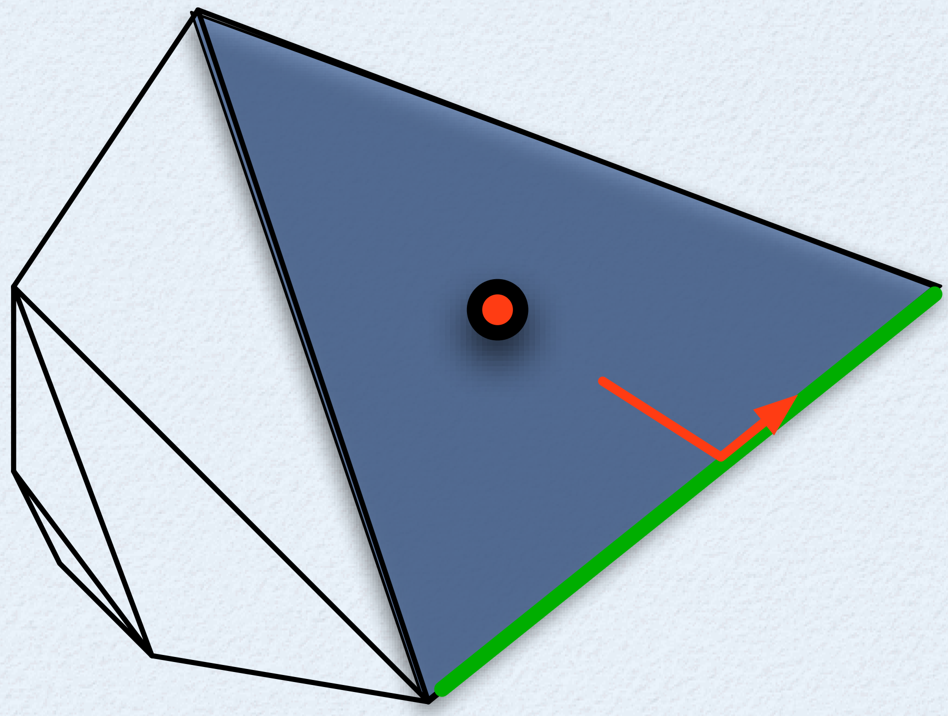

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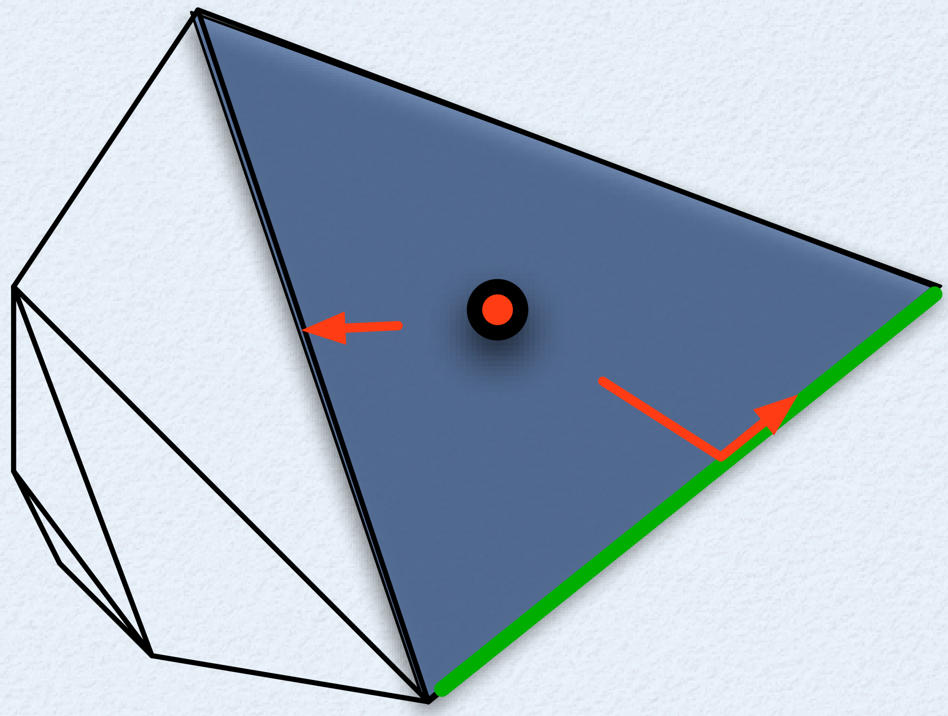

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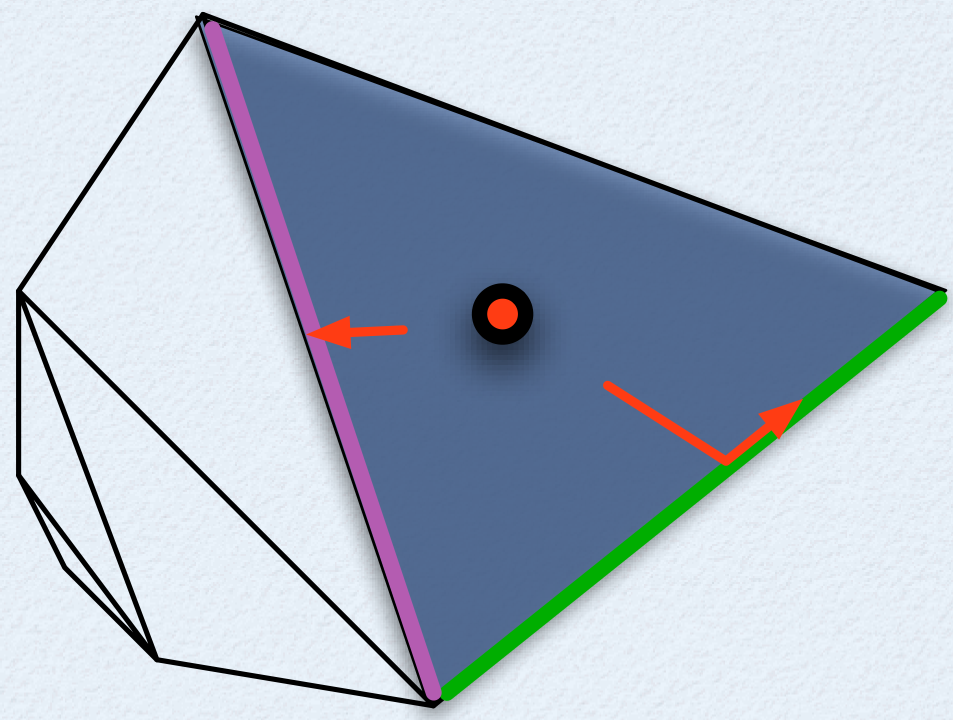

