Surface and Medial Axis Topology

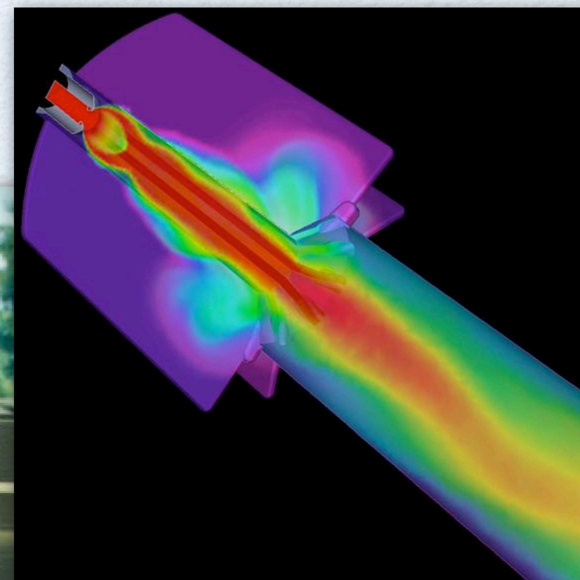
through

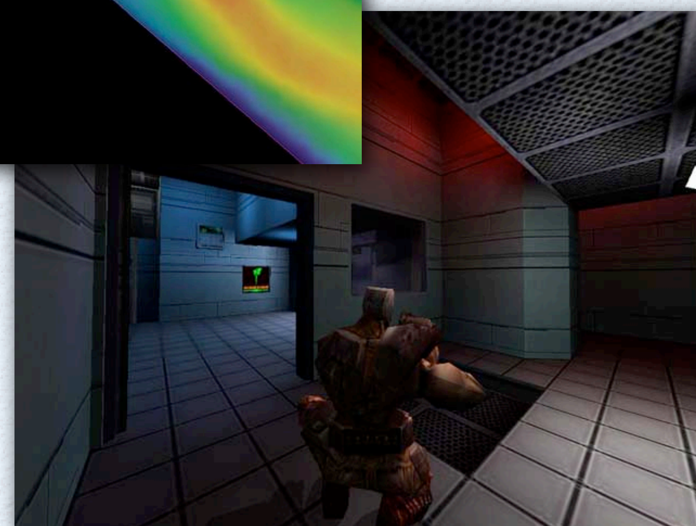
Distance Flows Induced by Discrete Samples

– PhD Final Examination –

Bardia Sadri
University of Illinois at Urbana-Champaign

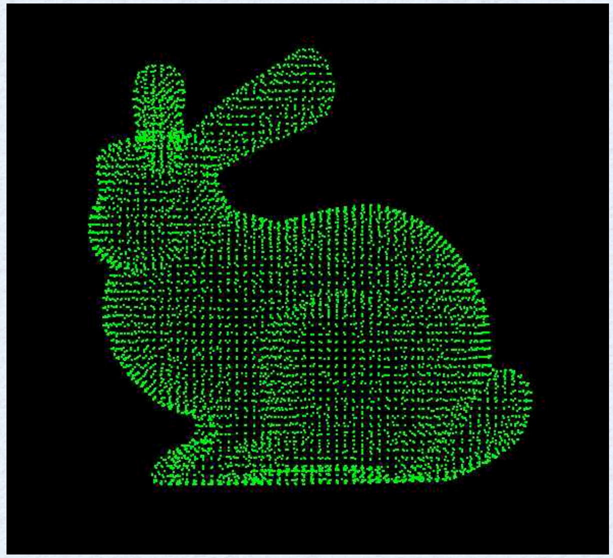
Geometric Modeling of Objects





Engineering: simulation, visualization, CAD, ...
Entertainment: animation, games, virtual reality, ...
Sciences: medicine, biology, ...

The Surface Reconstruction Problem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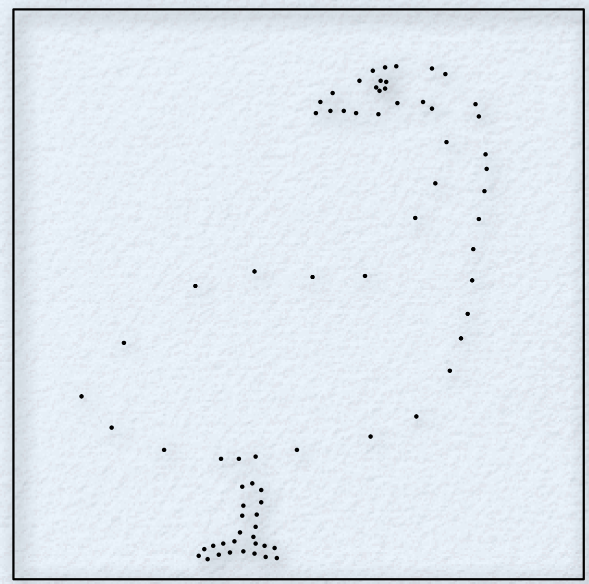

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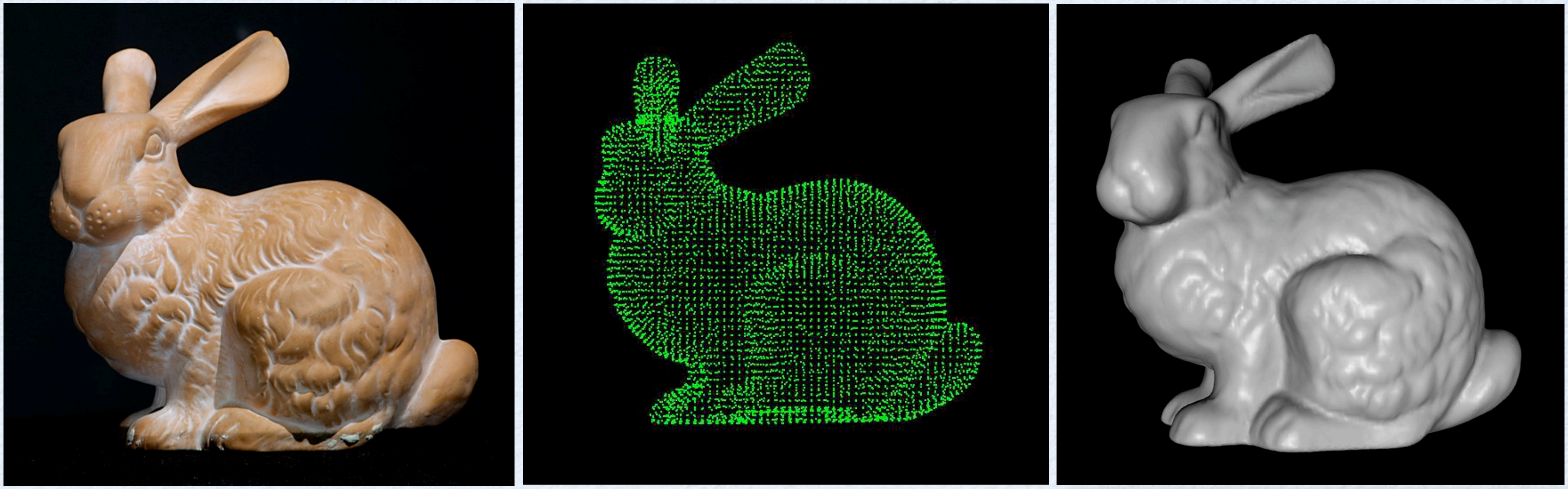

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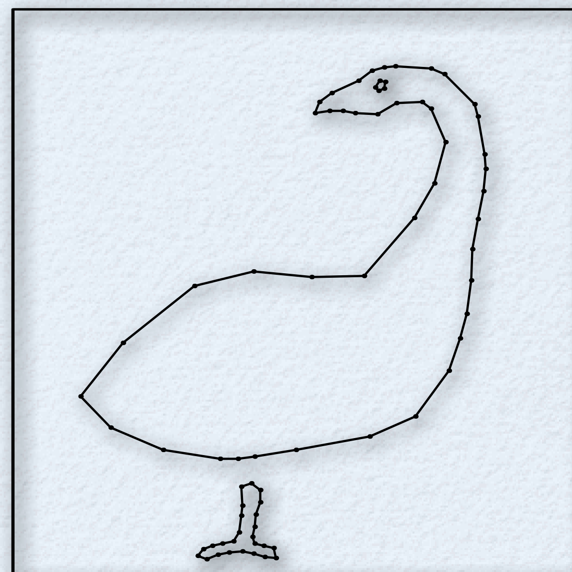

There are many surface reconstruction methods!

◇ As a 0-set of an approximate signed distance function:

- [Hoppe et al. '92, Curless et al.'96, . . .]

◇ As other iso-surfaces:

- NN Interpolation [Boissonnat-Cazals '02]
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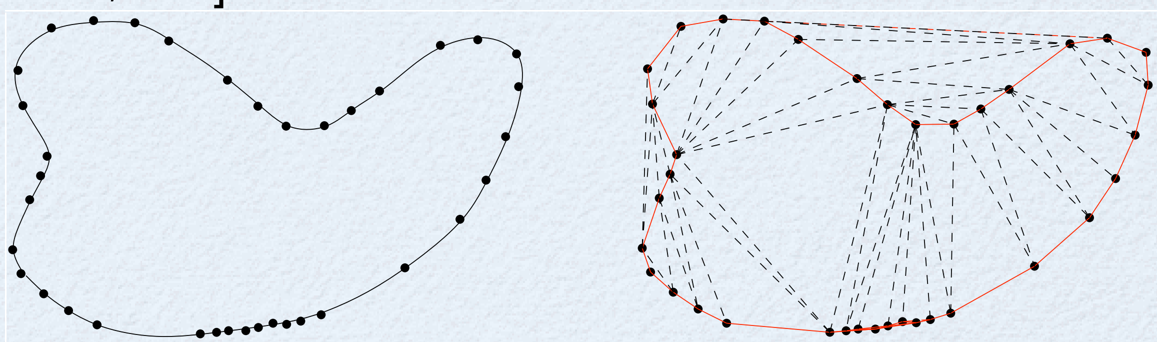

◇ Delaunay Methods:

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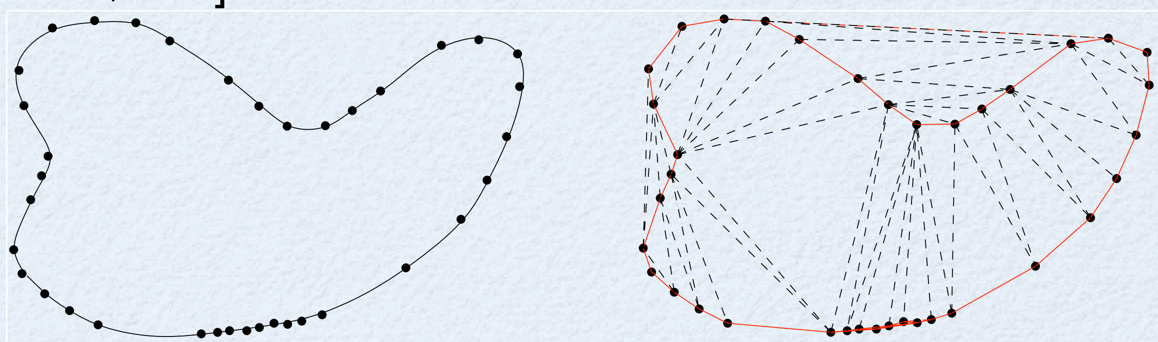

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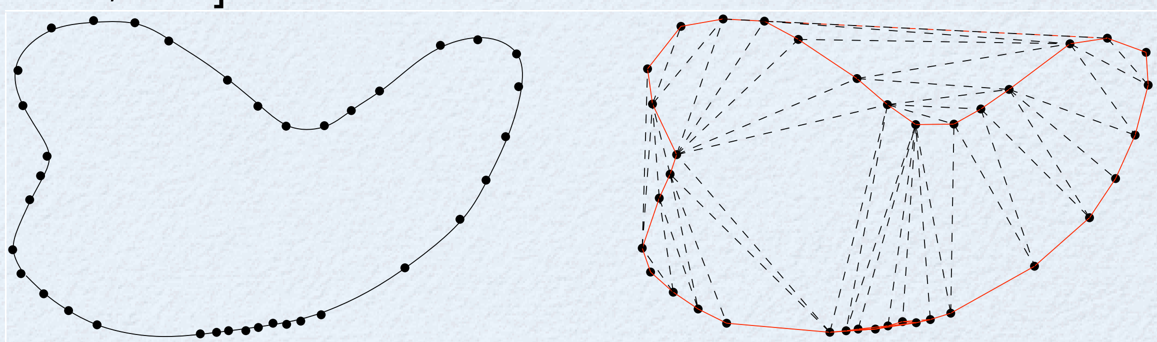

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Shapes, Surfaces, and their Medial Axes

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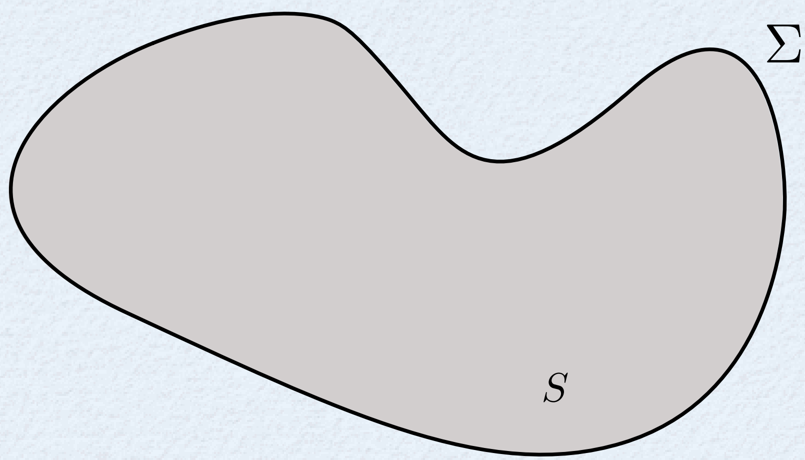

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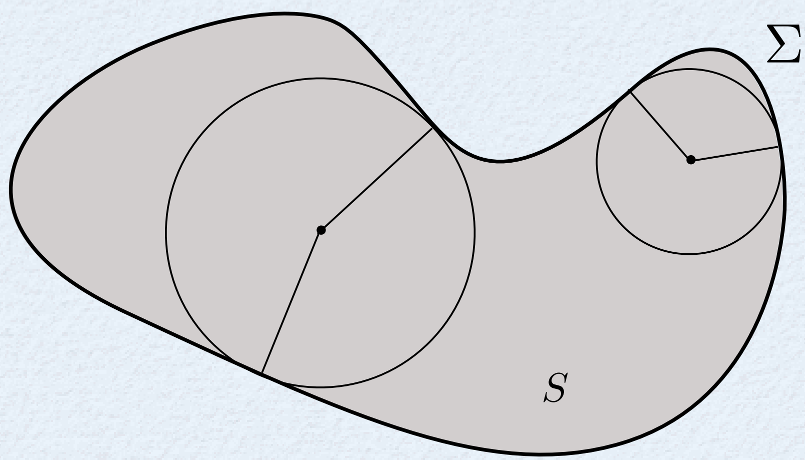
The medial axis (MA) of a shape S is the set of points in S that have ≥ 2 closest points in Σ.



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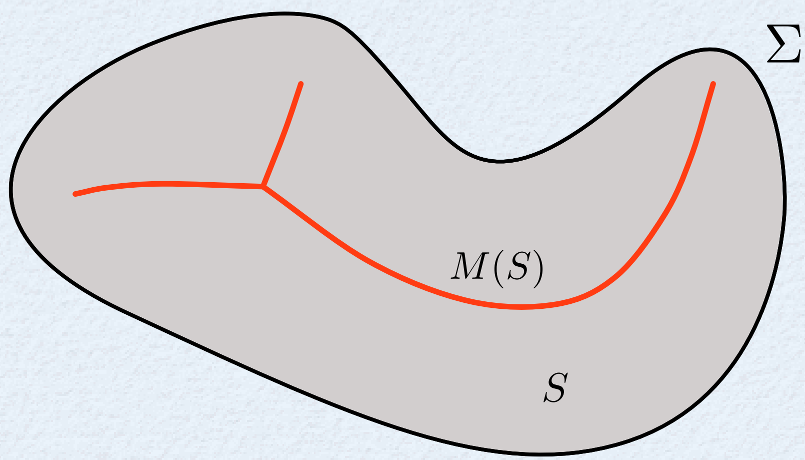

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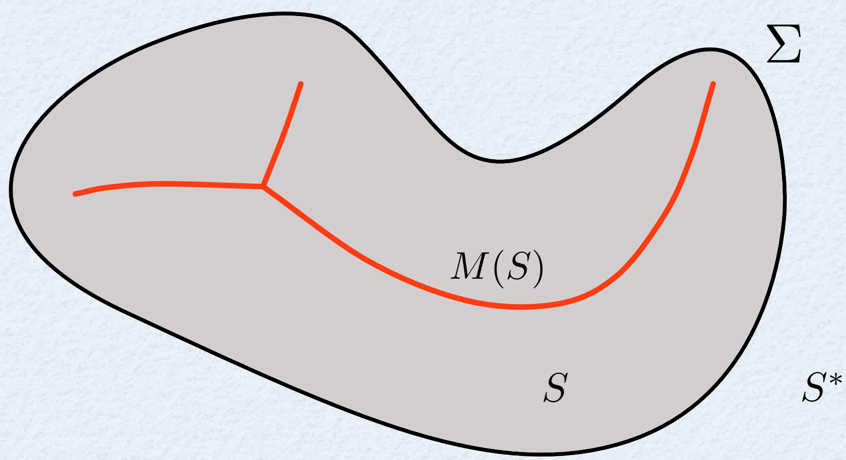

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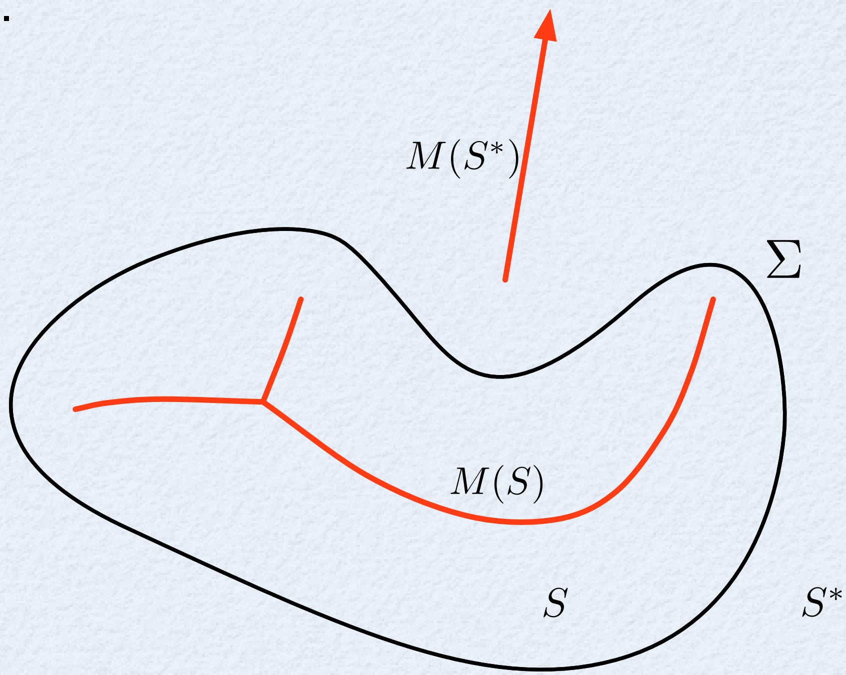
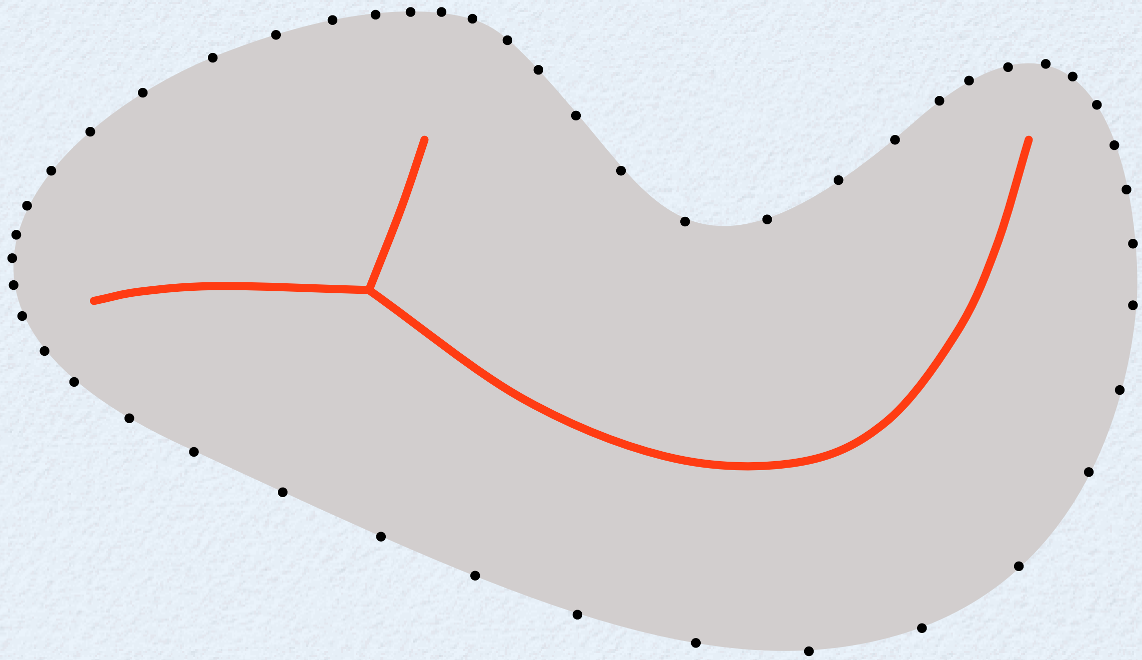


The medial axis of a surface Σ is the union of medial axes of all components of $\mathbb{R}^n \setminus \Sigma$.