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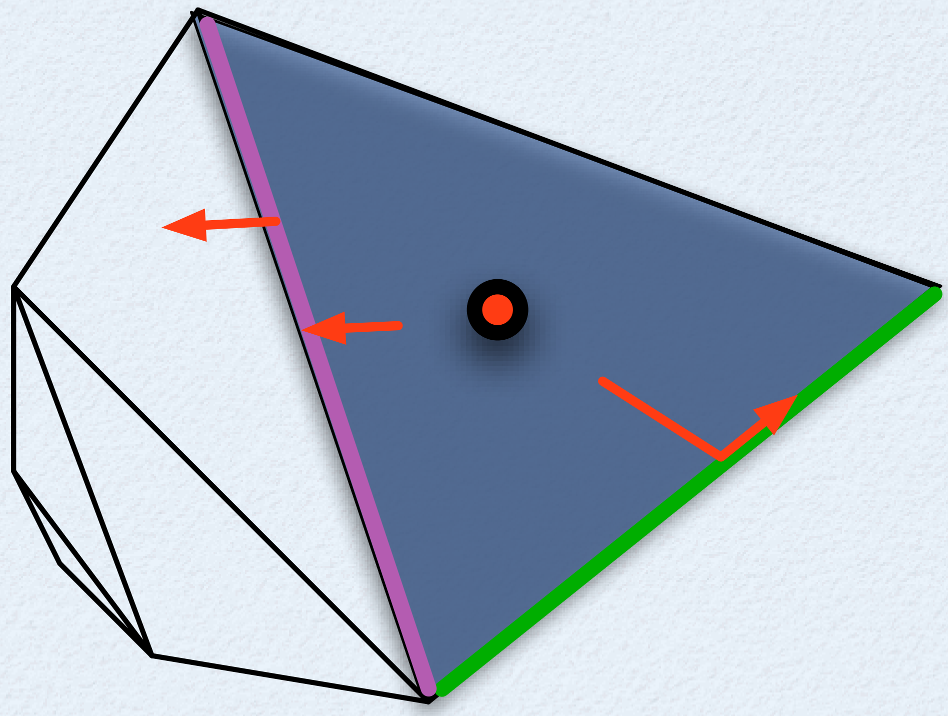

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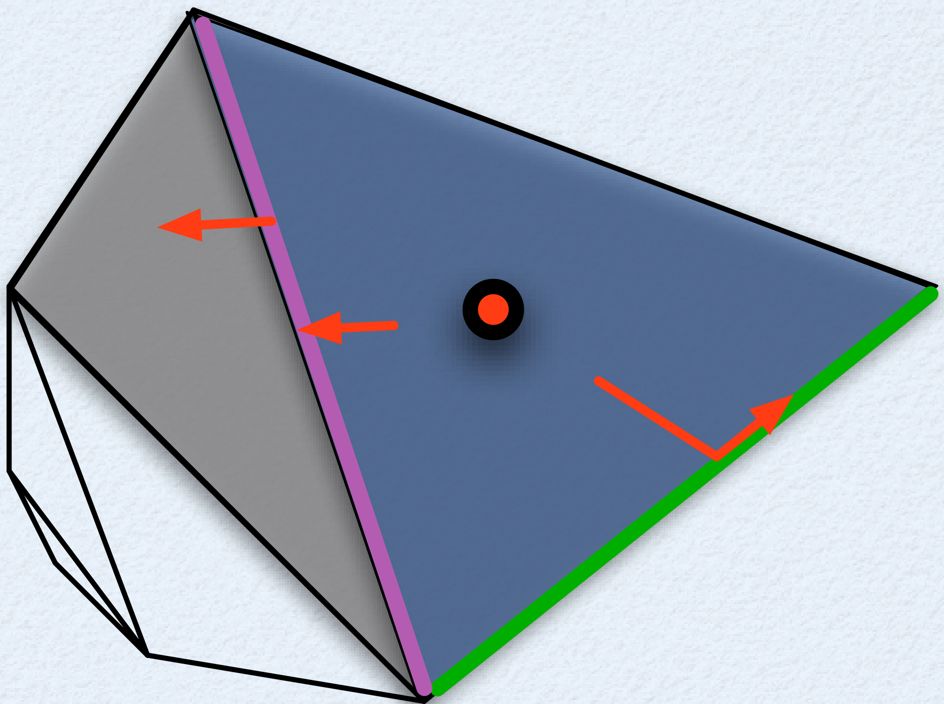

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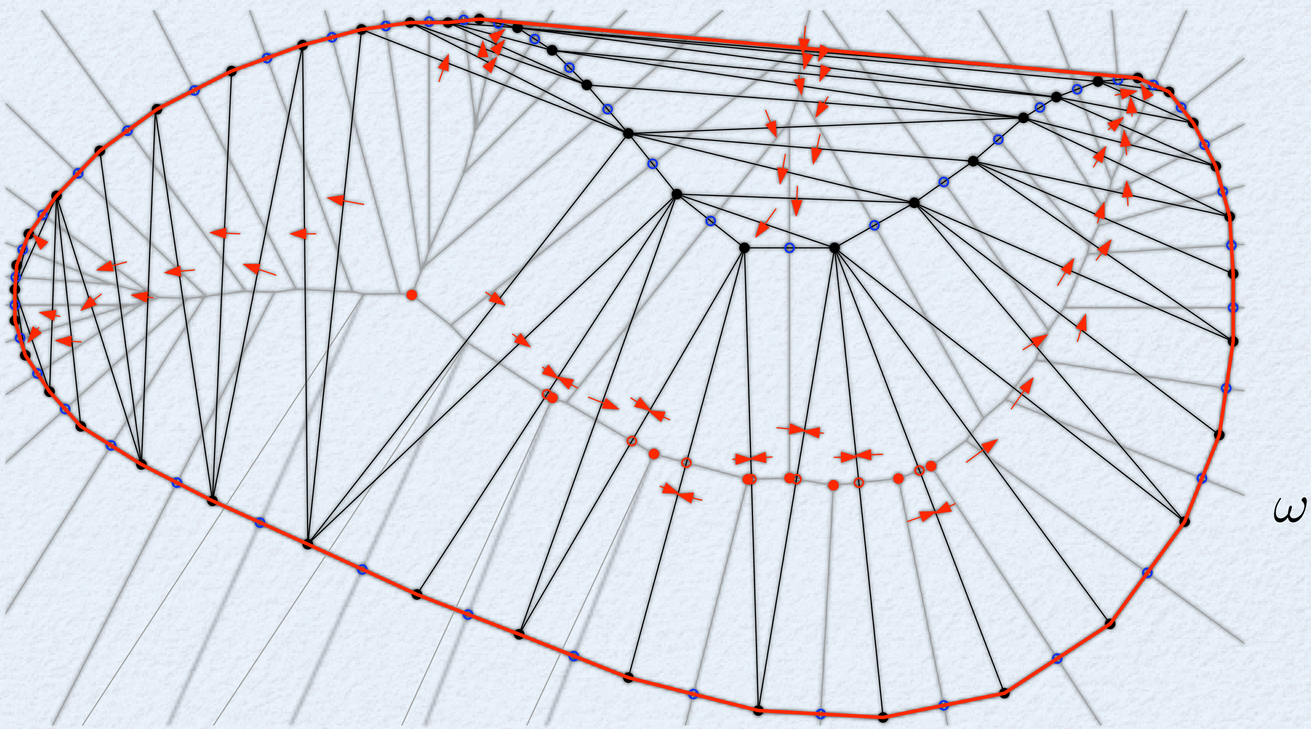
The WRAP Algorithm [Edelsbrunner'04]

1. For every $\tau \in \mathrm{Del}\ P$, if the only critical simplex that precedes τ is the abstract critical simplex ω, then remove τ.