Problem of Medial Axis Approximation

Given a sample of the smooth surface enclosing a shape, we want to approxiamate the MA of shape geometrically and capture its topology.



Applications: shape analysis, motion planning, mesh partitioning, medical imaging, ...

Theorem. [Lieutier'04] Any bounded open subset of $\mathbb{R}^n$ has the same homotopy type as its medial axis.

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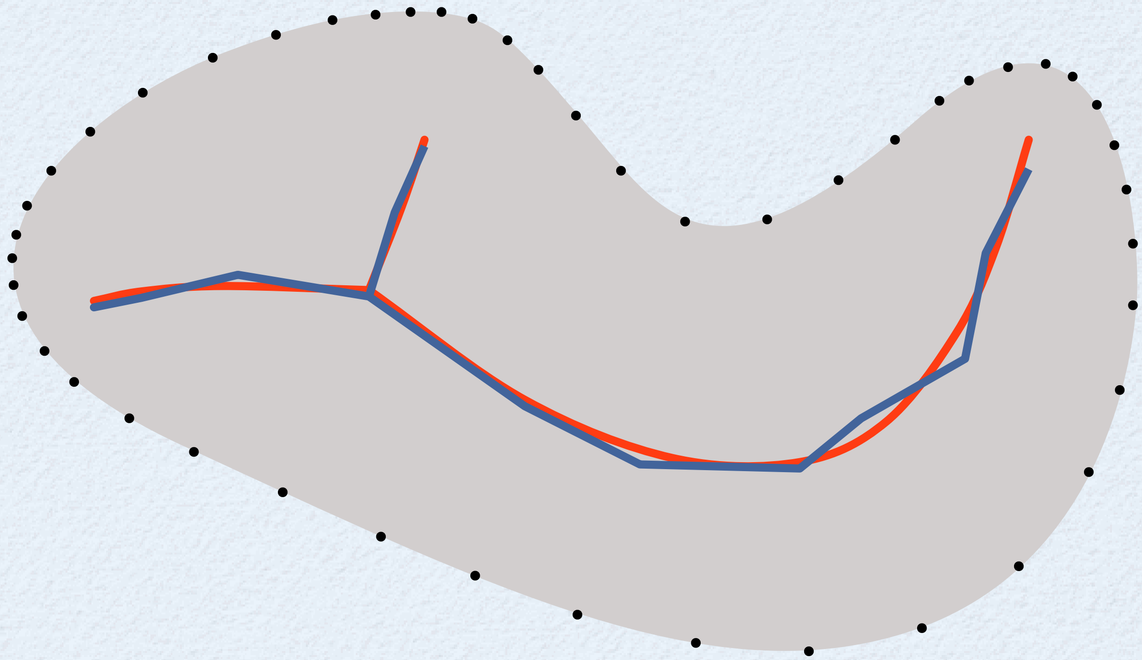

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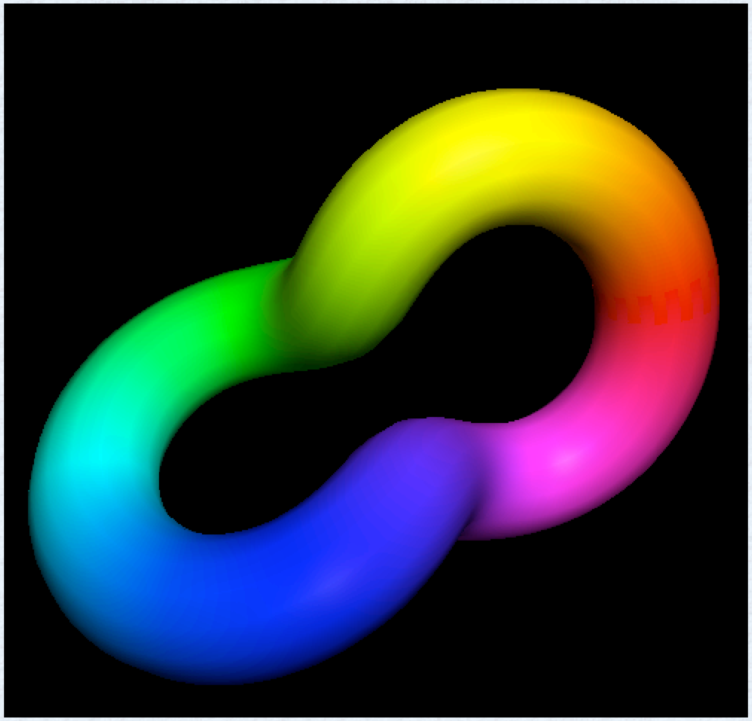

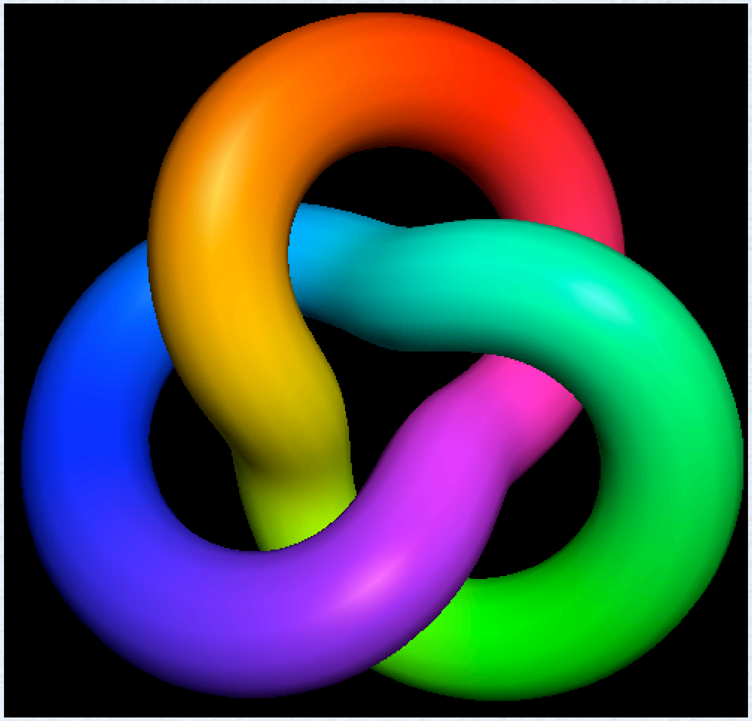

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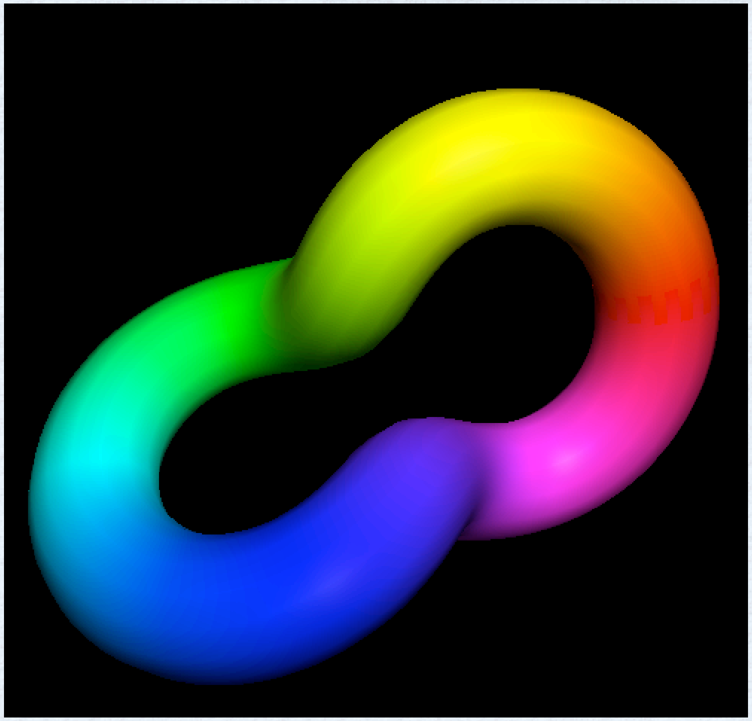
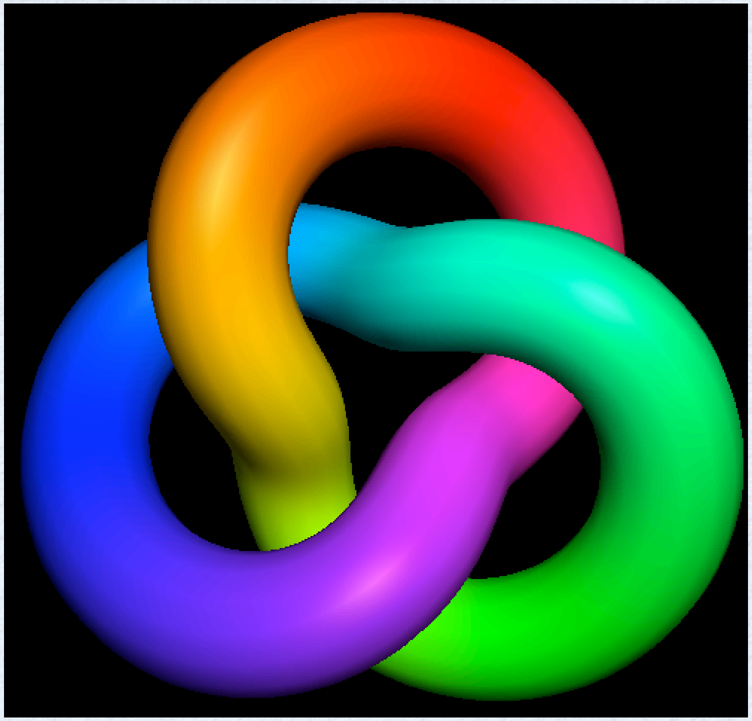
Medial Axis and Embedding of Surface

The outer shapes of torus and knotted torus have different homotopy types.





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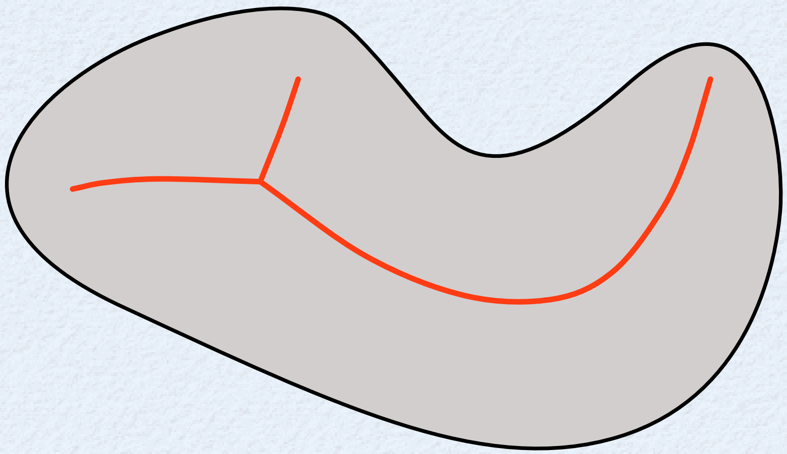
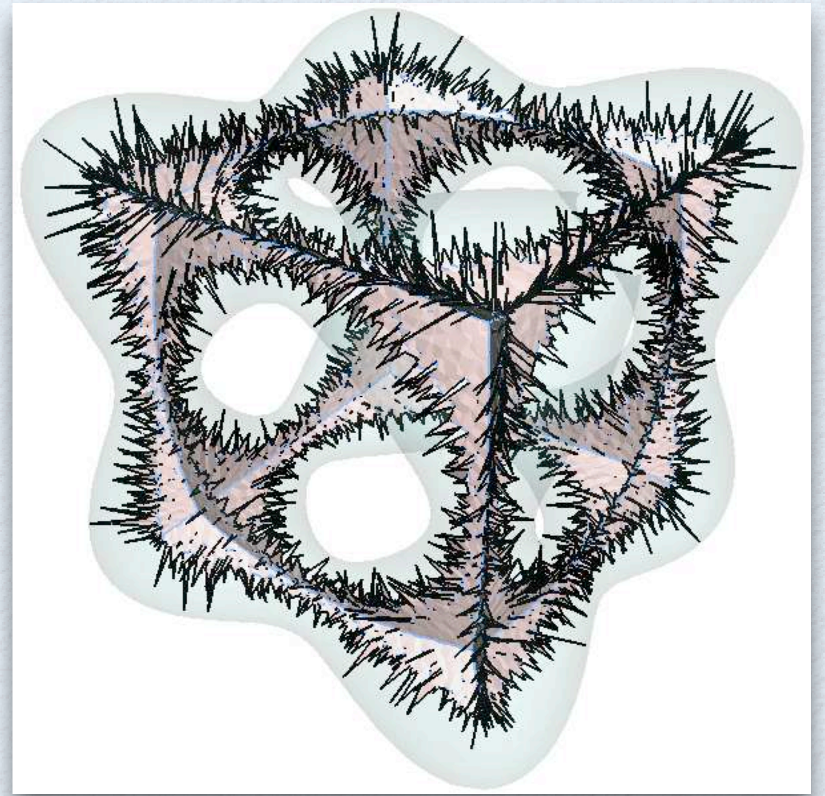
A small change in S can keep a sample valid but change $M(S)$ dramatically.





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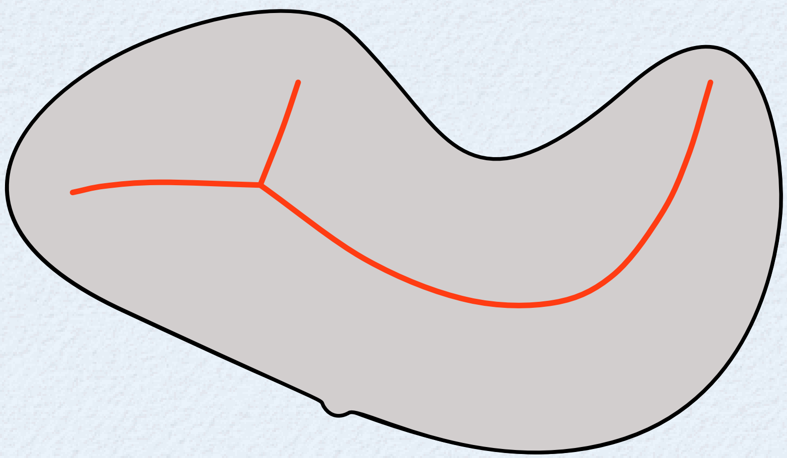

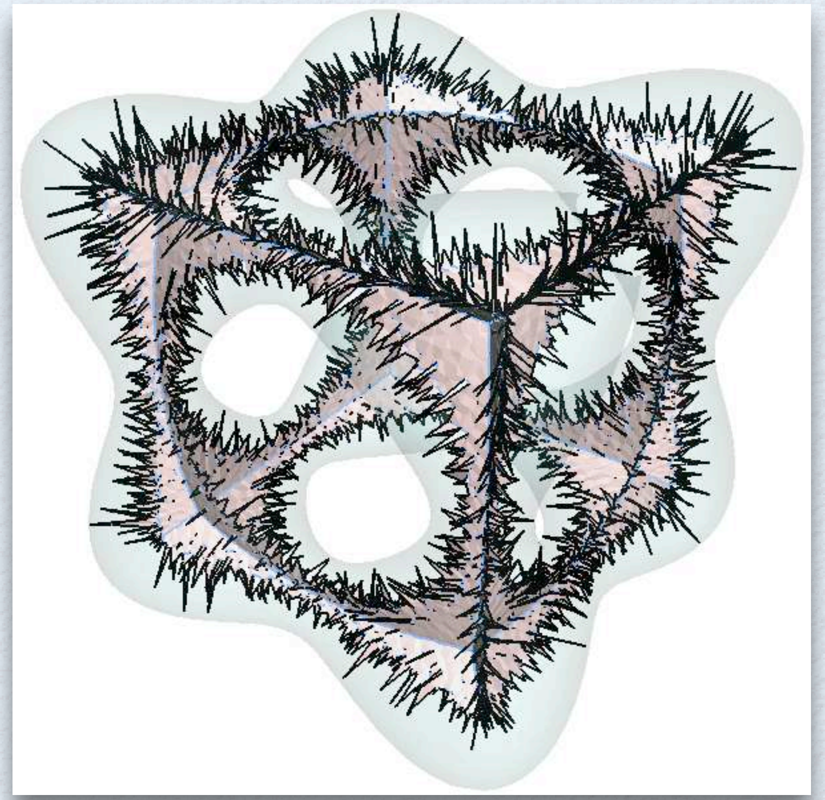

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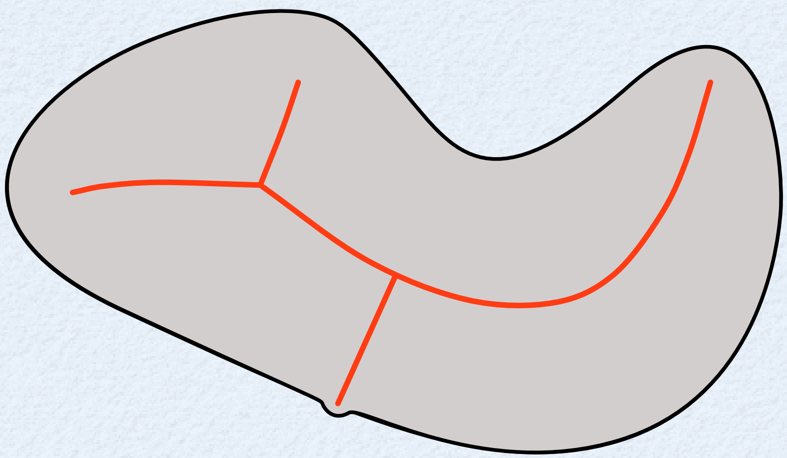

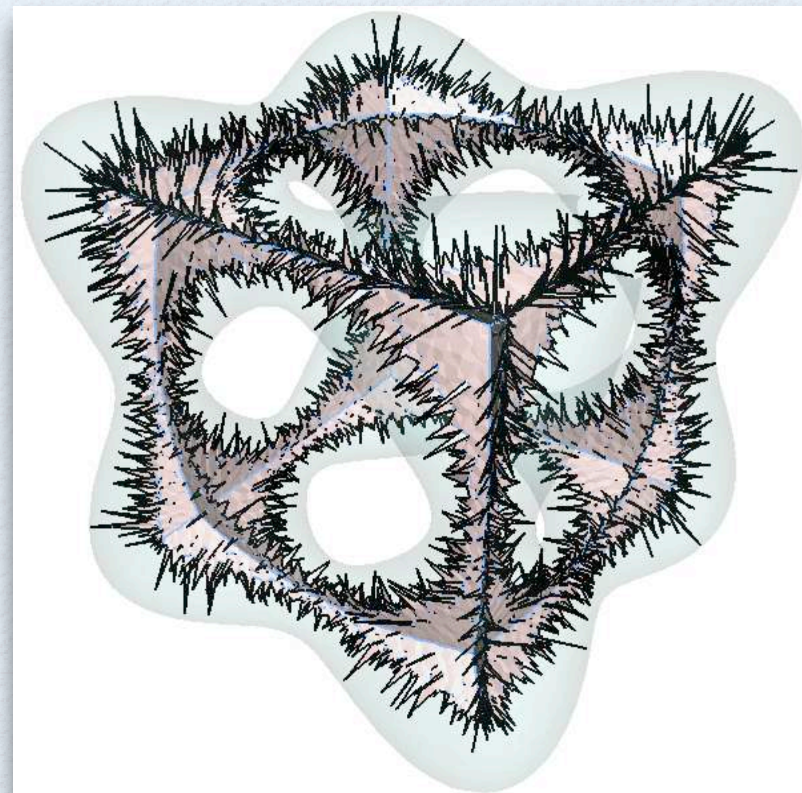

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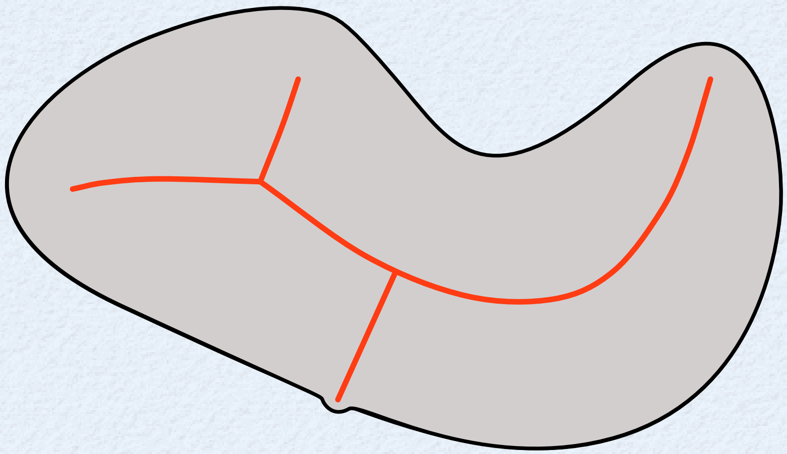

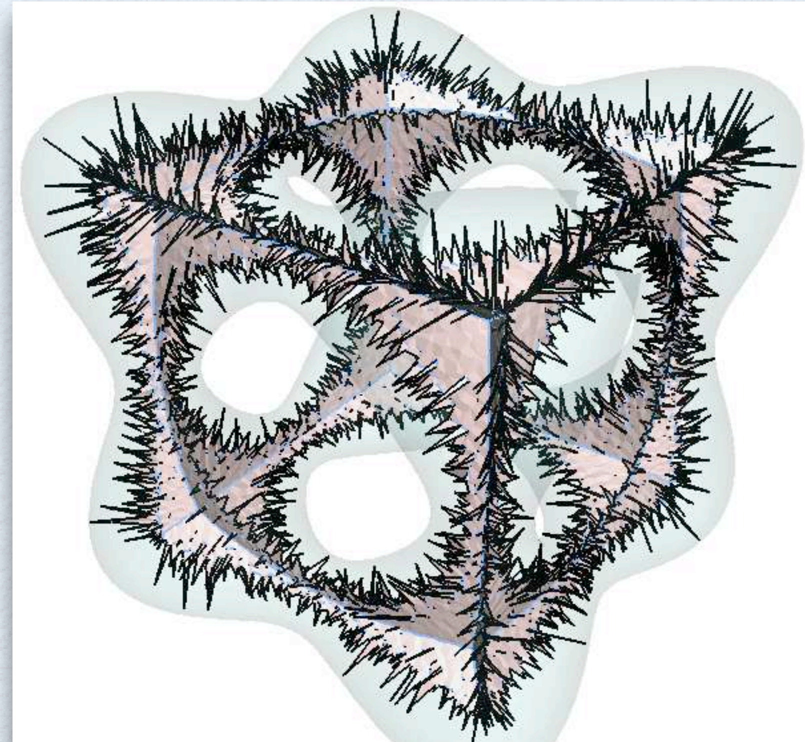

Picture courtesy of Attali, Boissonnat, & Edelsbrunner

In practice, a filtered medial axis can be more interesting.

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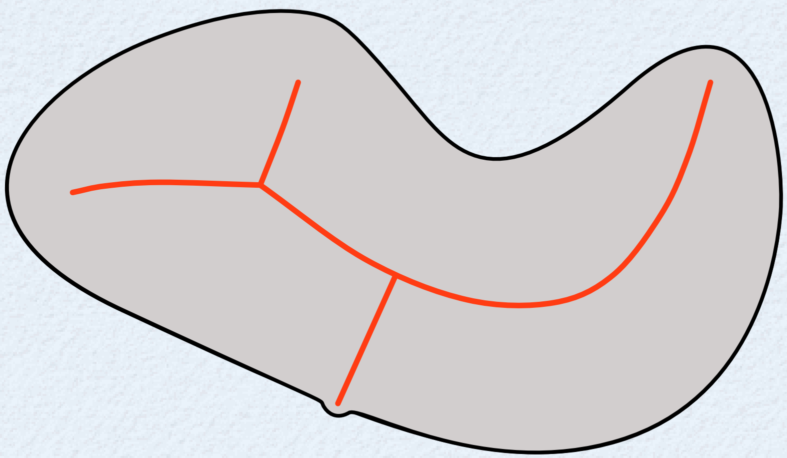

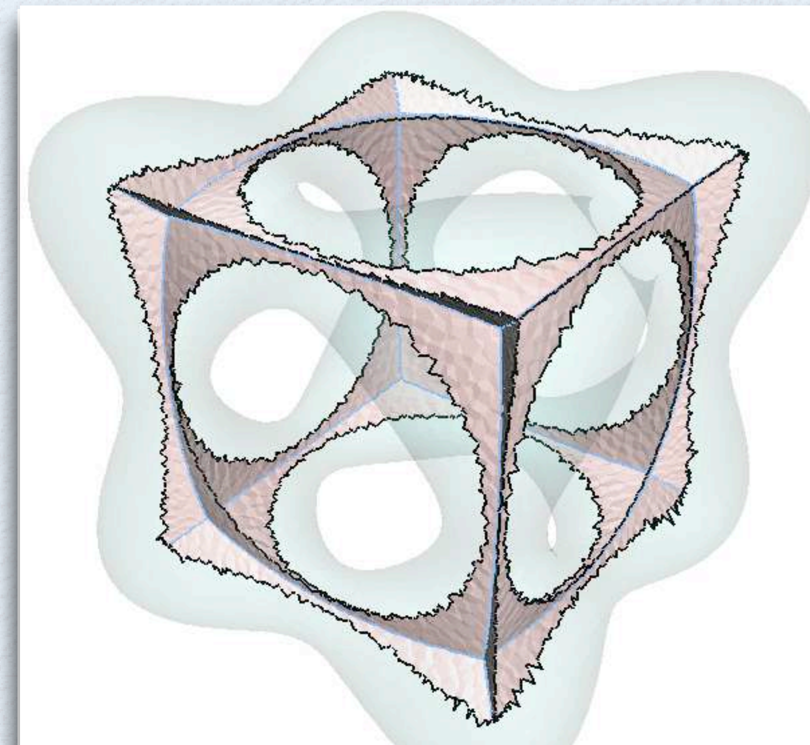

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Some History on Medial Axis Approximation

◇ Exact Methods: for limited classes of shapes
- [Manocha et al. '04] for polytopes
- [Zerroug et al. '94] for Generalized Cylinders

◇ Voronoi Filtering:
- [Amenta & Bern '99] 2d
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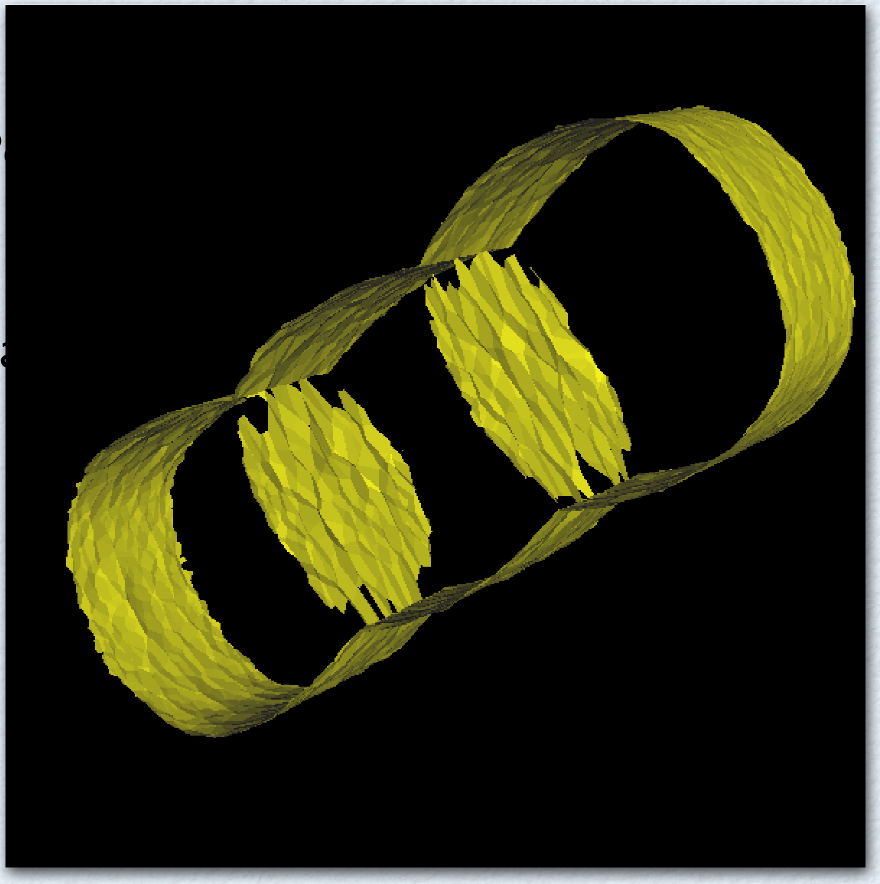

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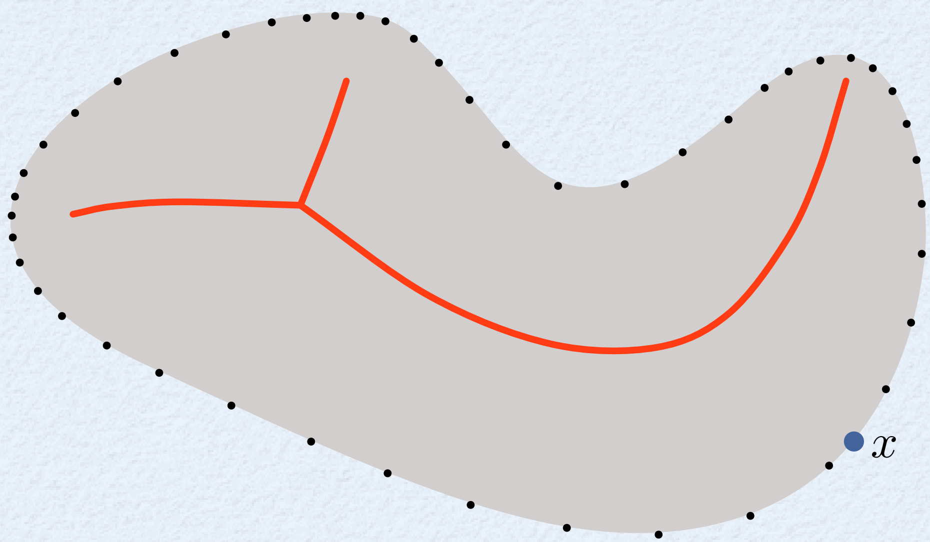
For a point $x \in \Sigma$, the local feature size of x is

$$\mathrm{lfs}(x) := d(x, M).$$



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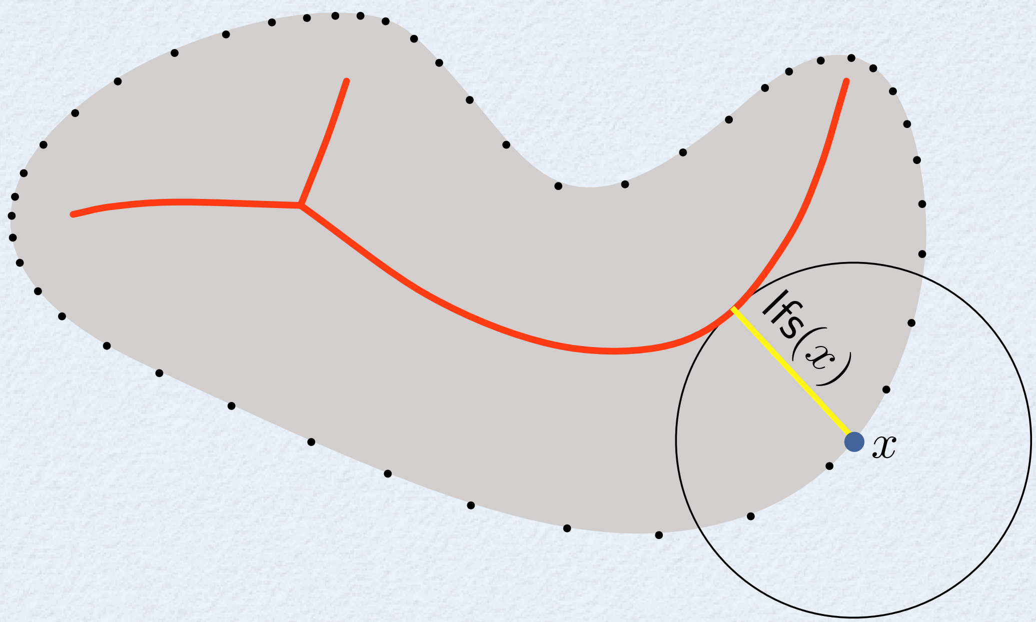

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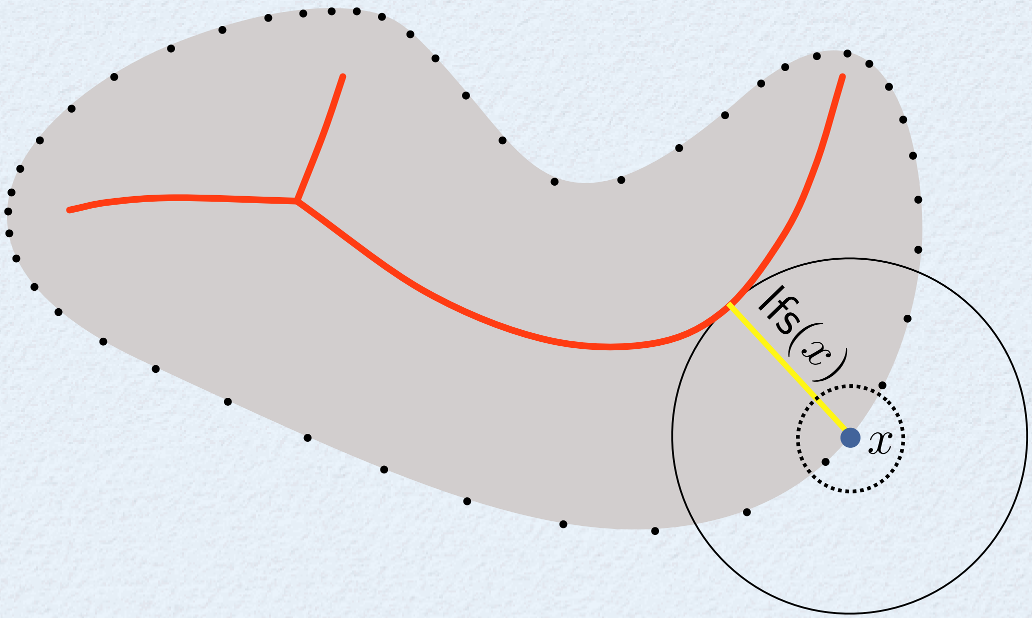

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Our Approach:
A Lot is Encoded in Distance Functions

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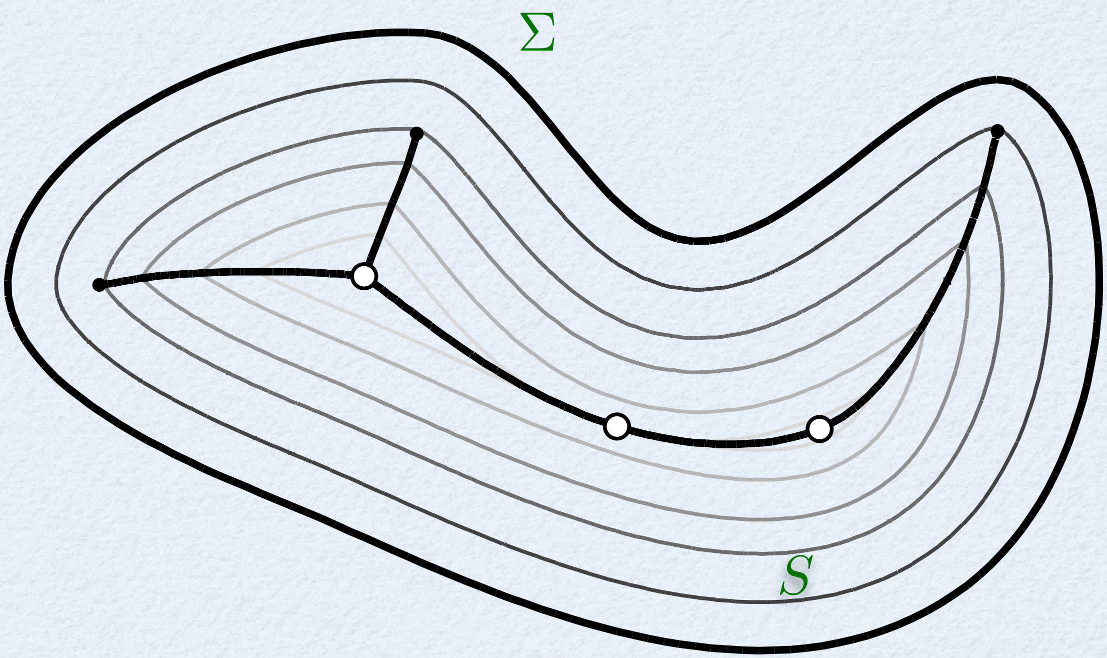

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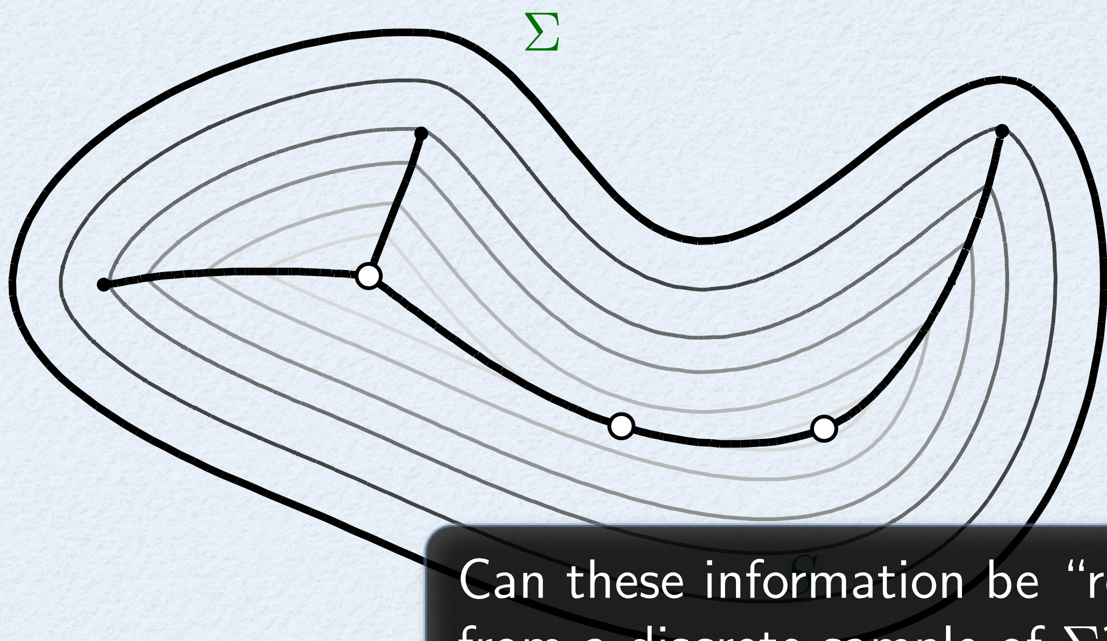

Can these information be “read” similarly from a discrete sample of Σ?