2. Return what is left as WRAP.



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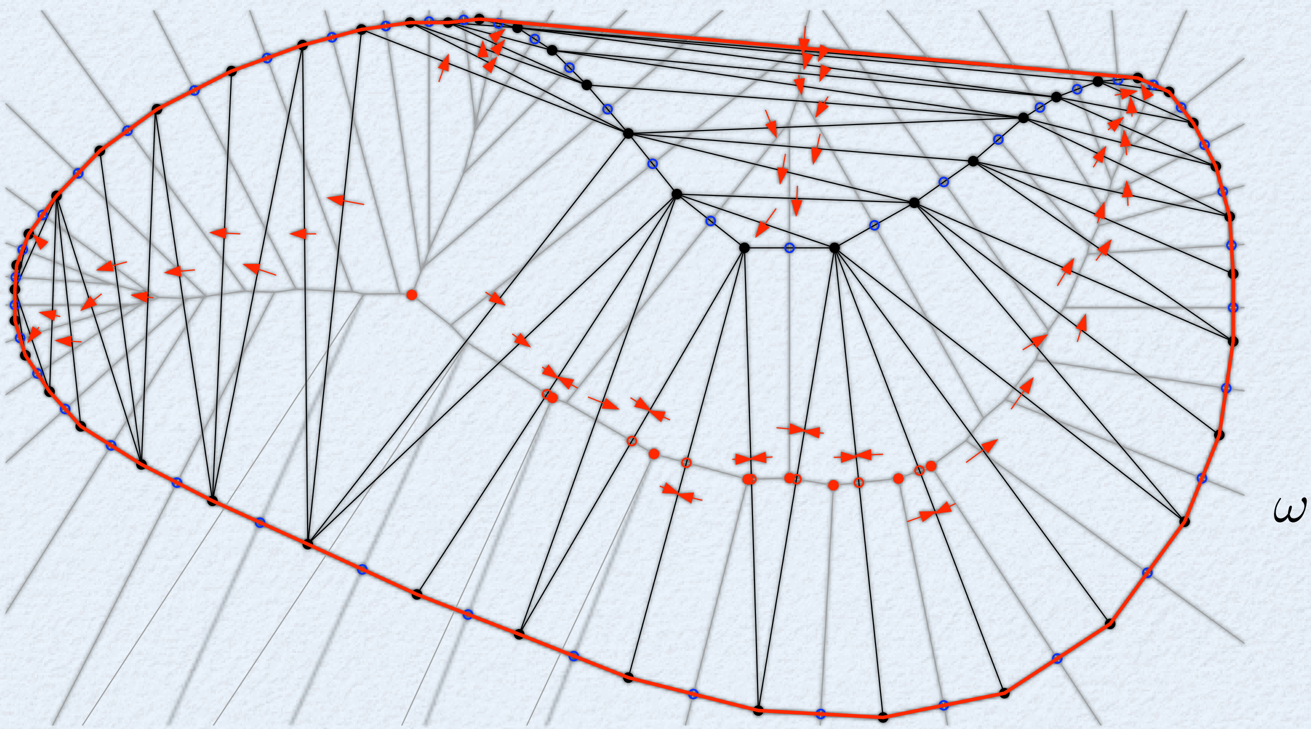

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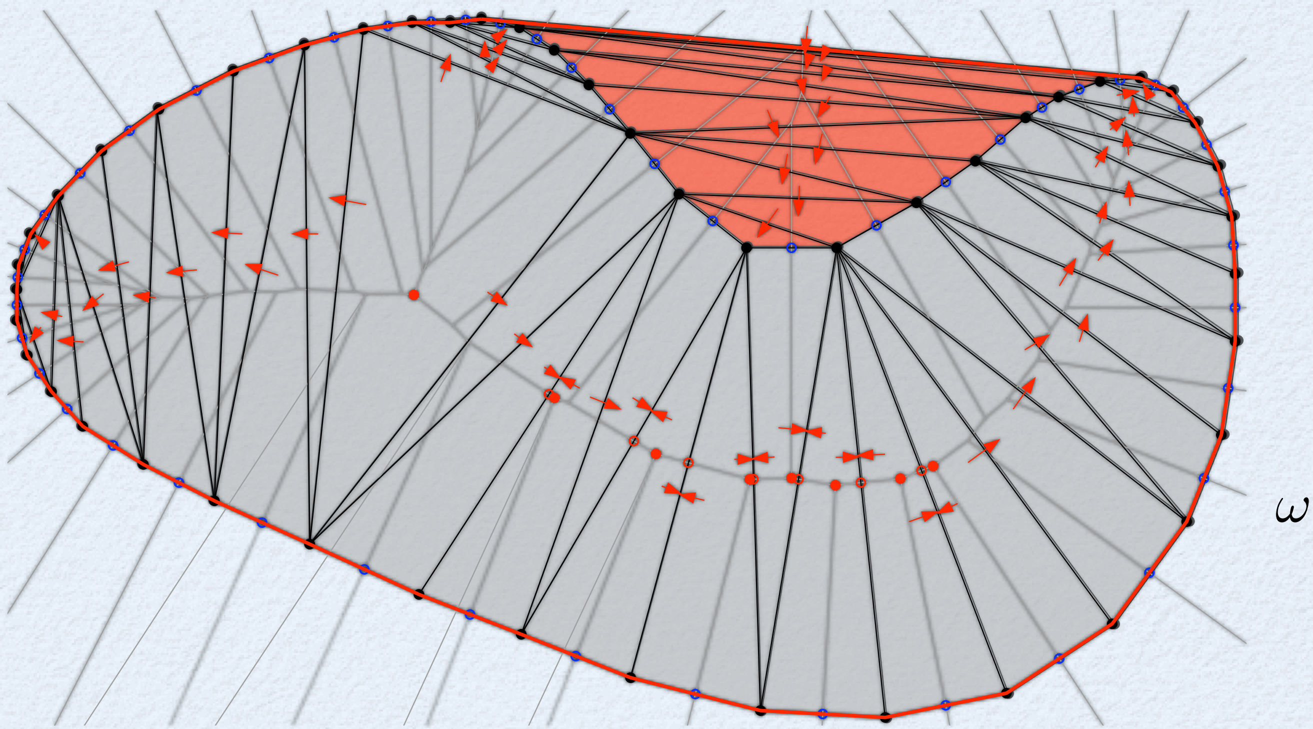