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Acknowledgments and Agenda

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2. Medial Axis Approximation.
3. Shape Reconstruction.
4. Analysis of WRAP.

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Covered in Prelim

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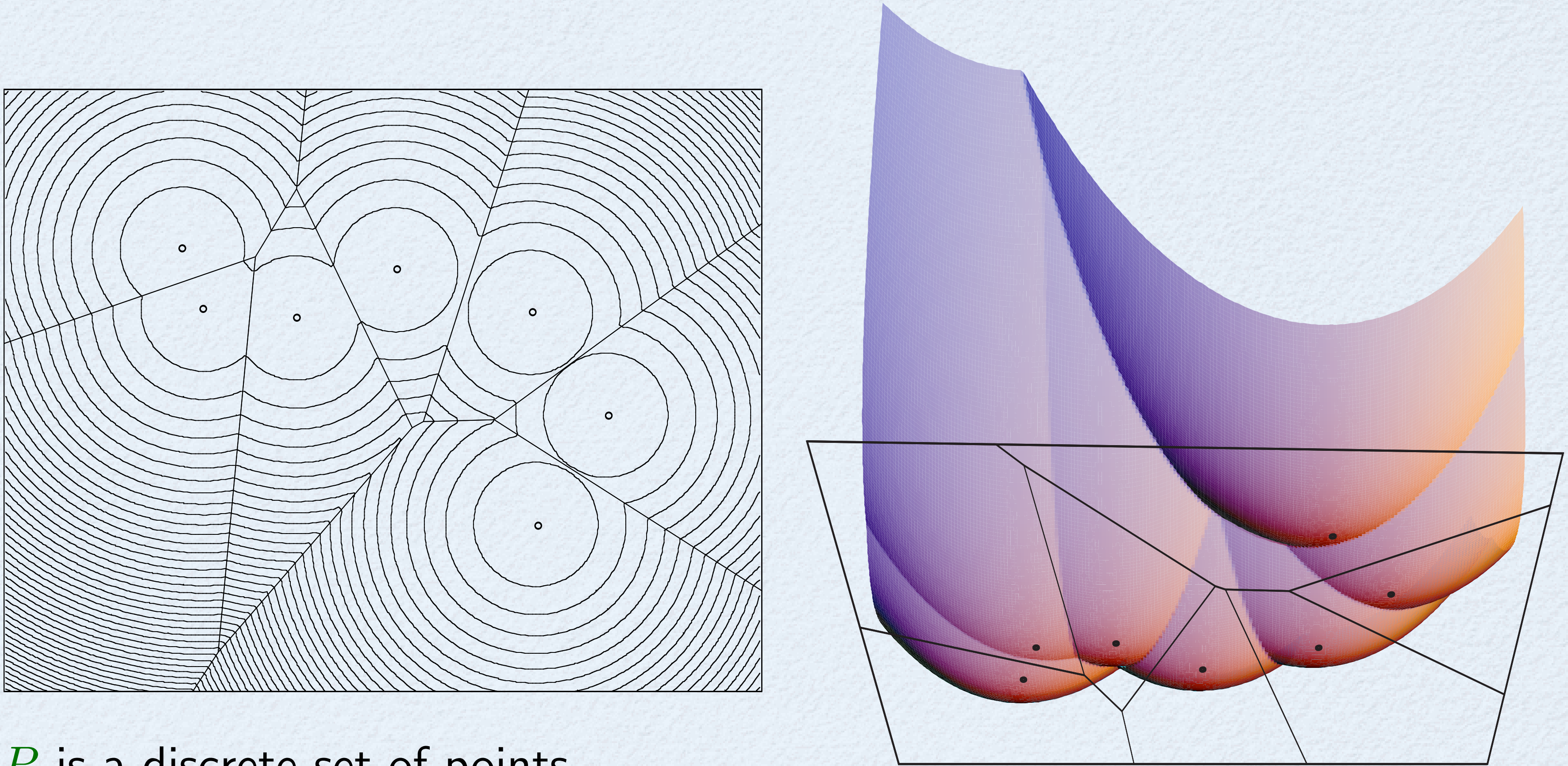
1
The Machinery

(Squared) Distance to Discrete Point Sets



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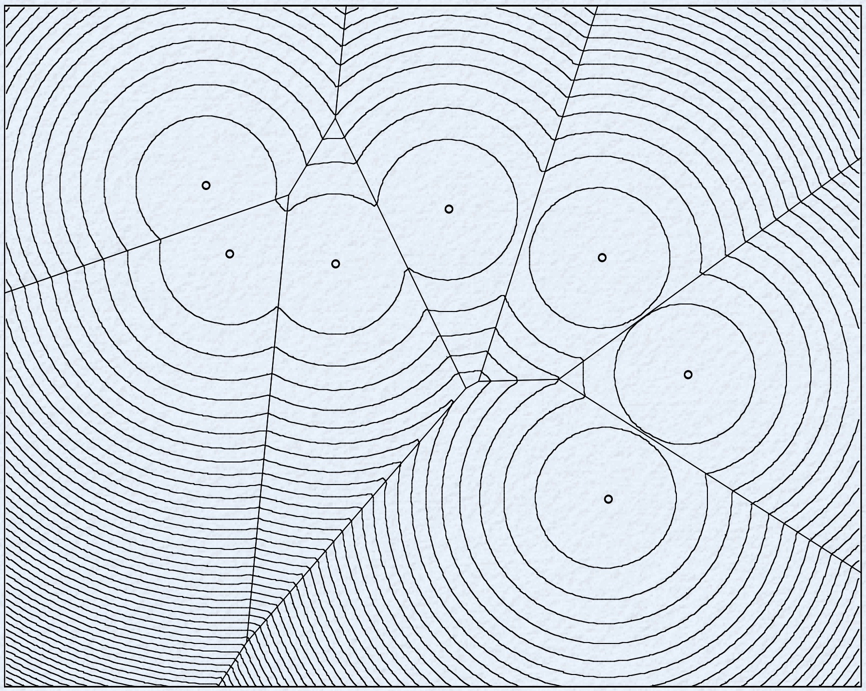

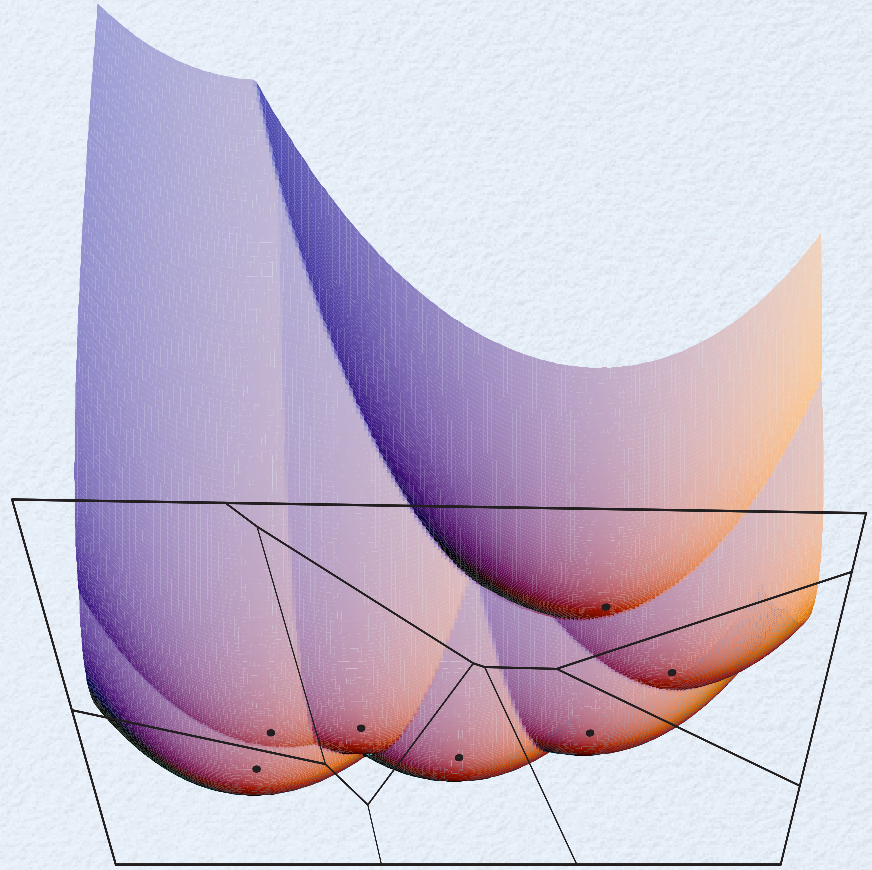

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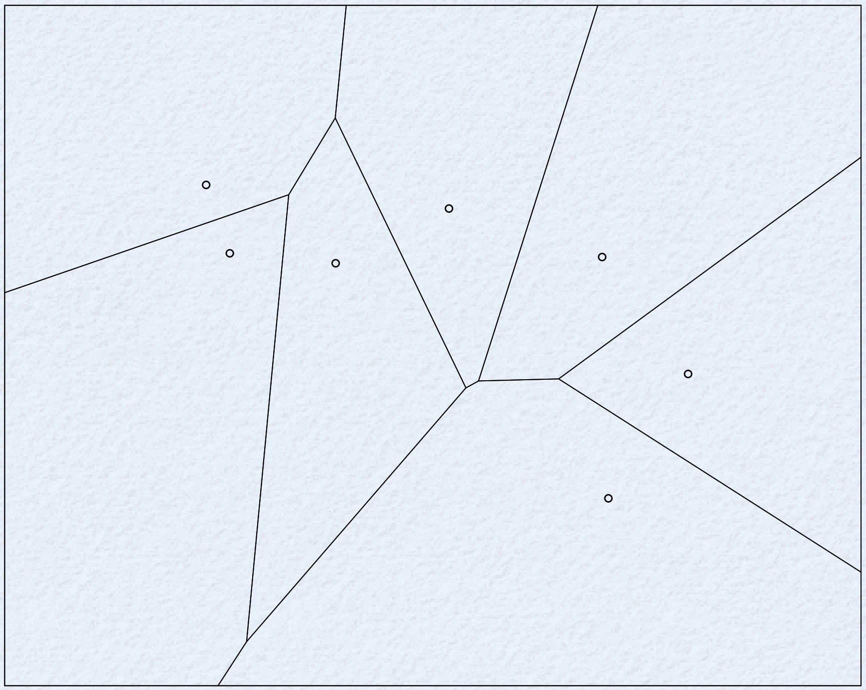
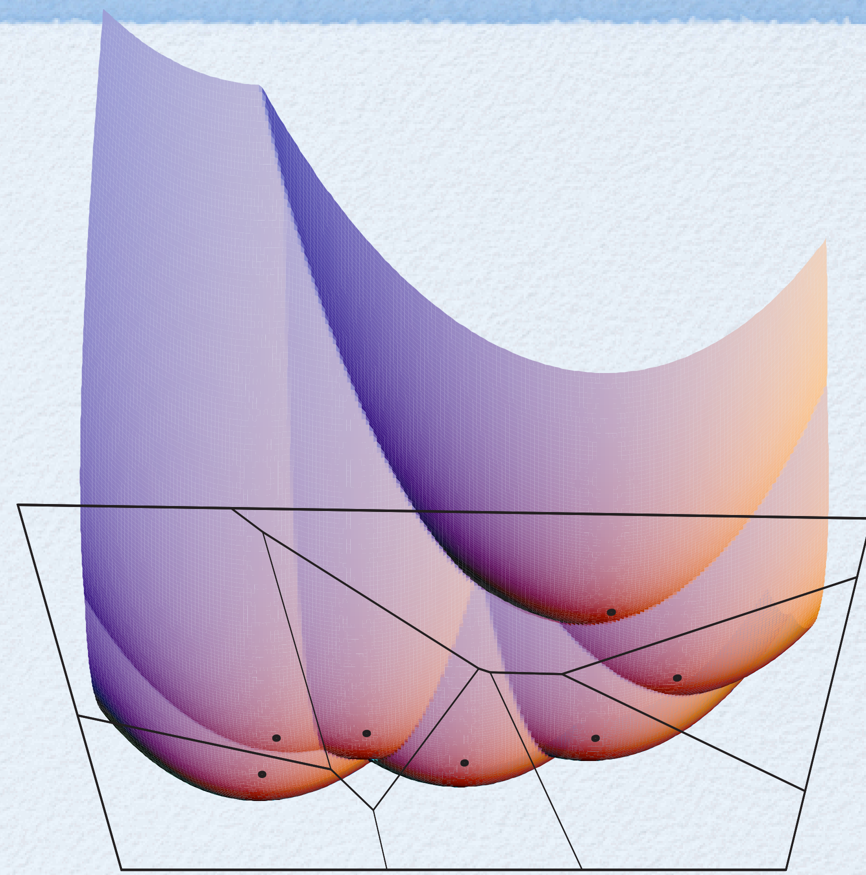
Observation. h is smooth at points with a unique closest point in P.

Generalized Gradient





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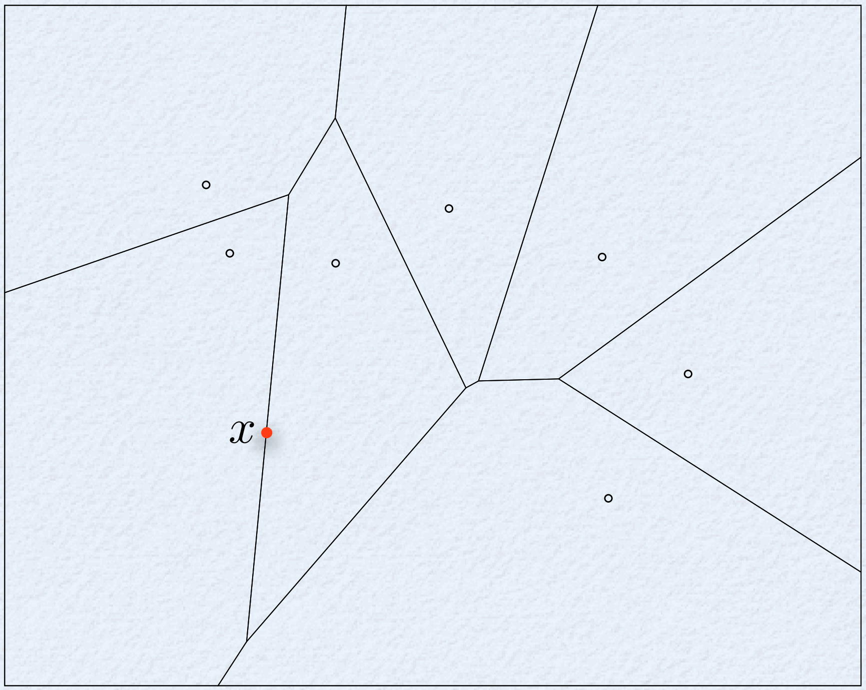

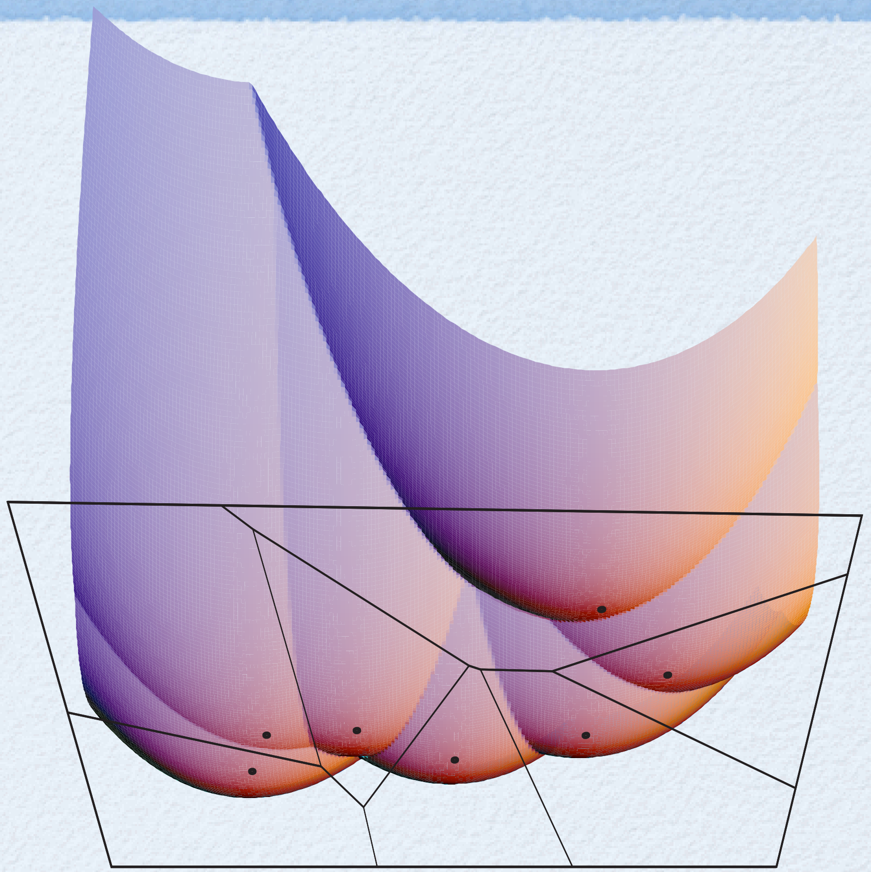

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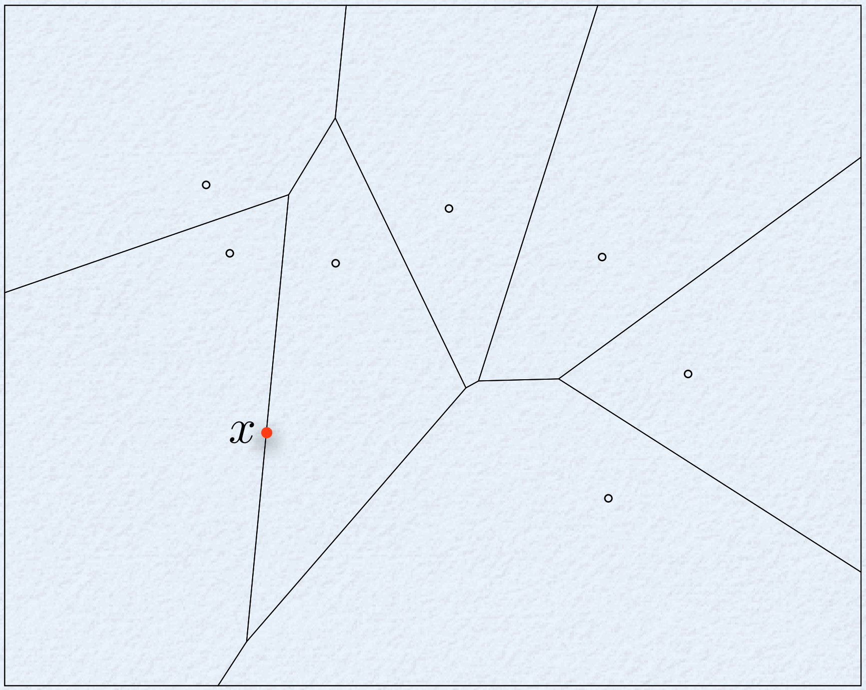

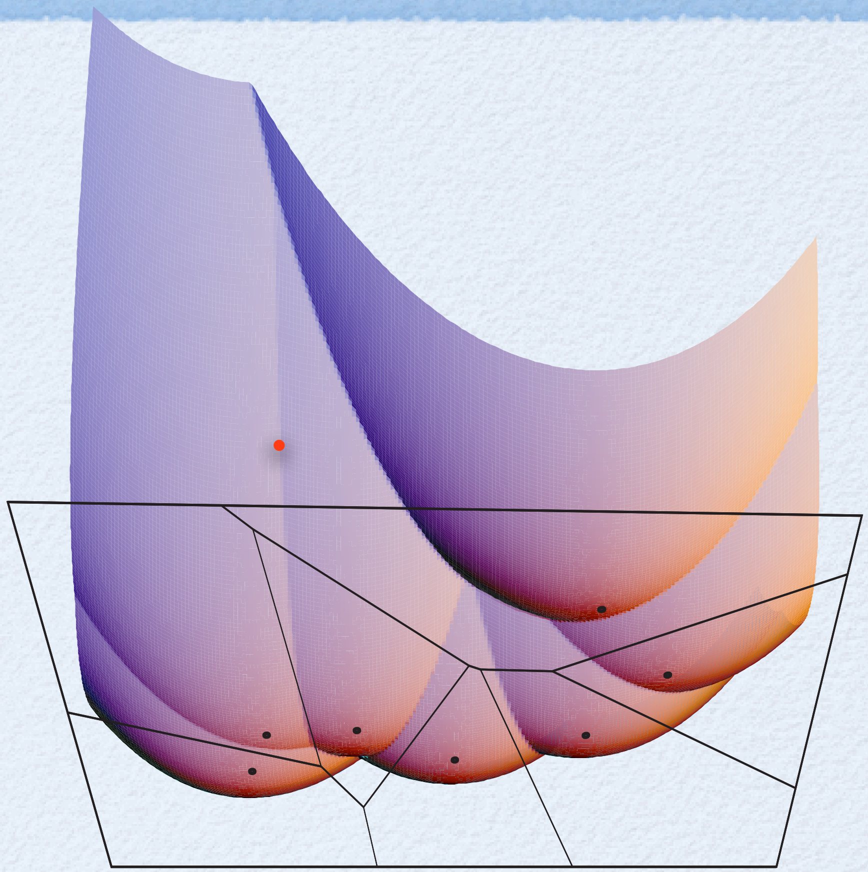

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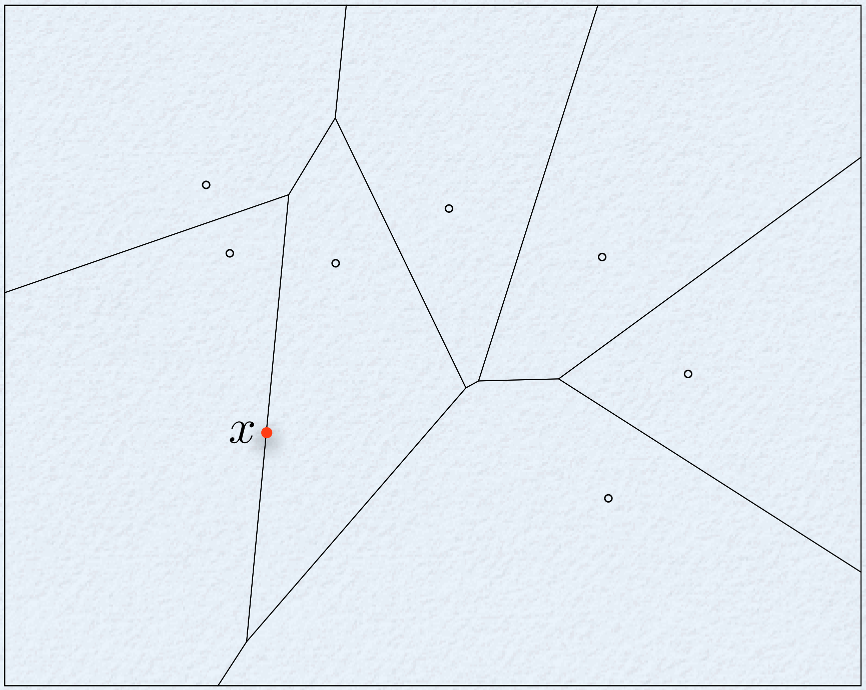

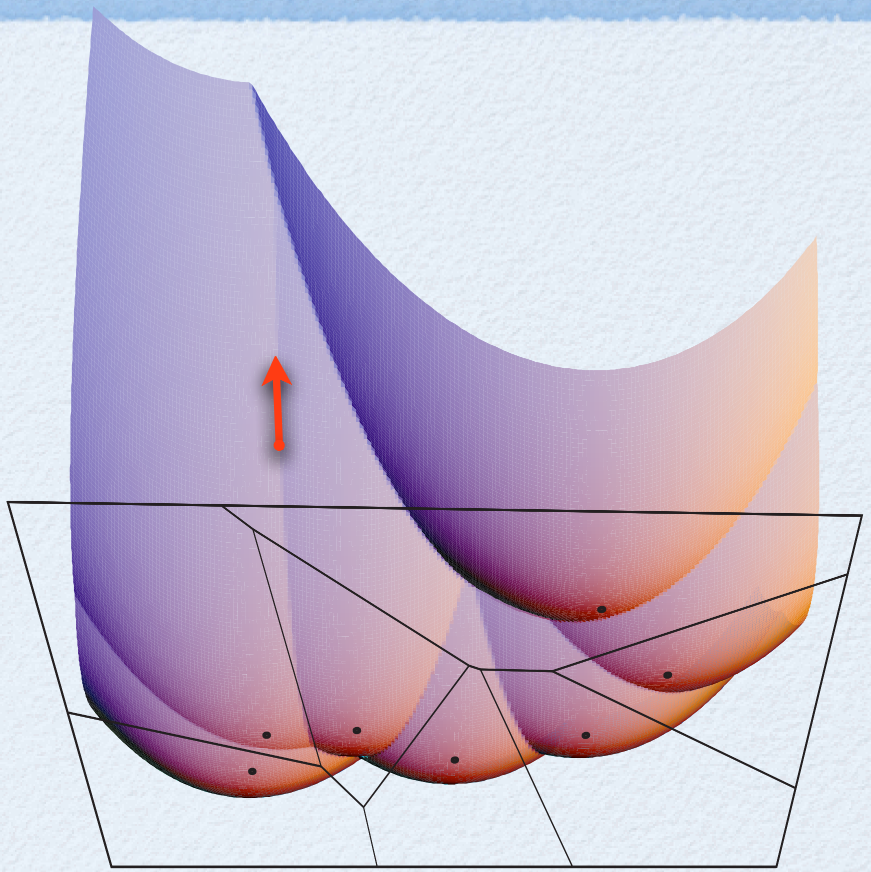

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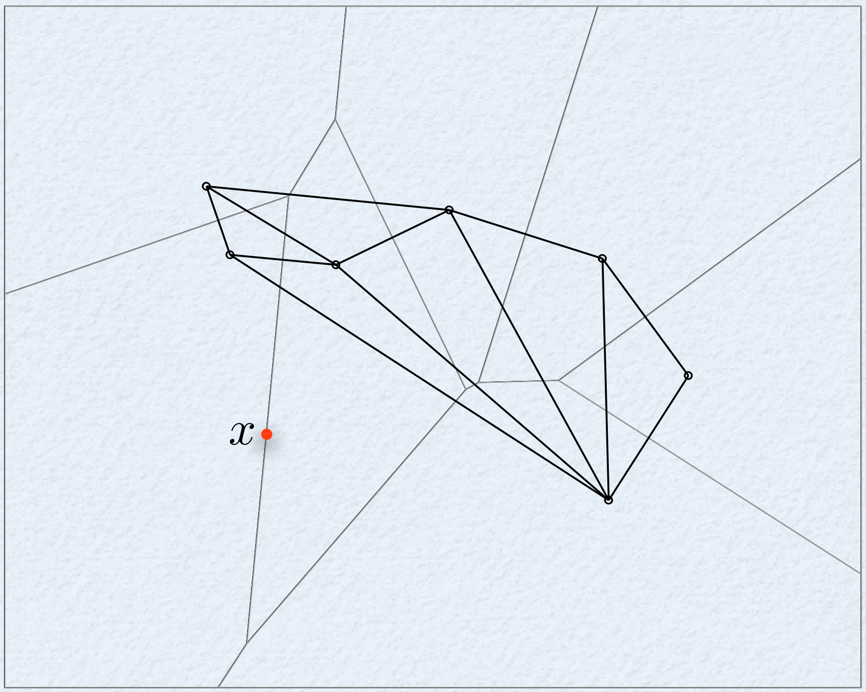

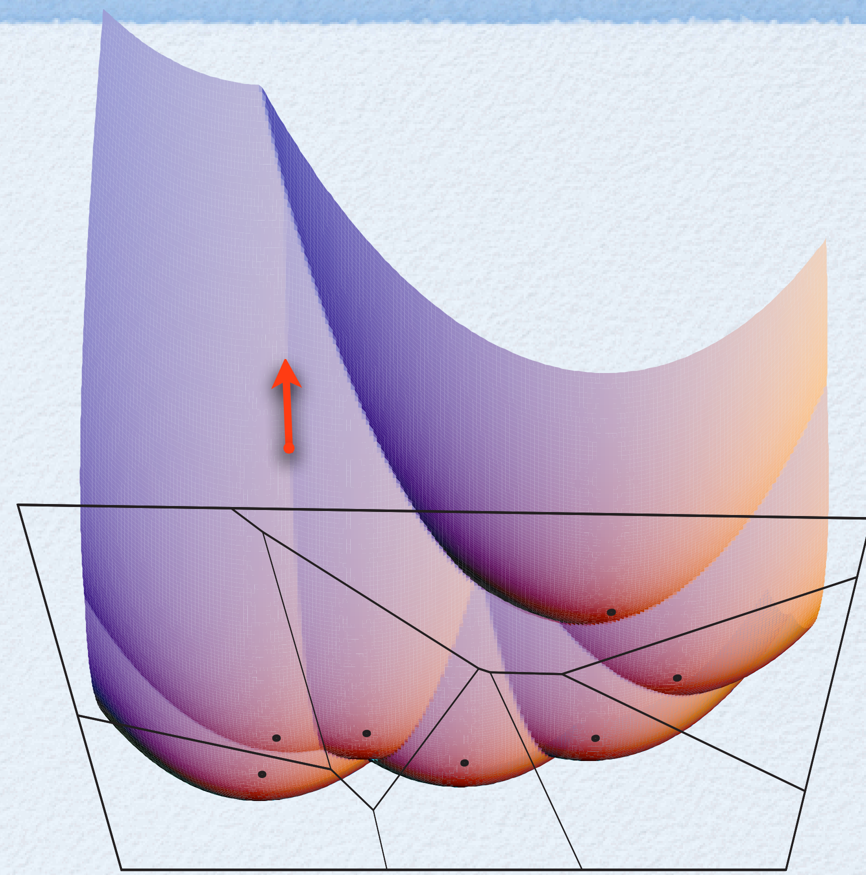

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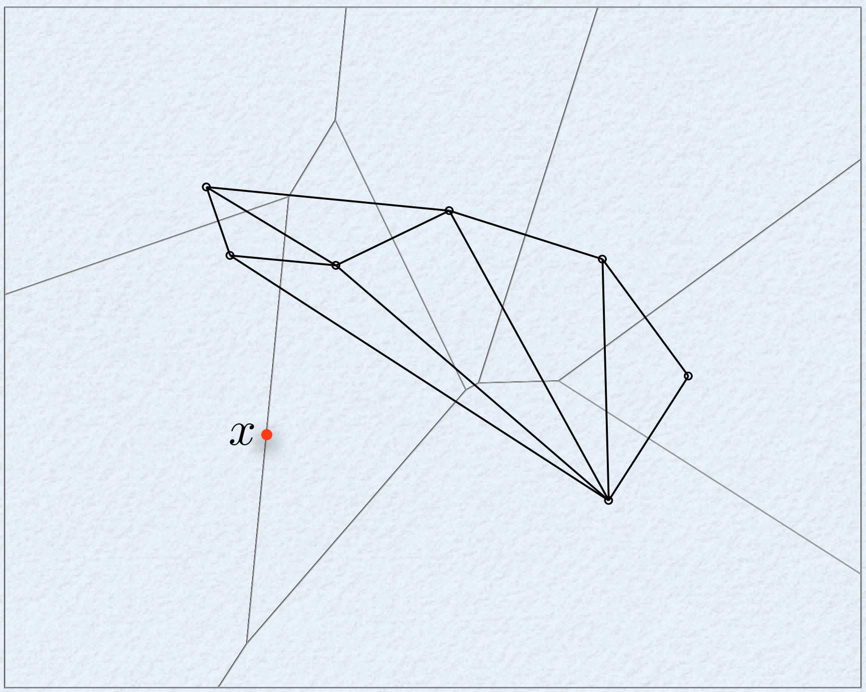

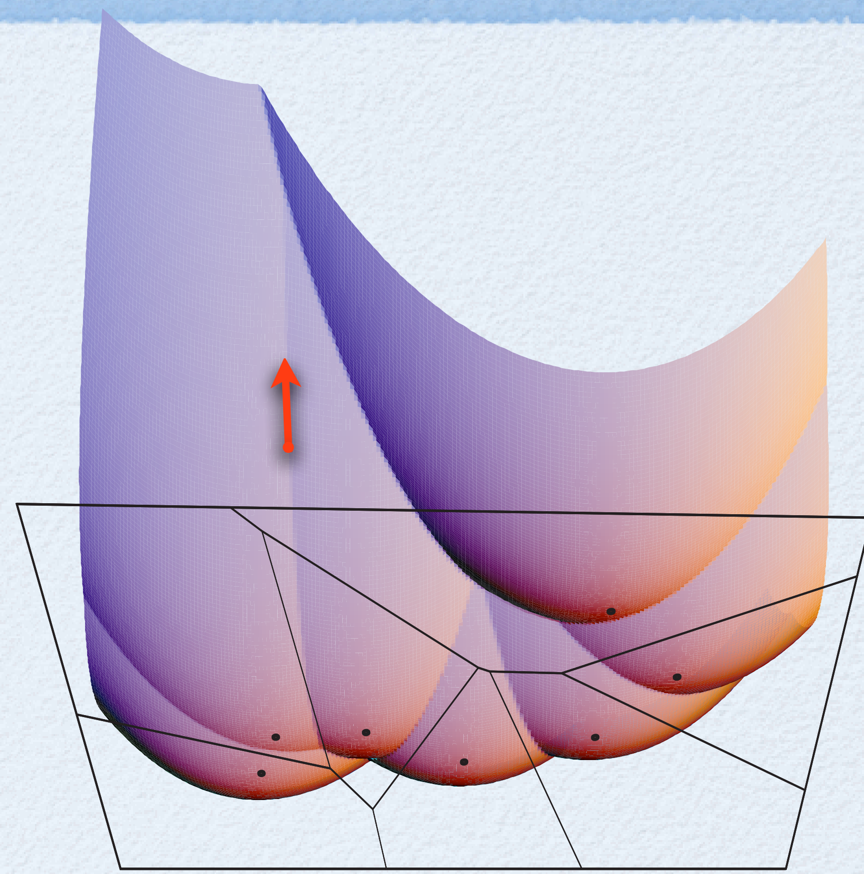

$V(x)$: lowest-dimensional Voronoi face containing x.

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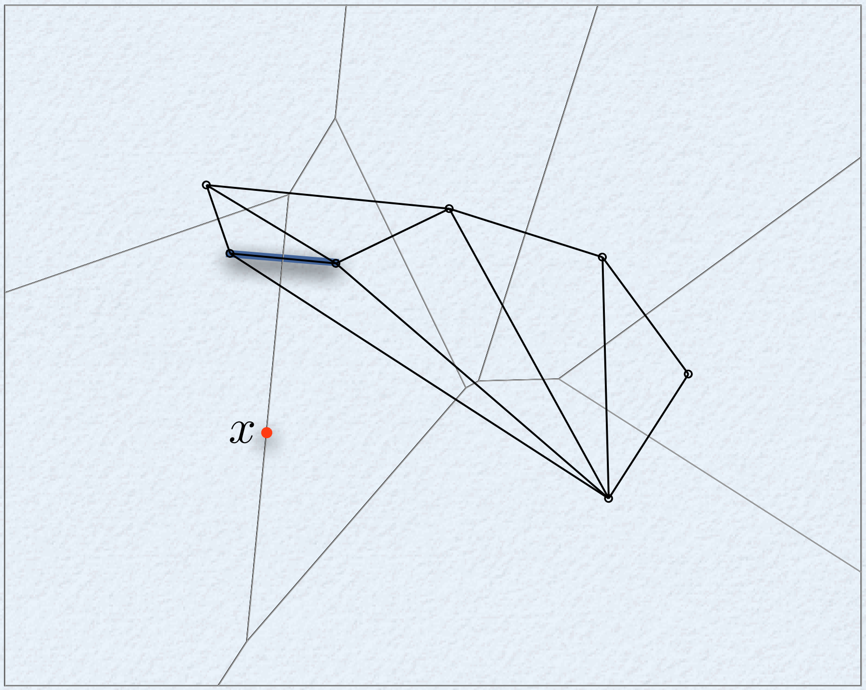

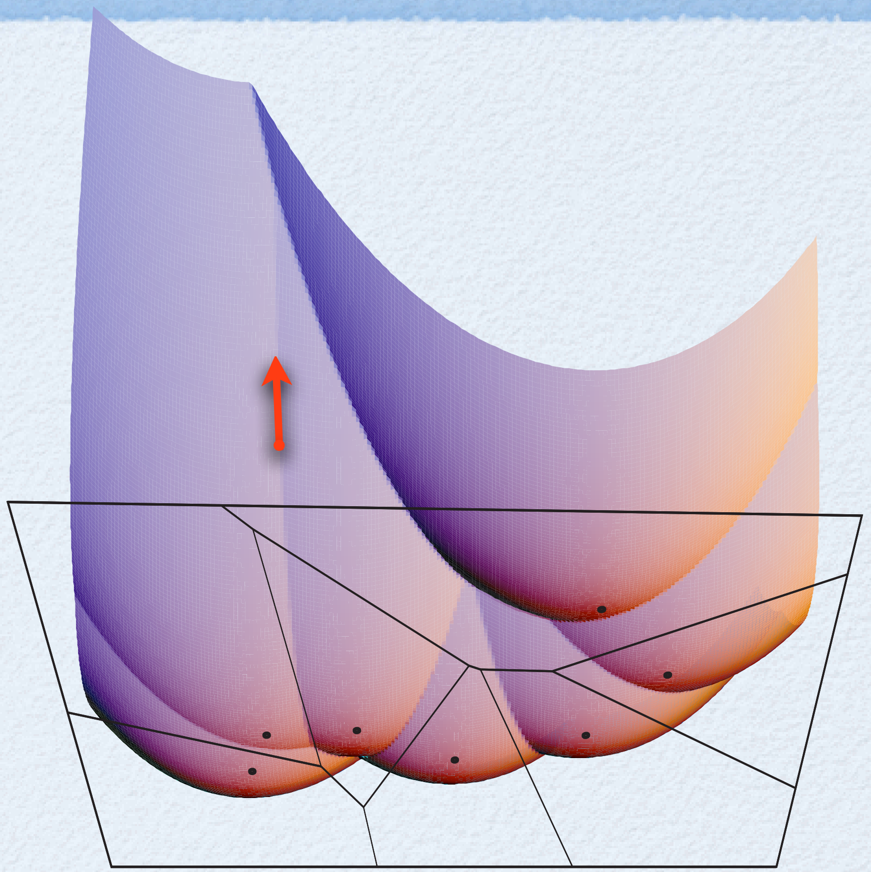

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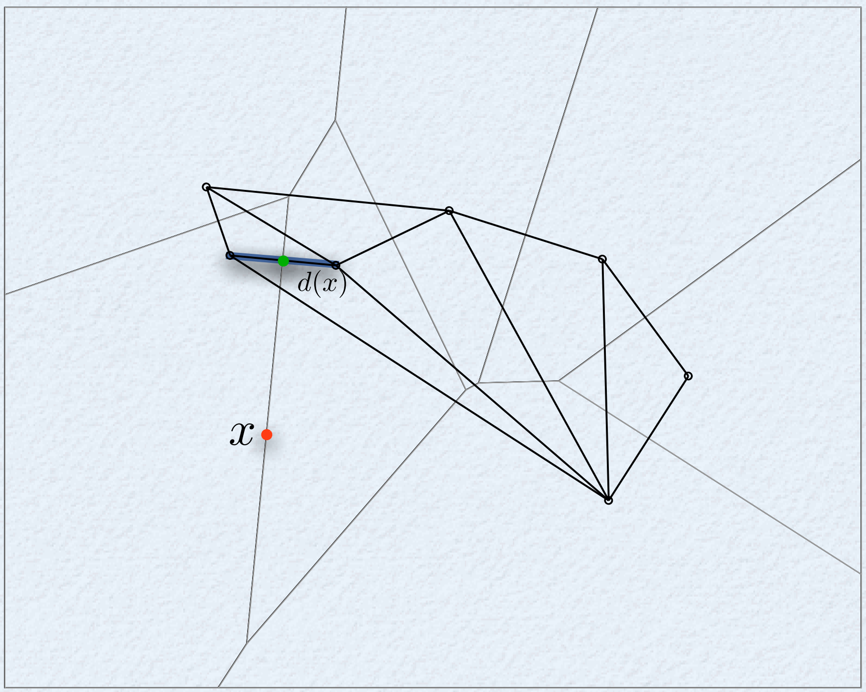