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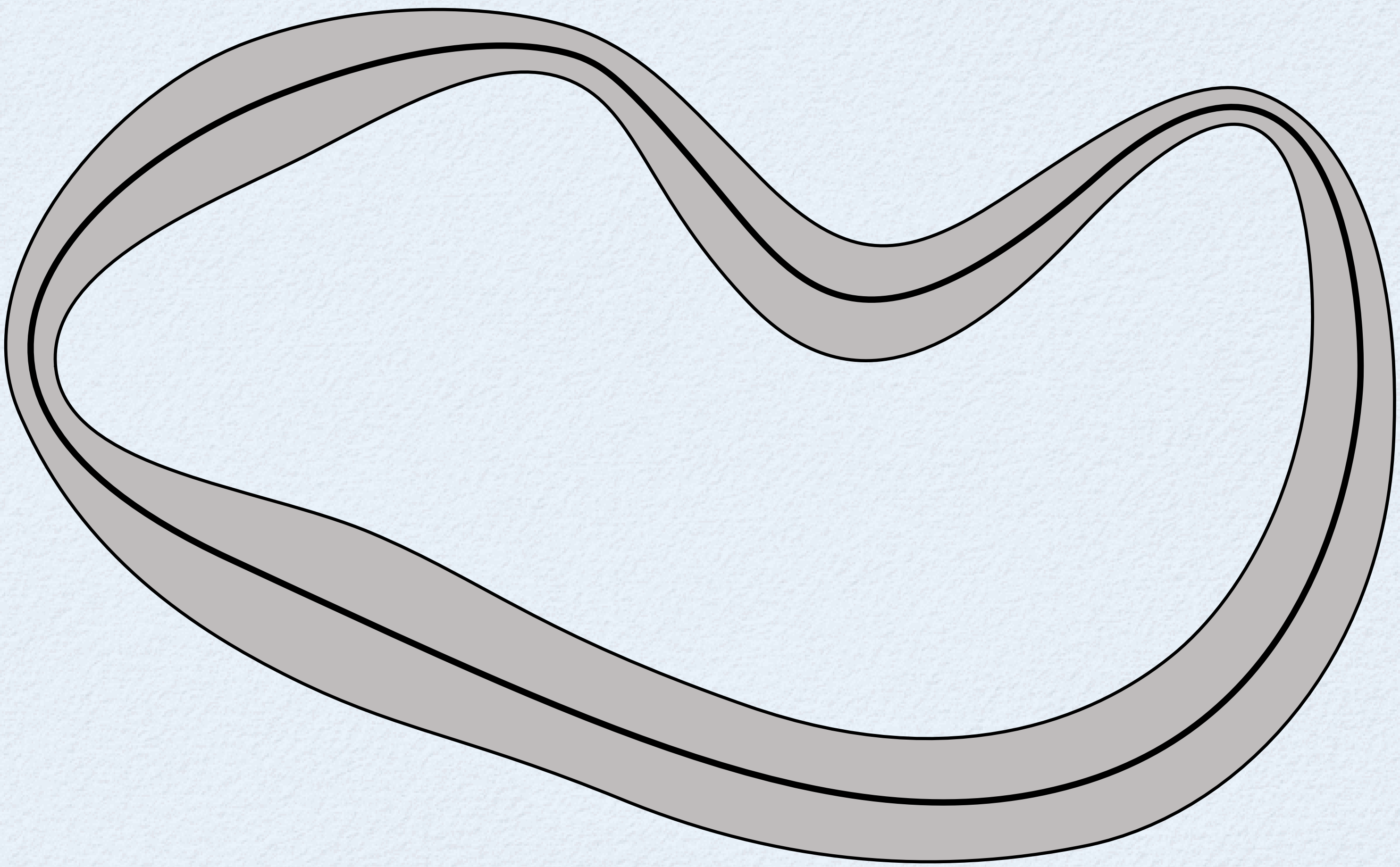
The Guarantees

Analysis of WRAP

Theorem. WRAP and cl S are homotopy equivalent.

Lemma. $\phi^*(\Sigma_\delta) = \Sigma_\delta$.



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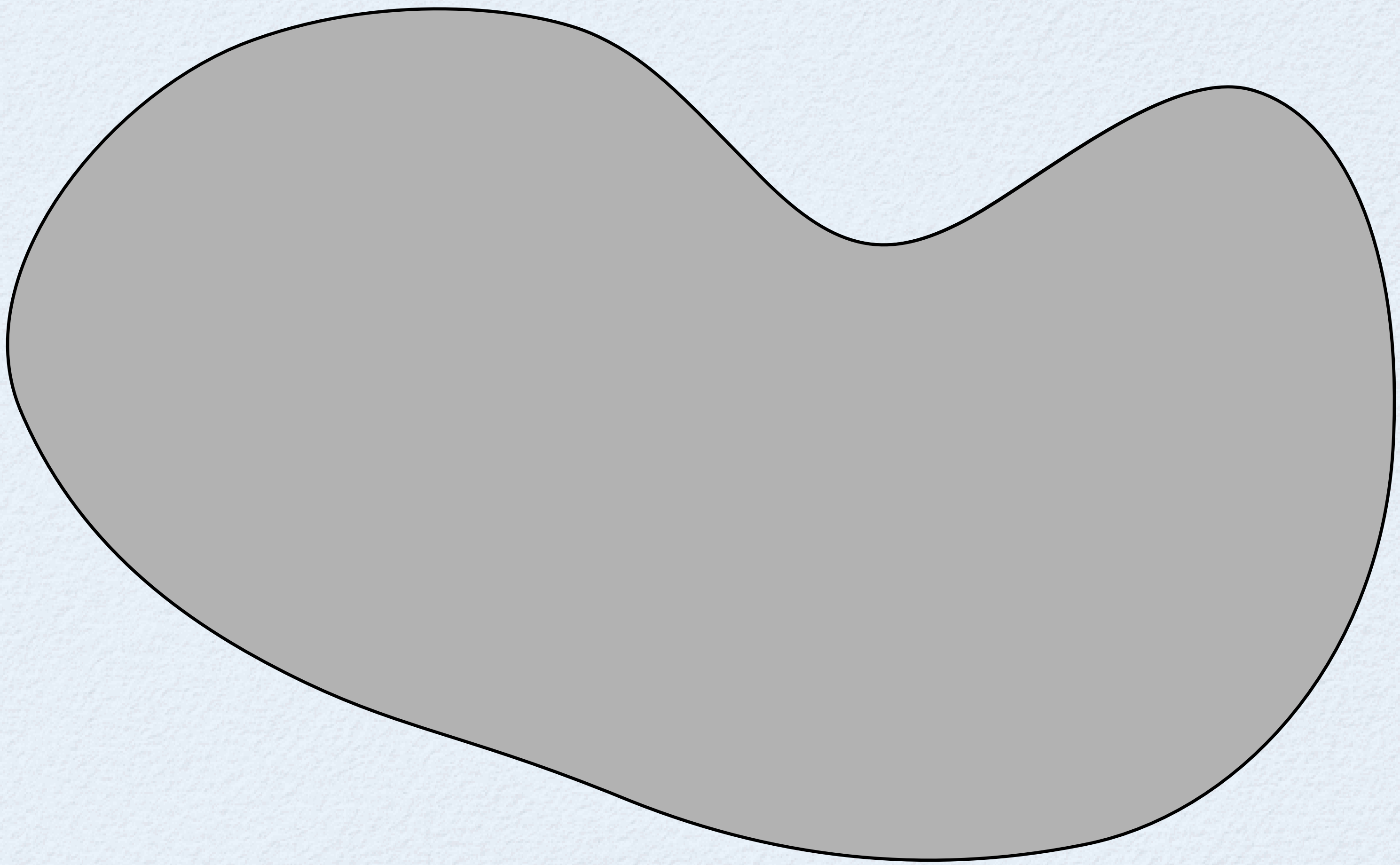

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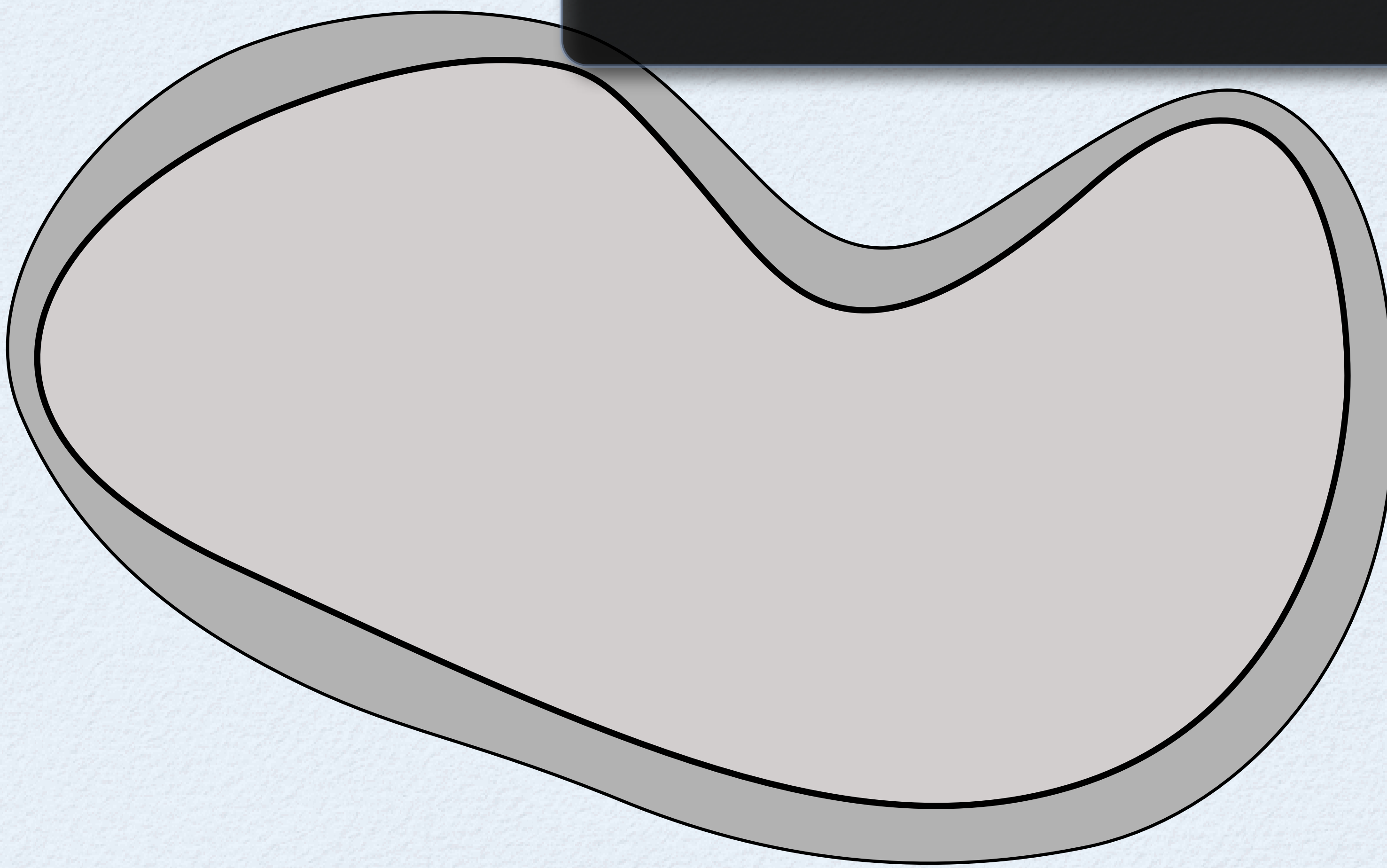

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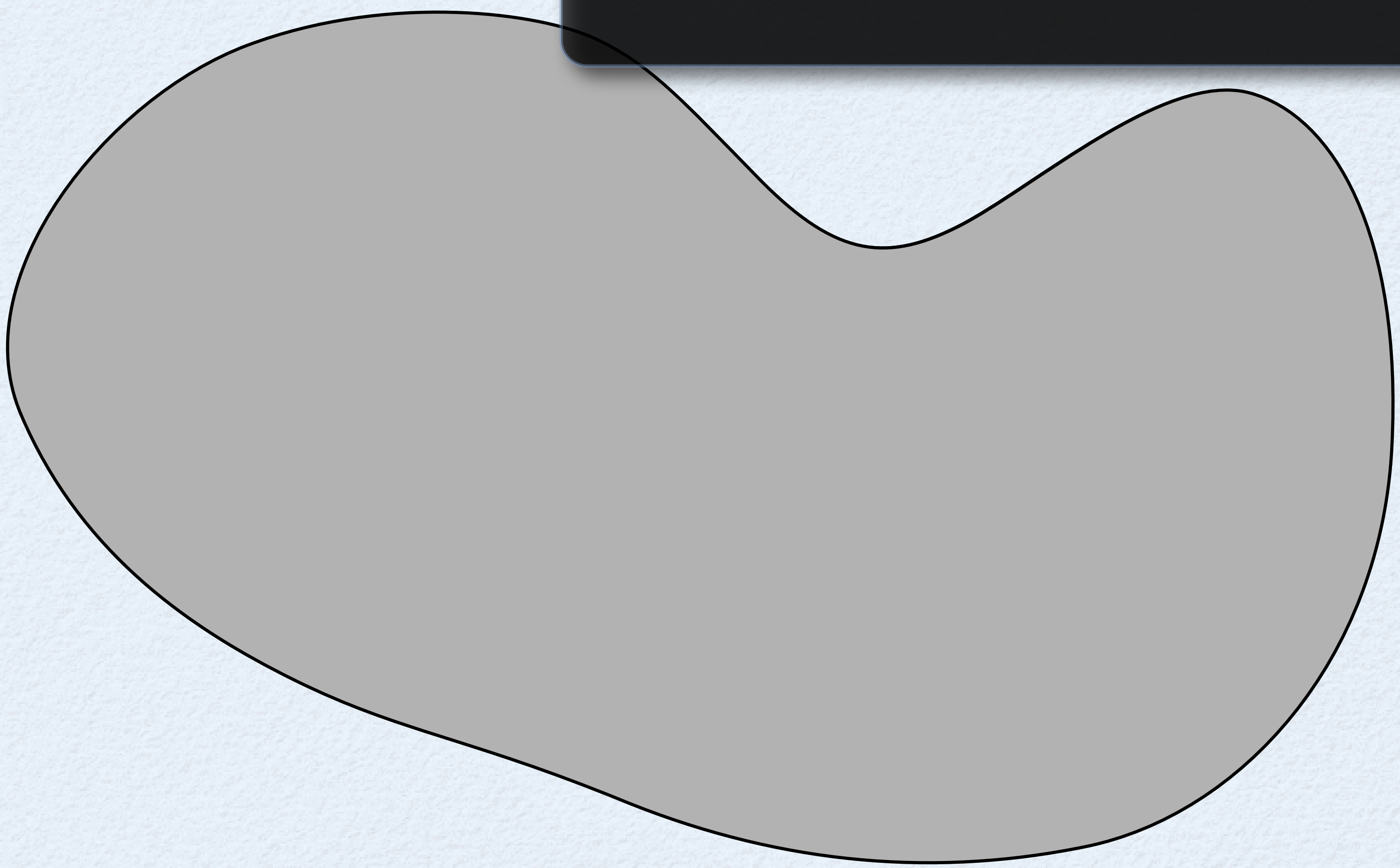