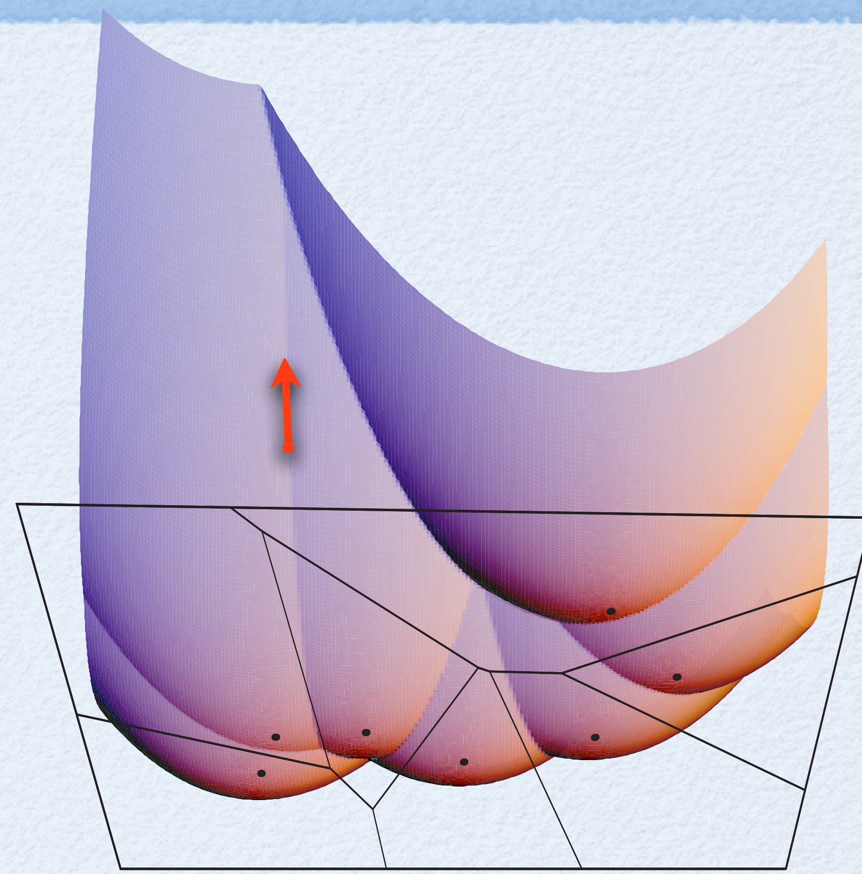

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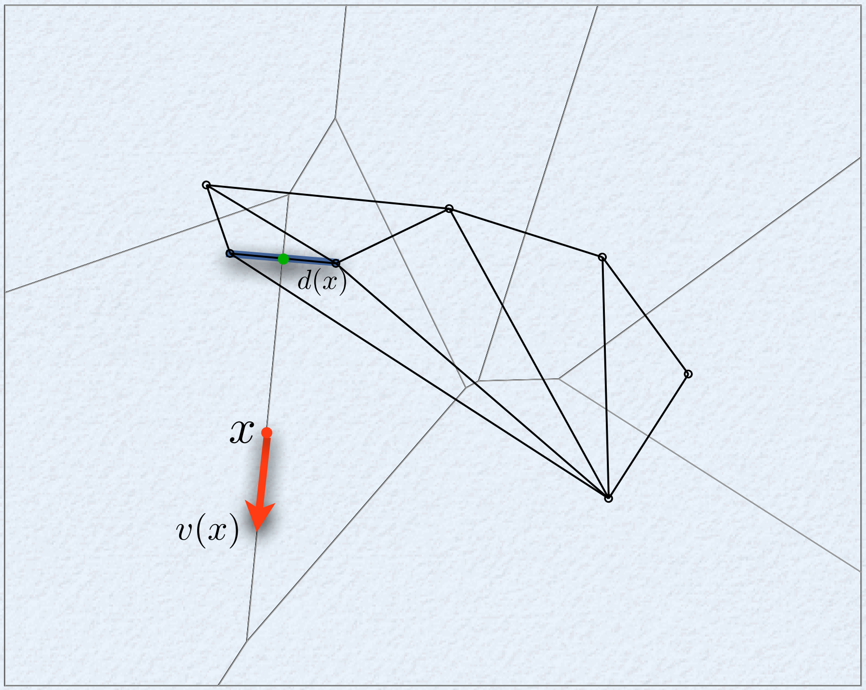

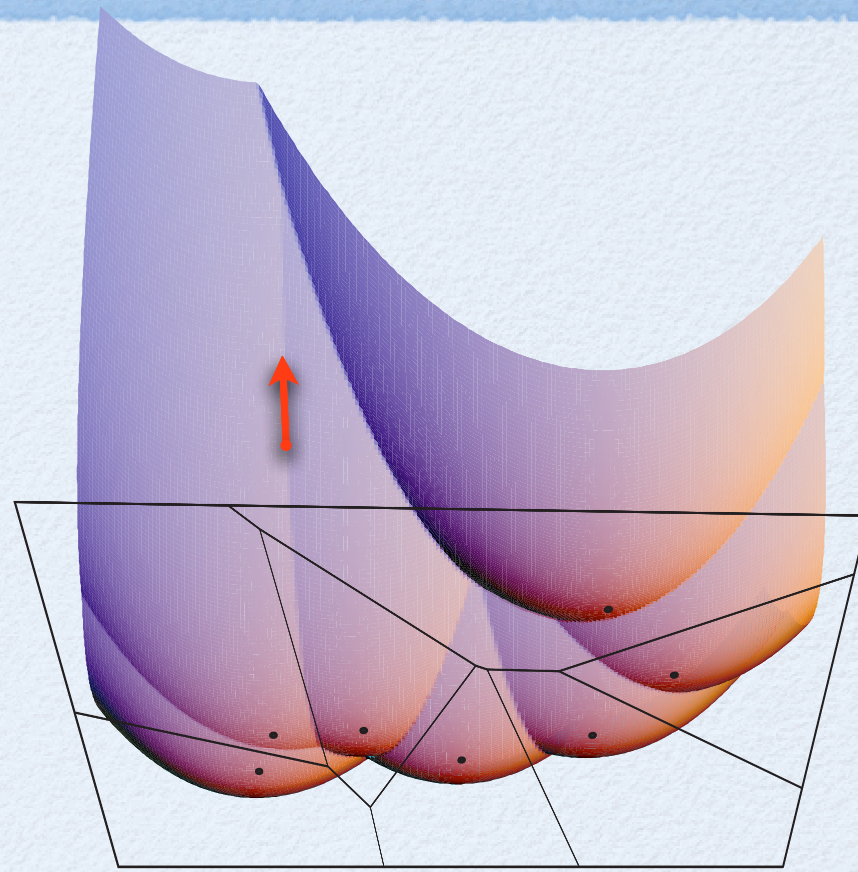

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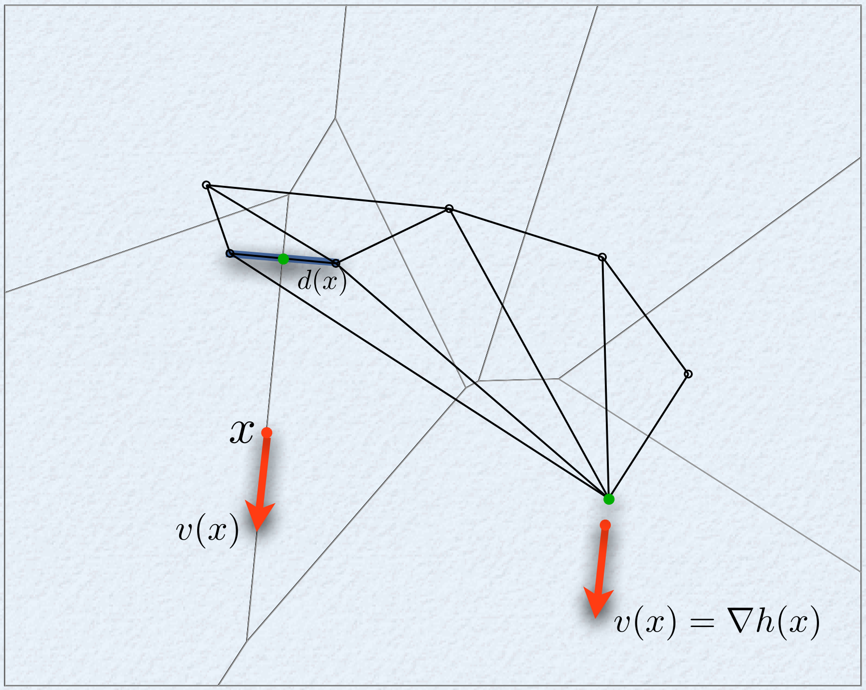

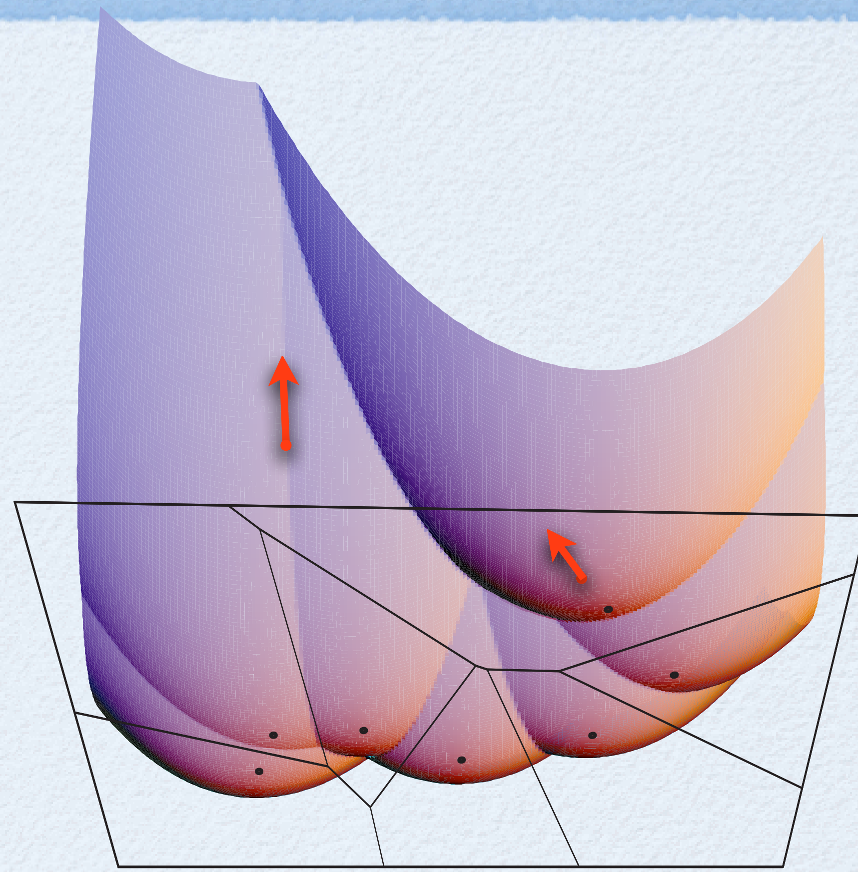

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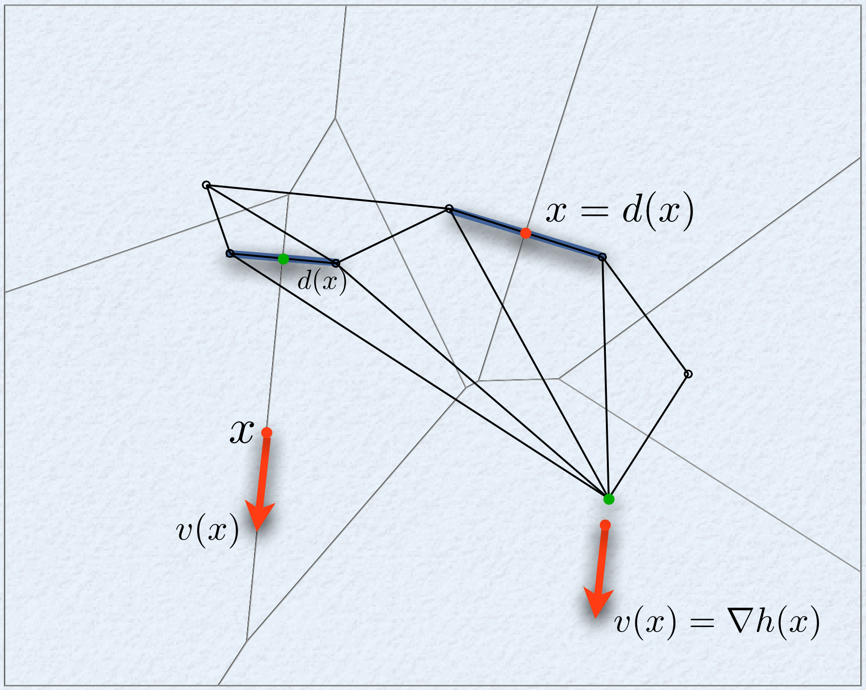

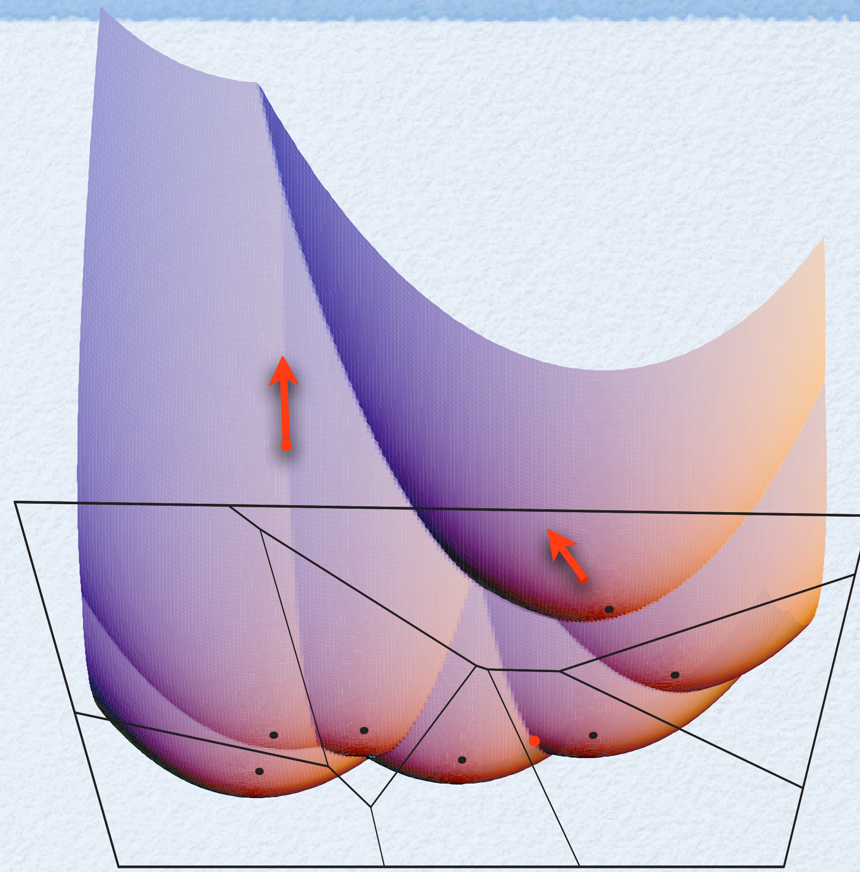

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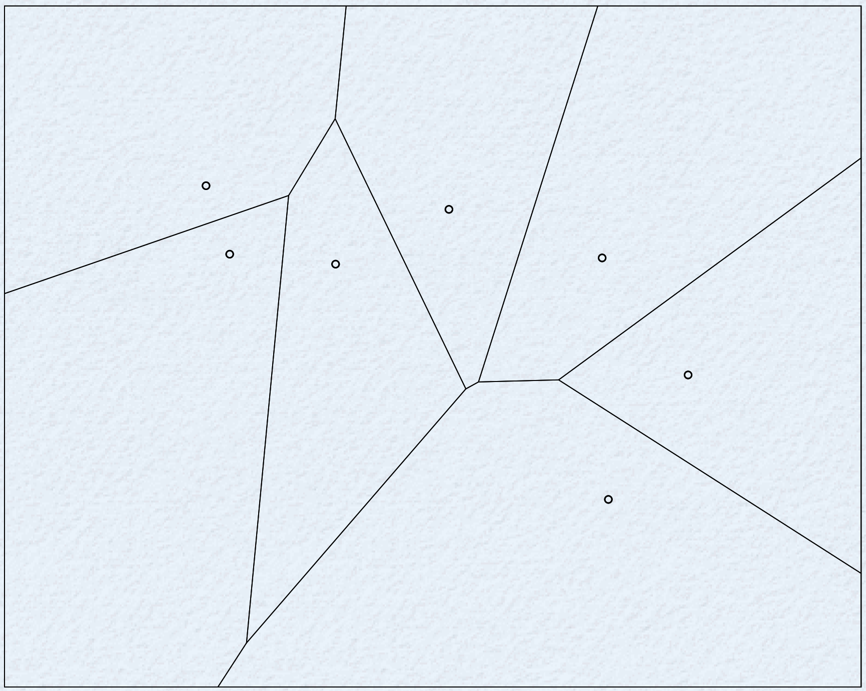

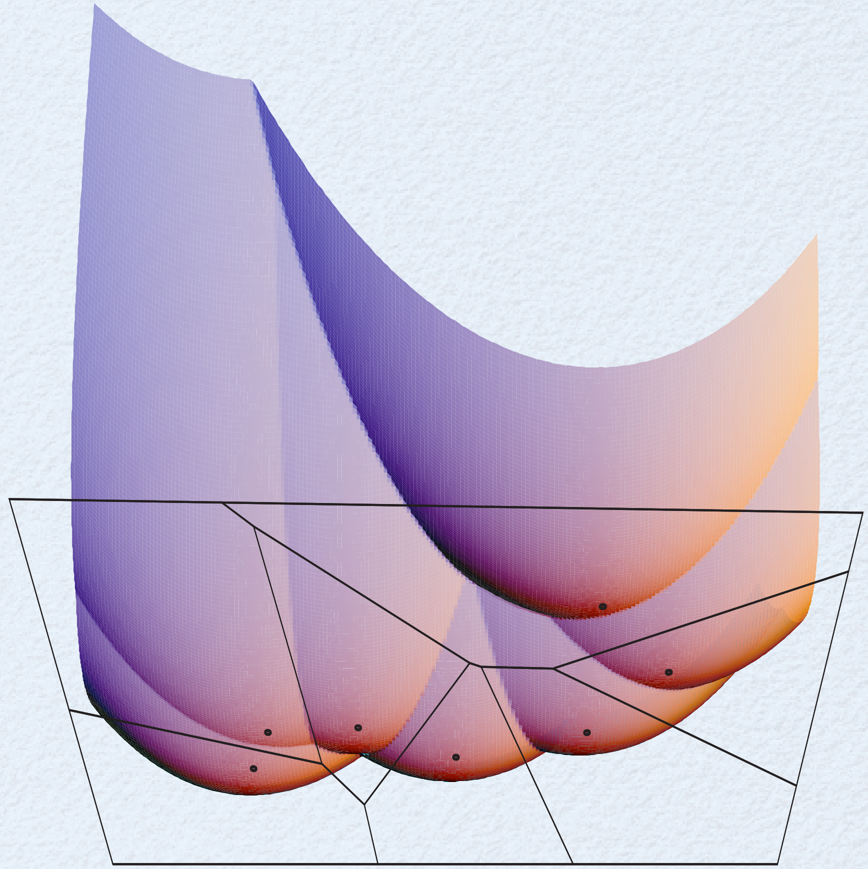

Moving at point x in with speed $v(x)$ results a flow map $\phi : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$.