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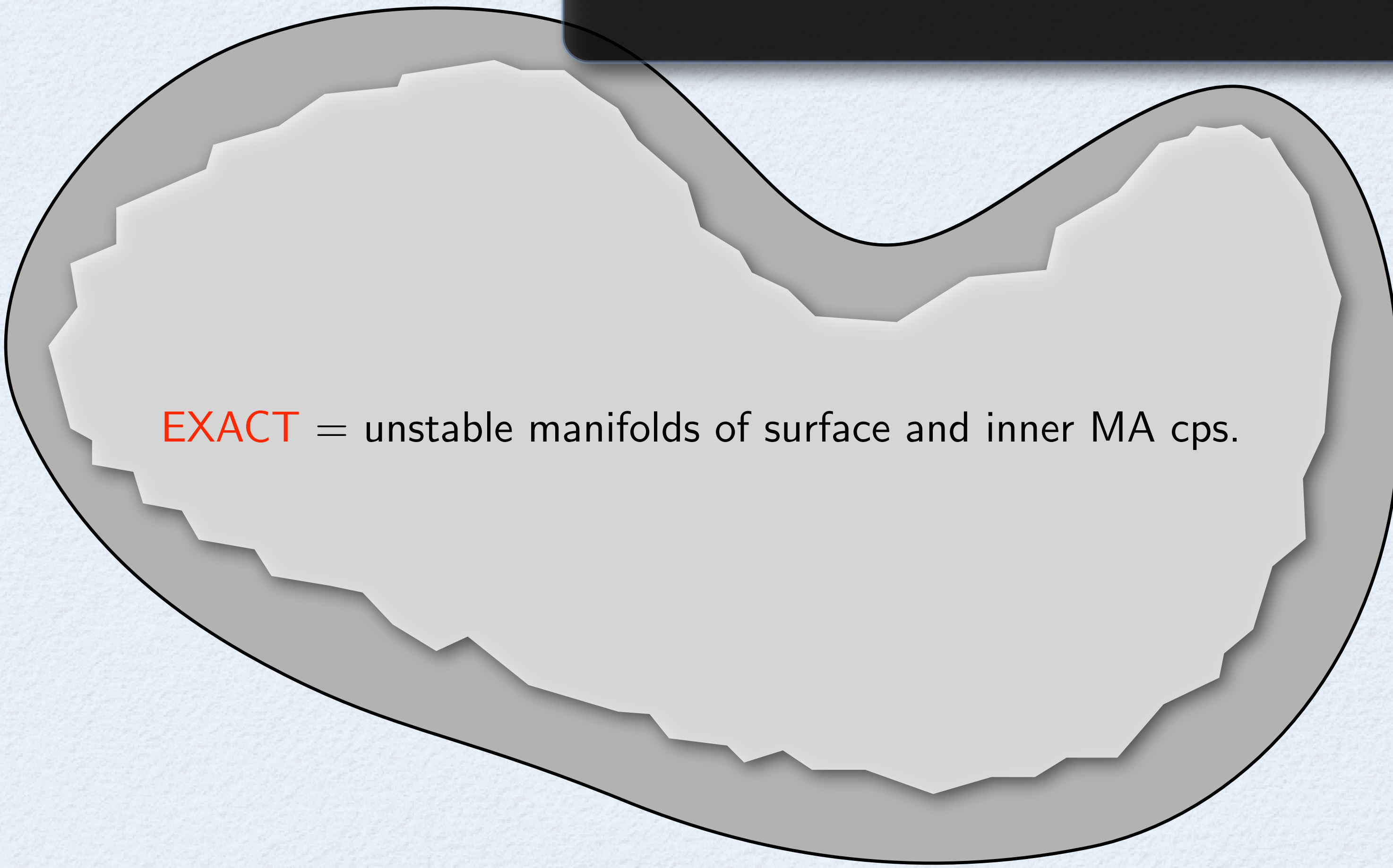

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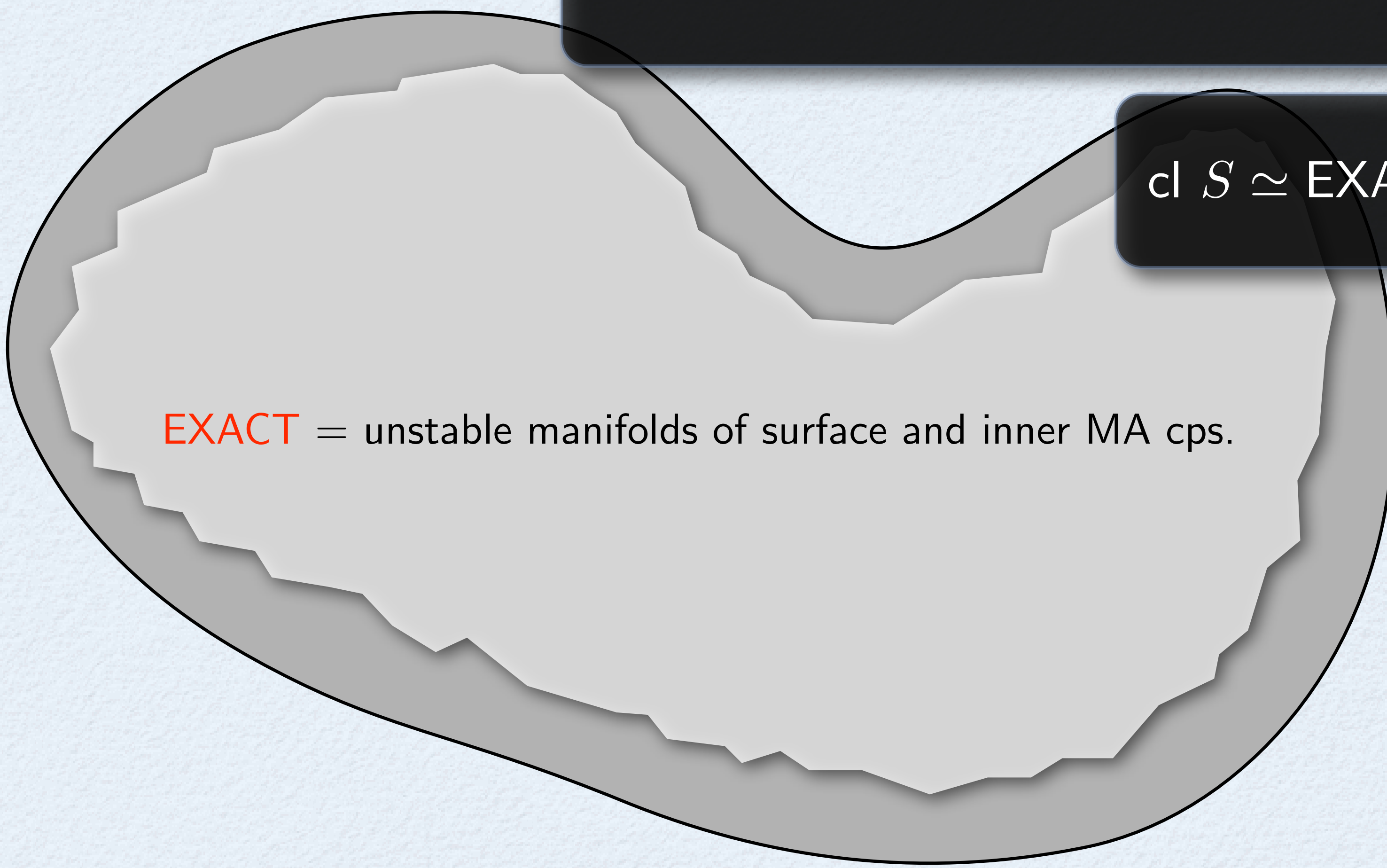

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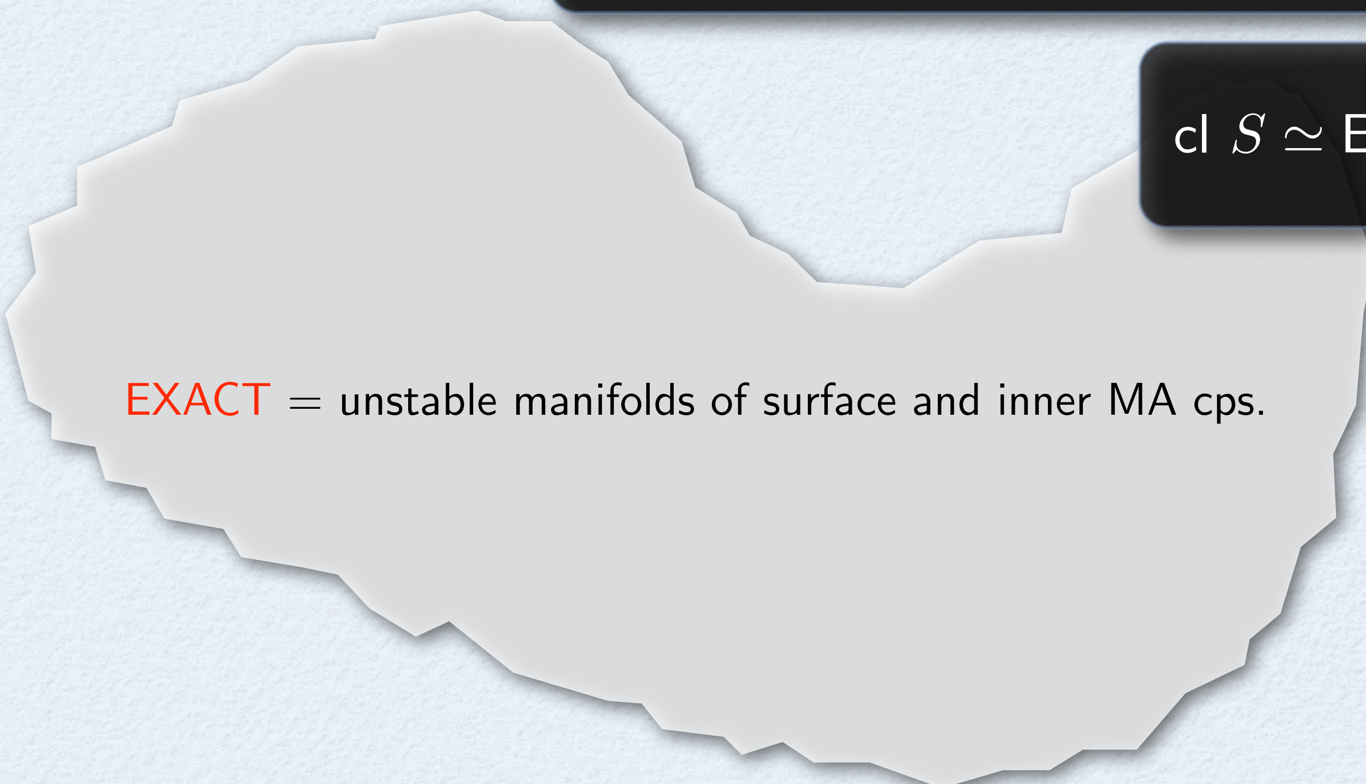

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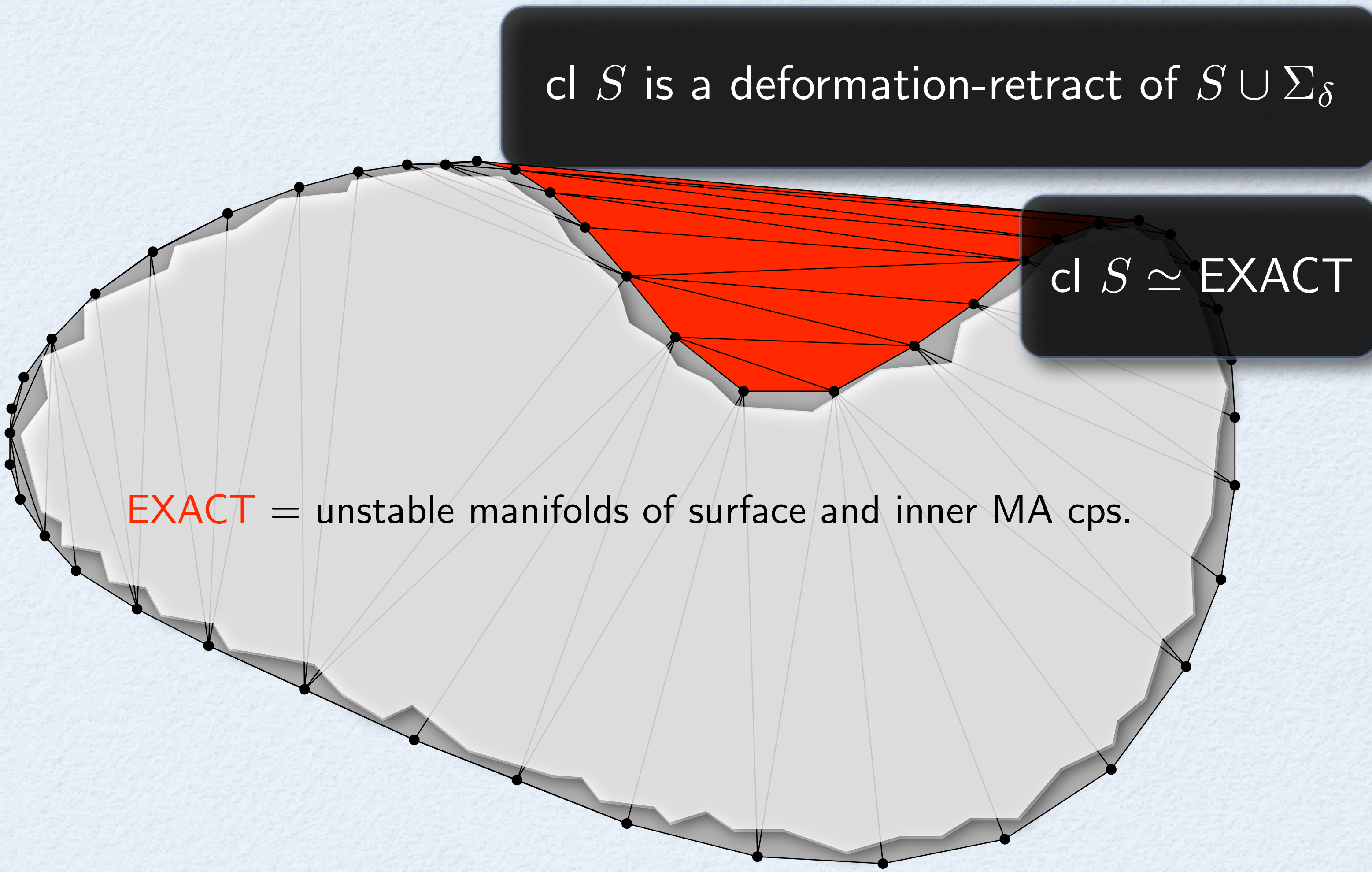