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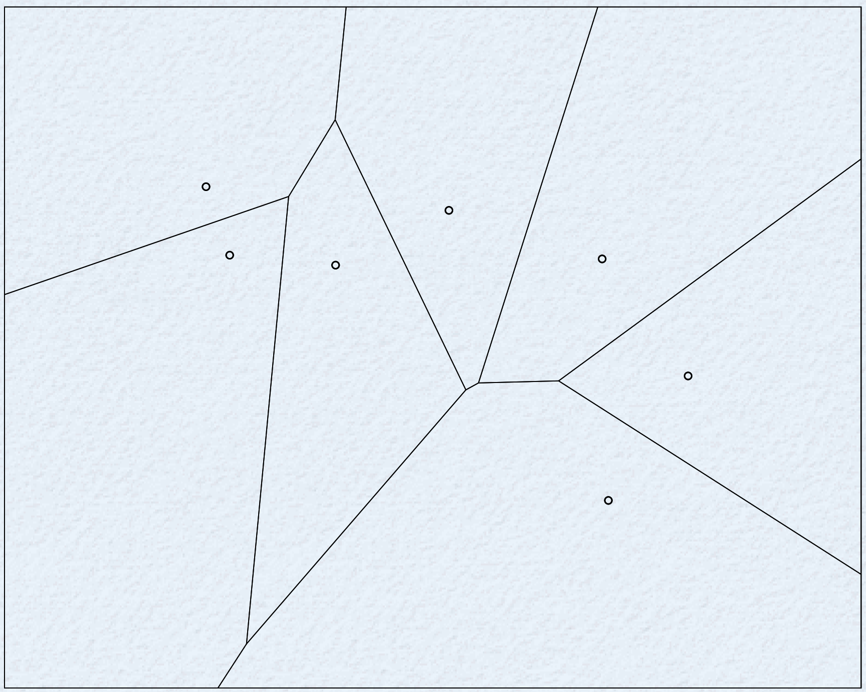

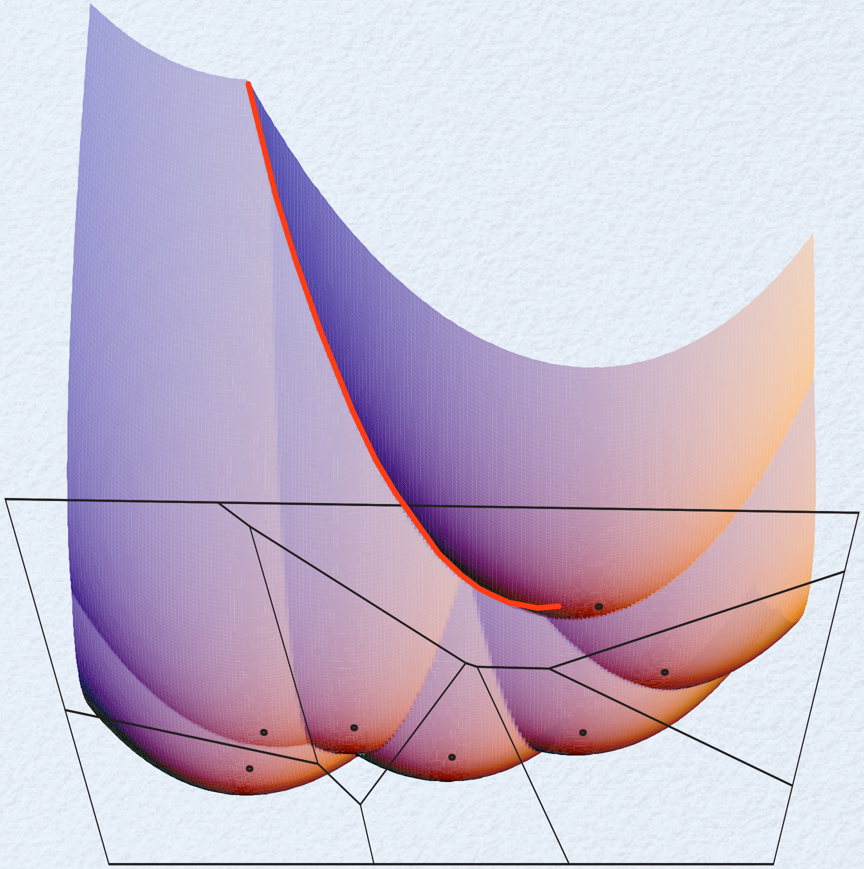

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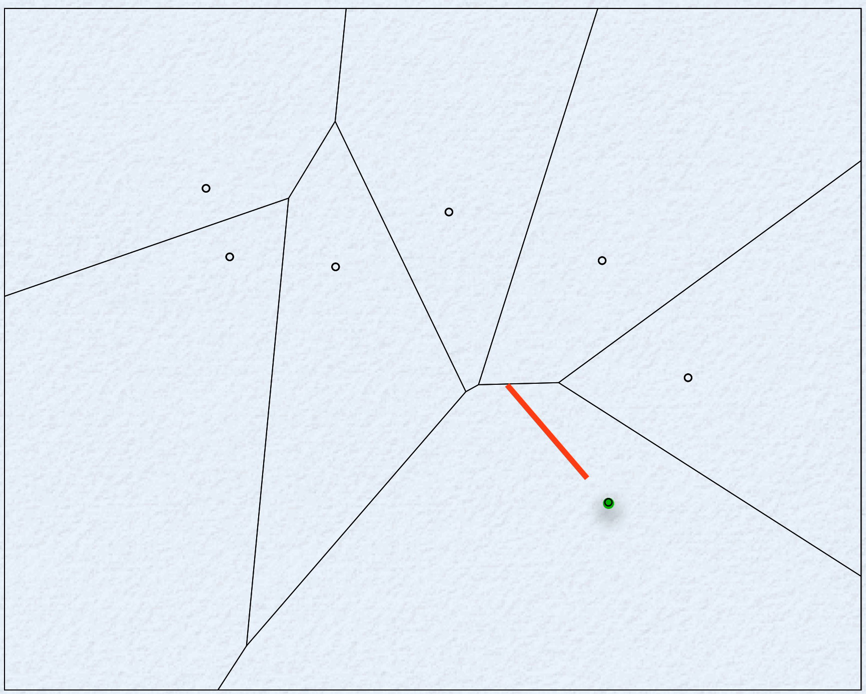

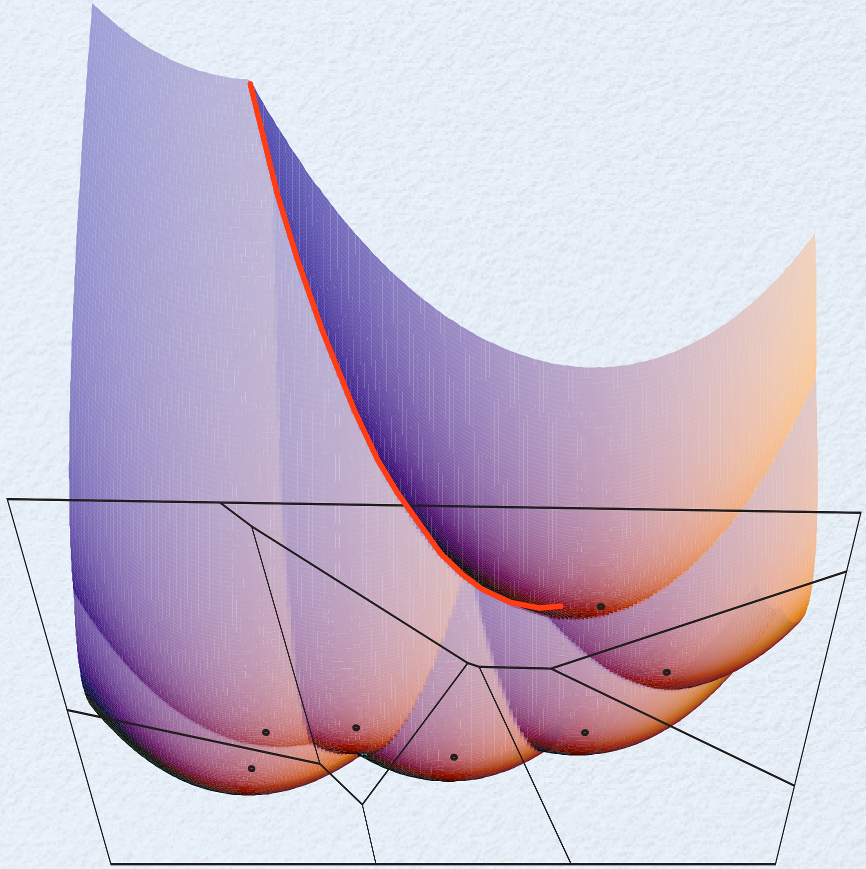

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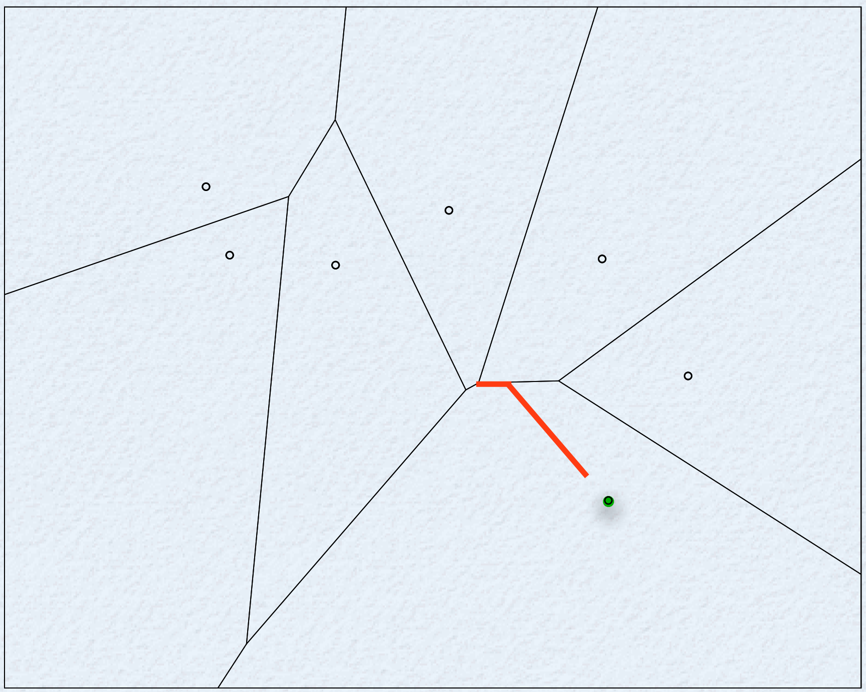

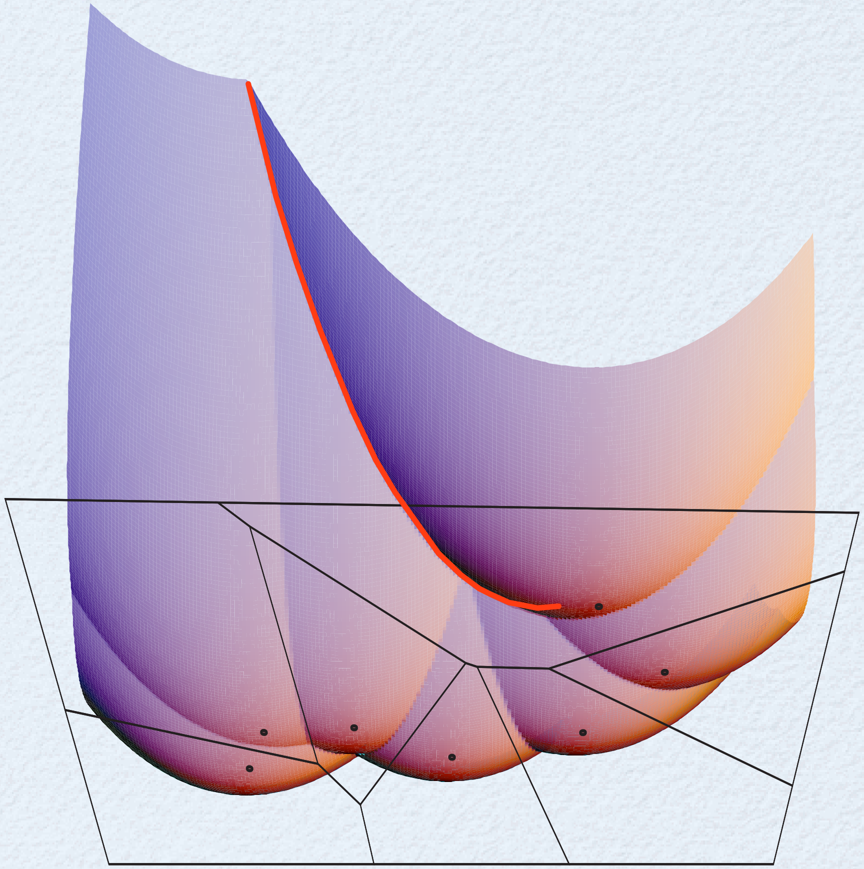

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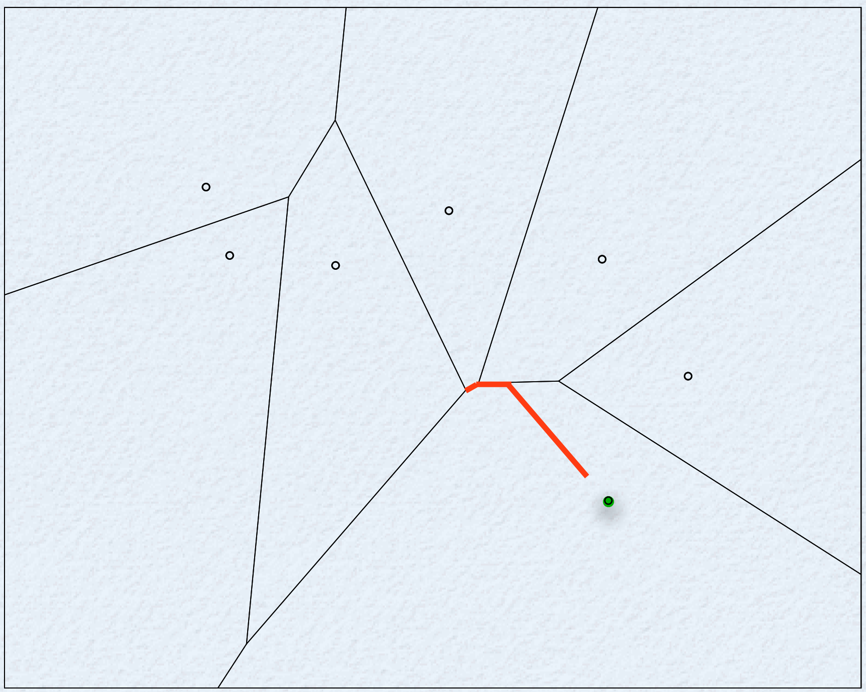

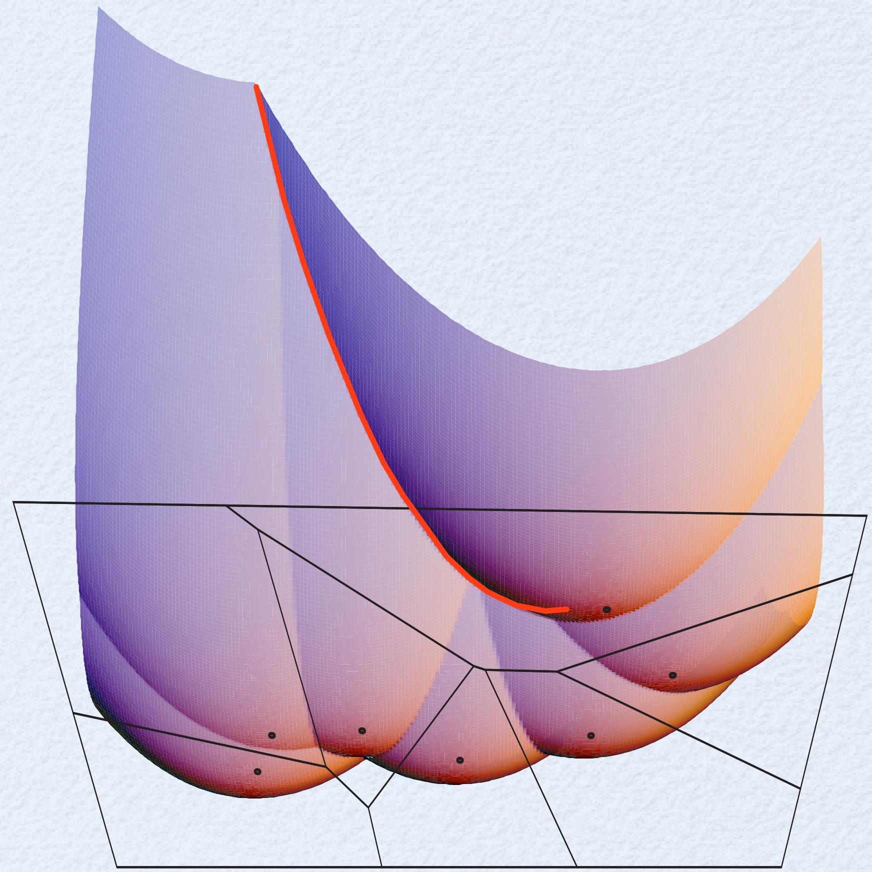

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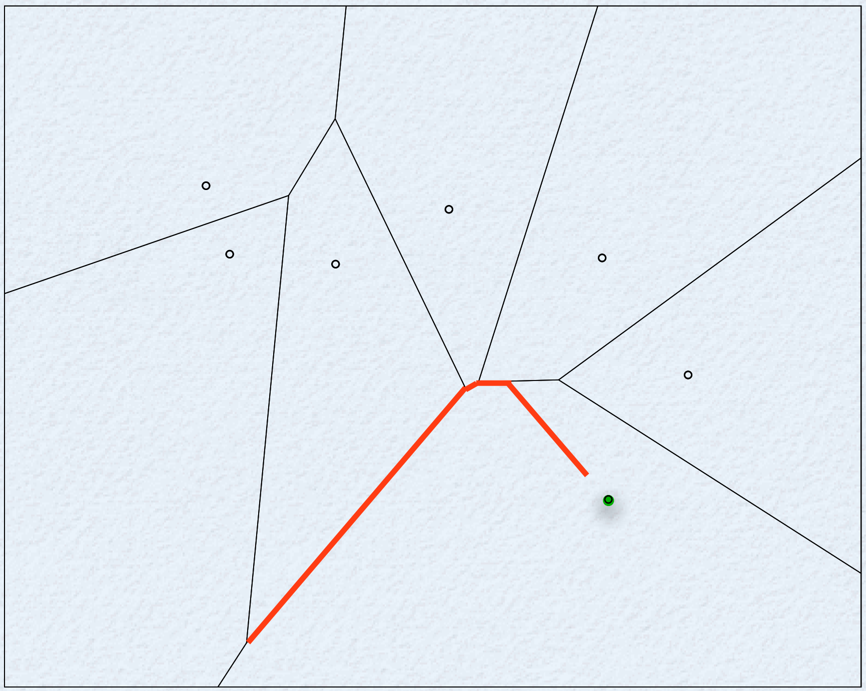