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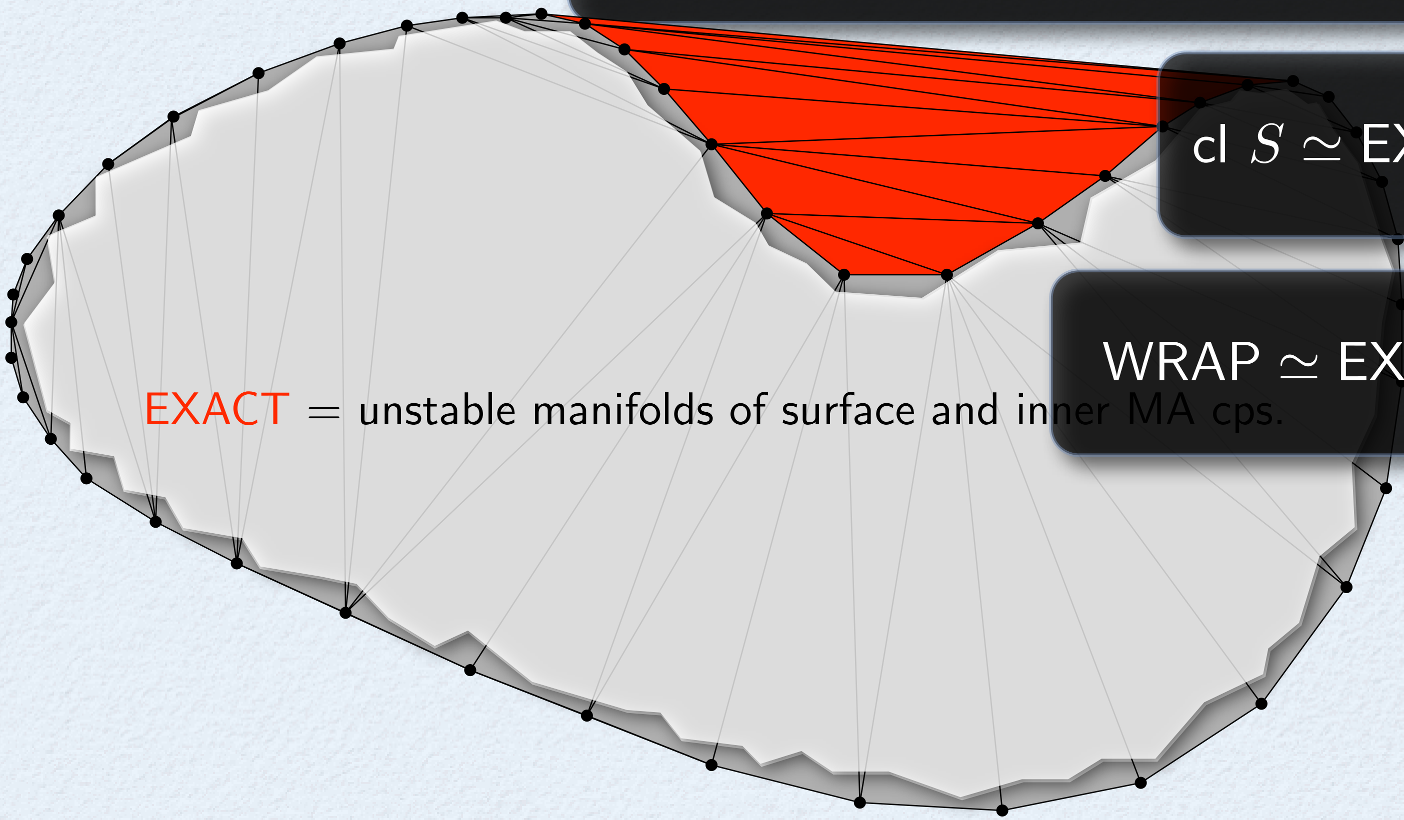

EXACT $=$ unstable manifolds of surface and inner MA cps.

Analysis of WRAP

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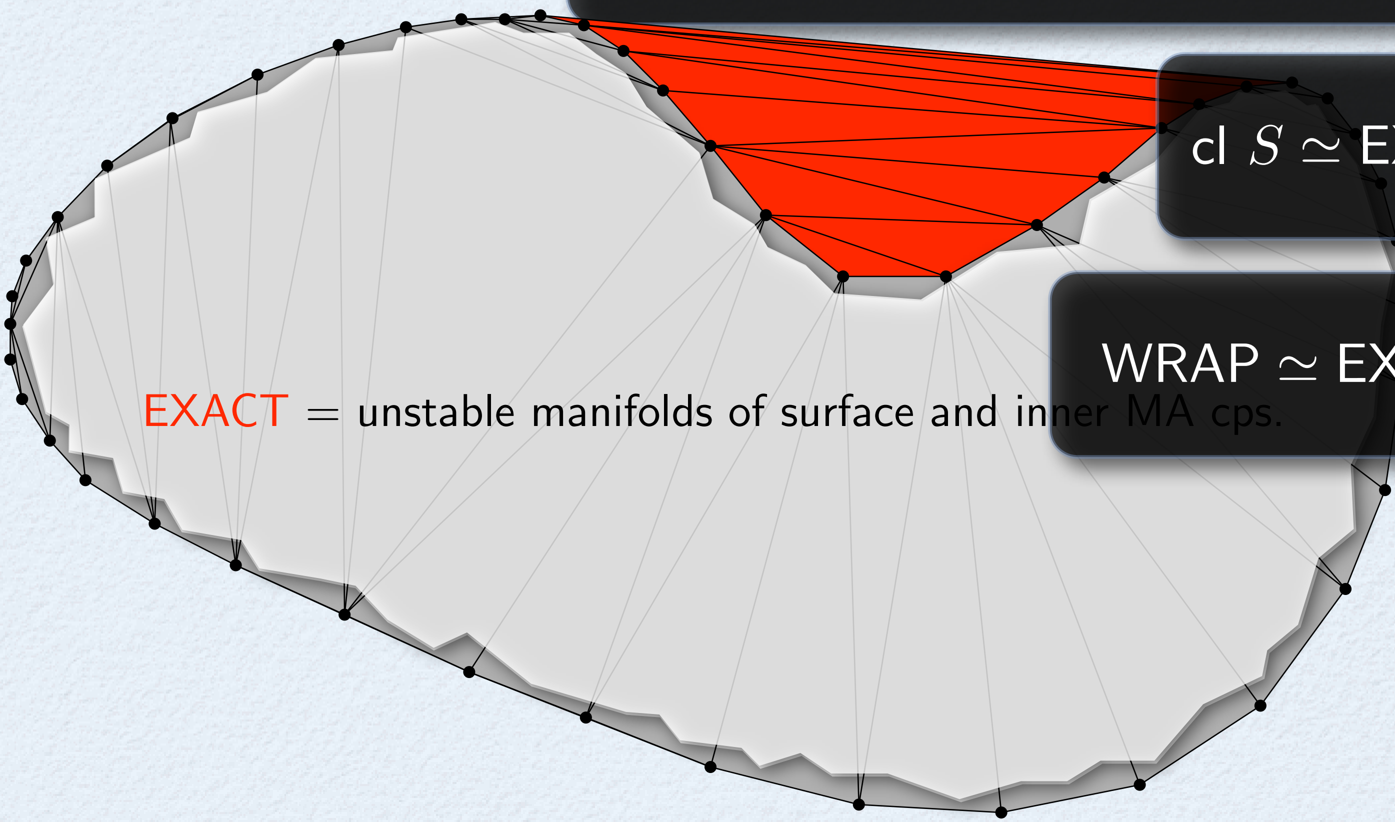

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Some Open Questions

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Thank You!