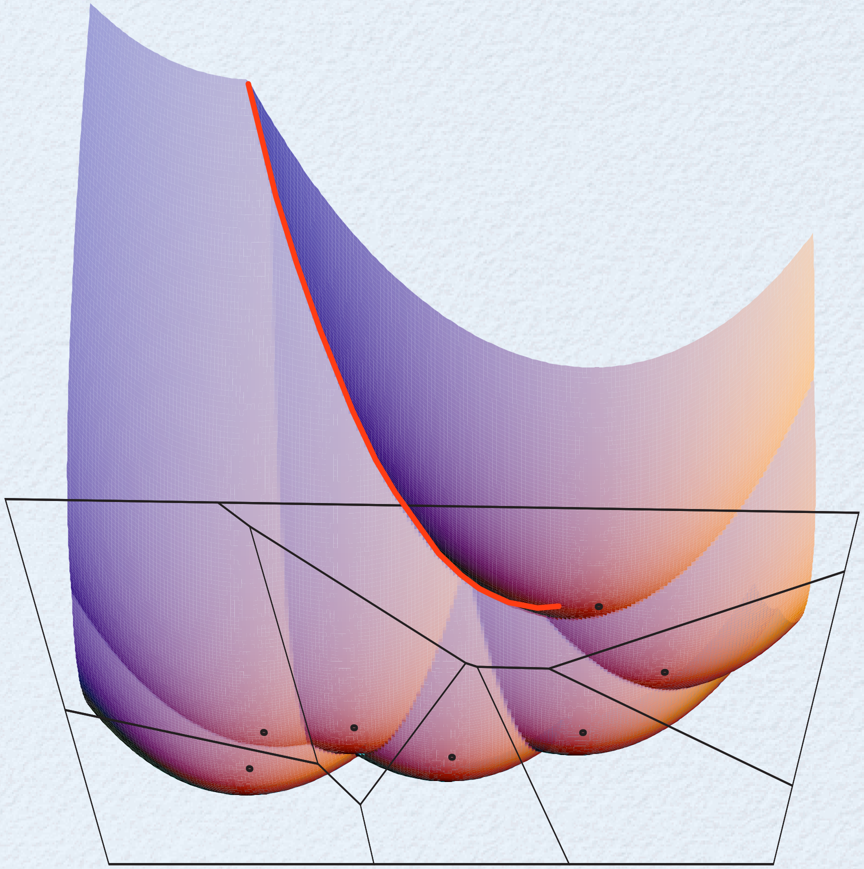

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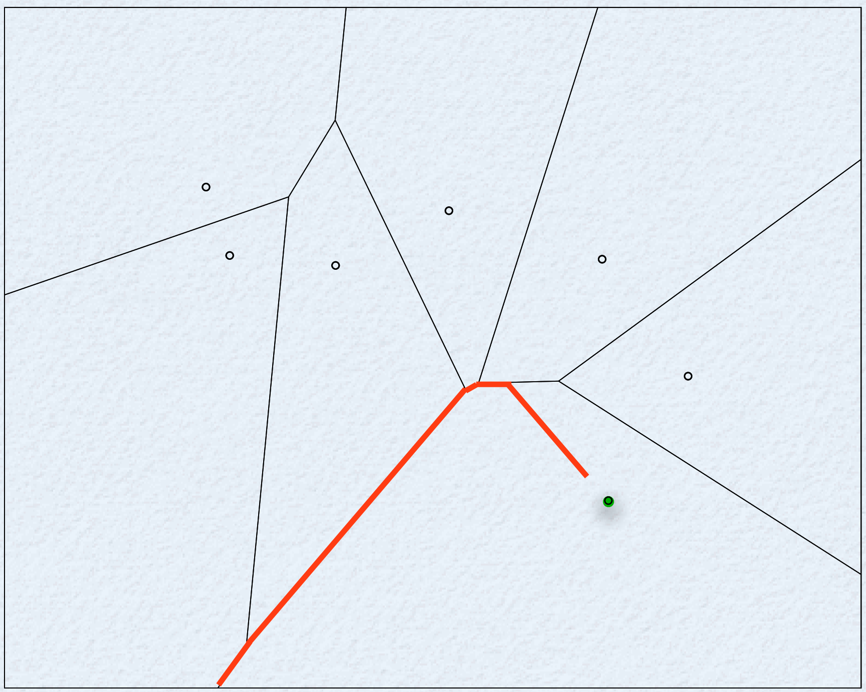

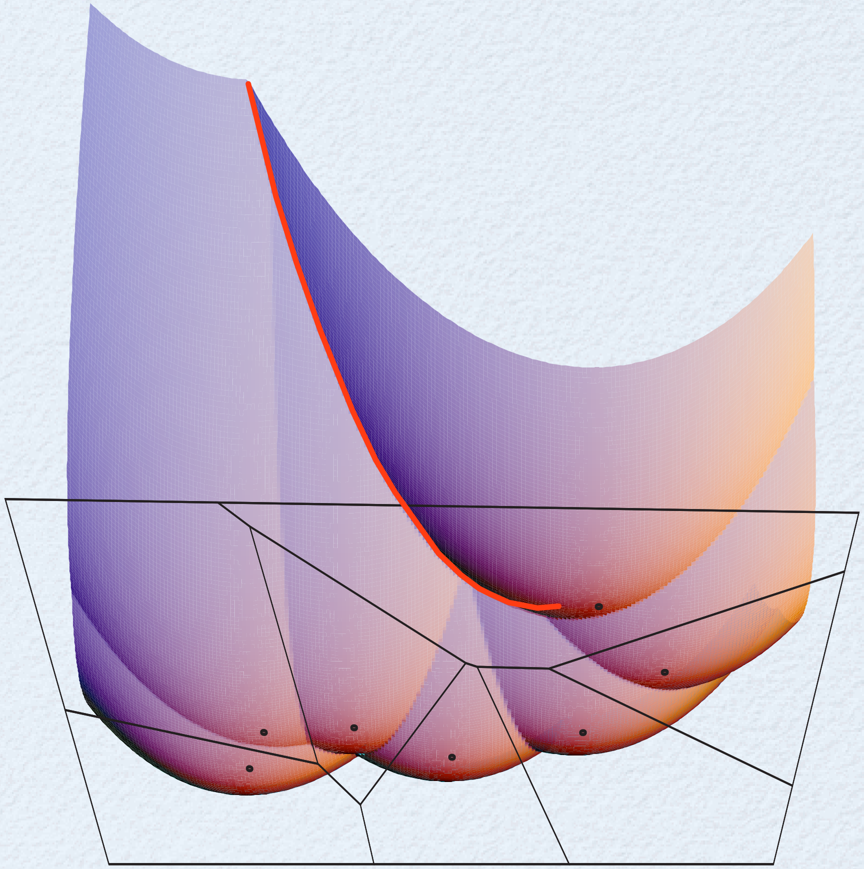

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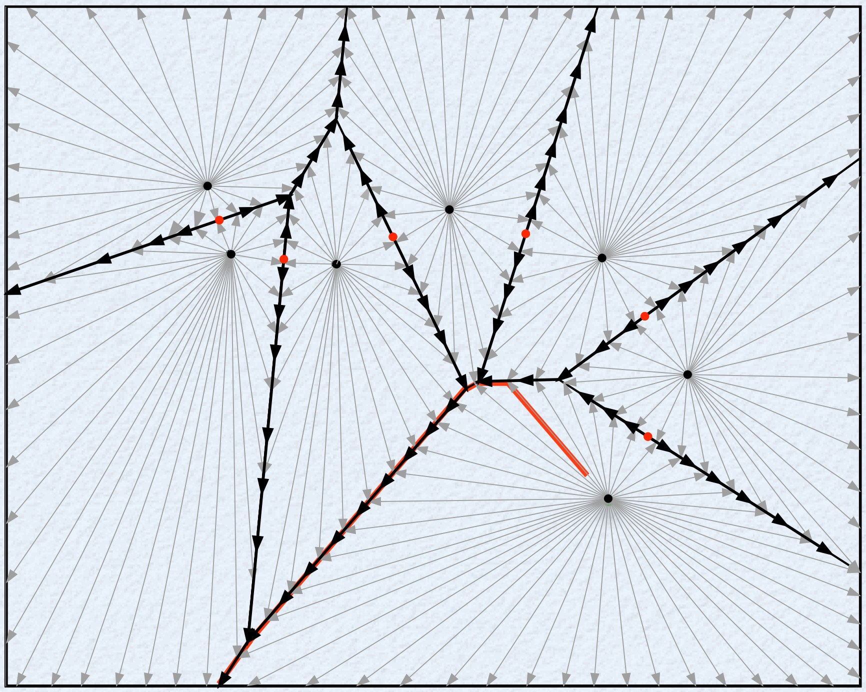

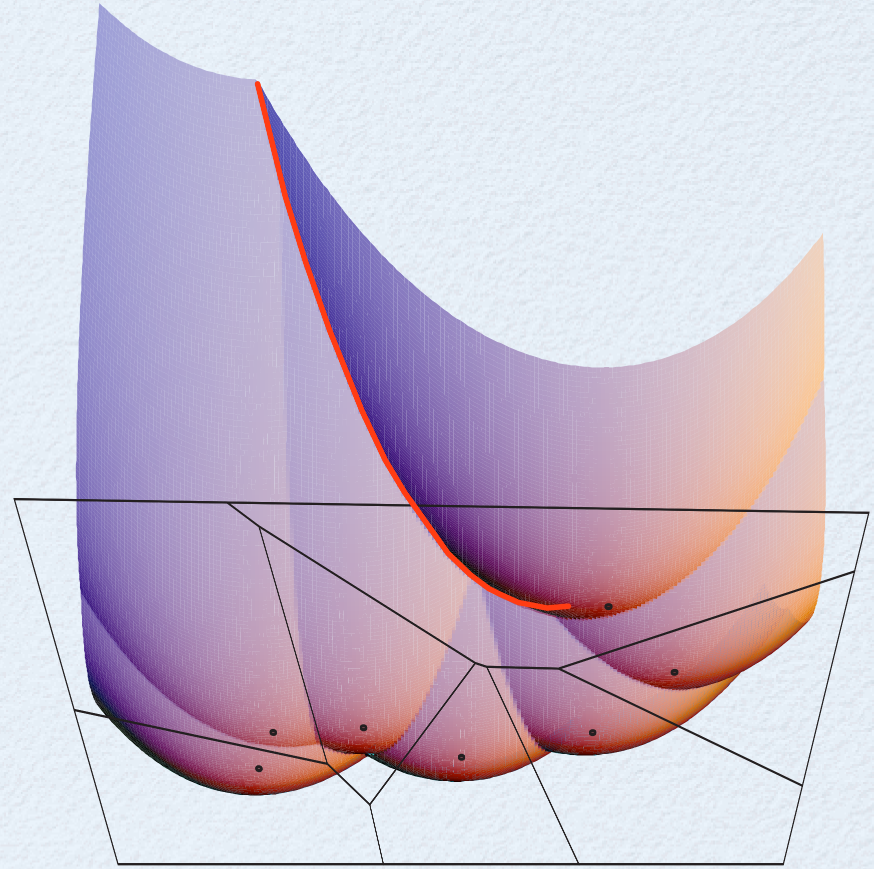

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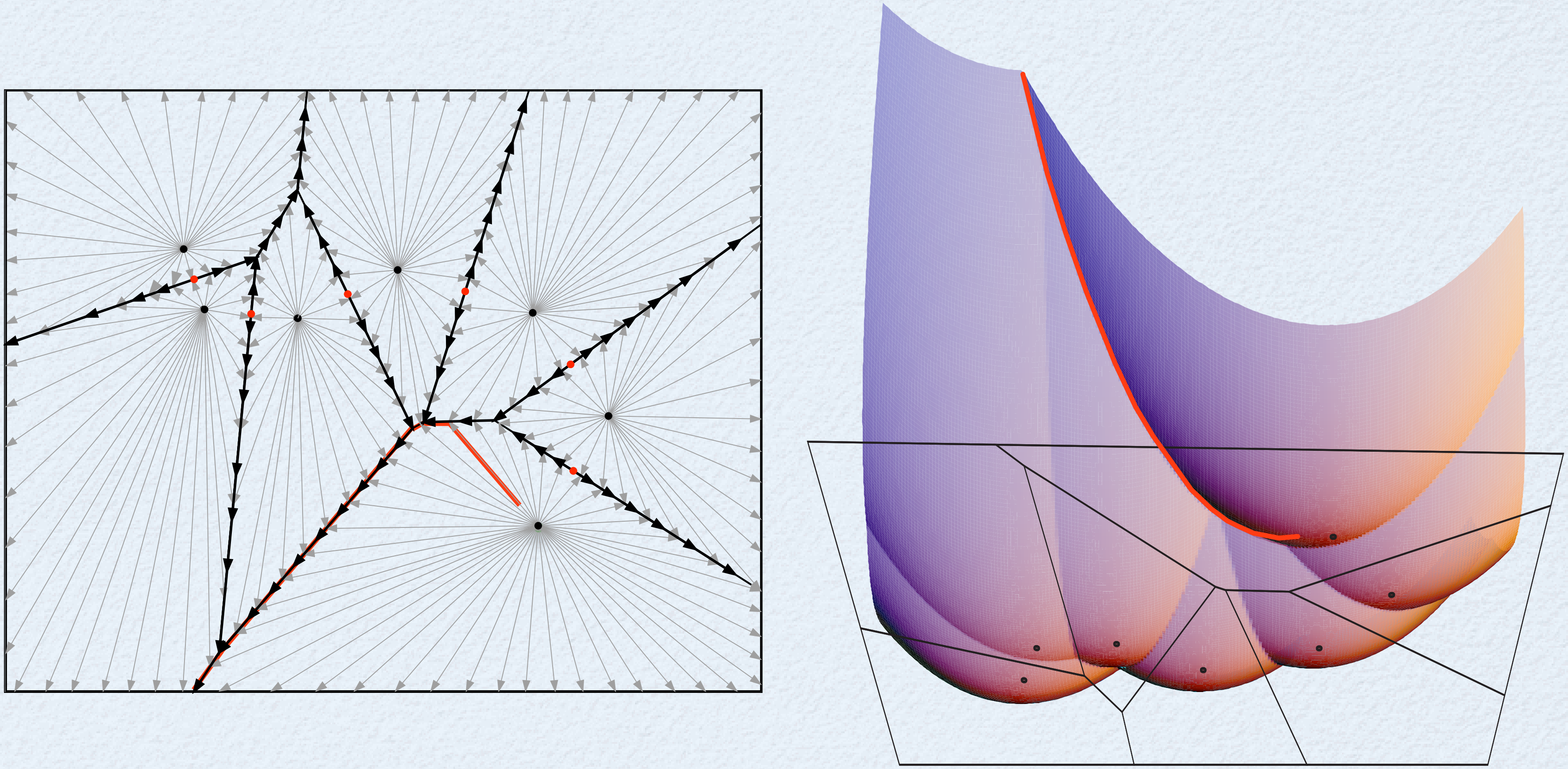

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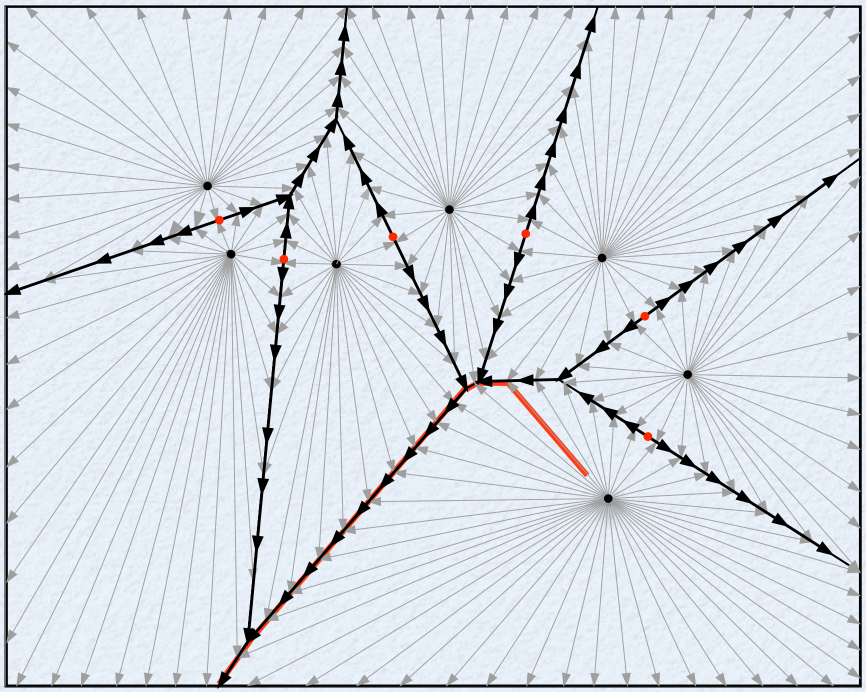

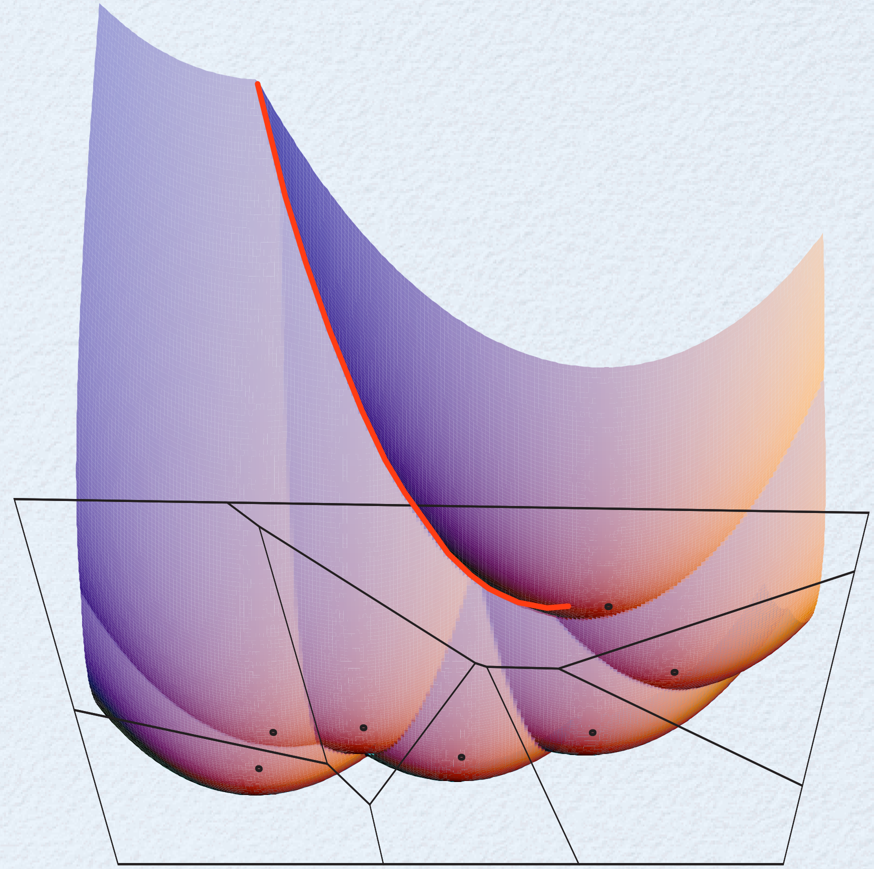

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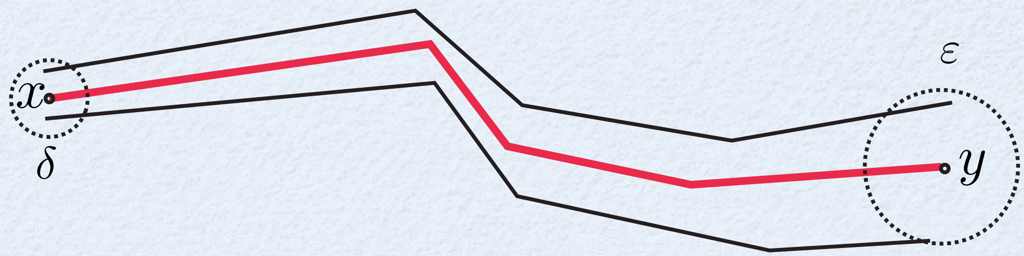
Continuity of the Induced Flow

Theorem. The flow map $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous on both variables.



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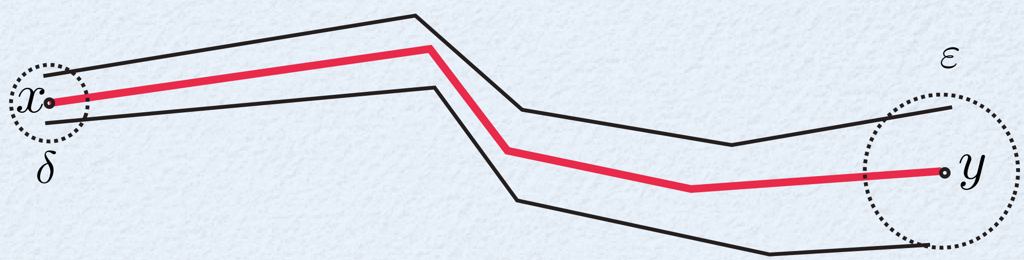
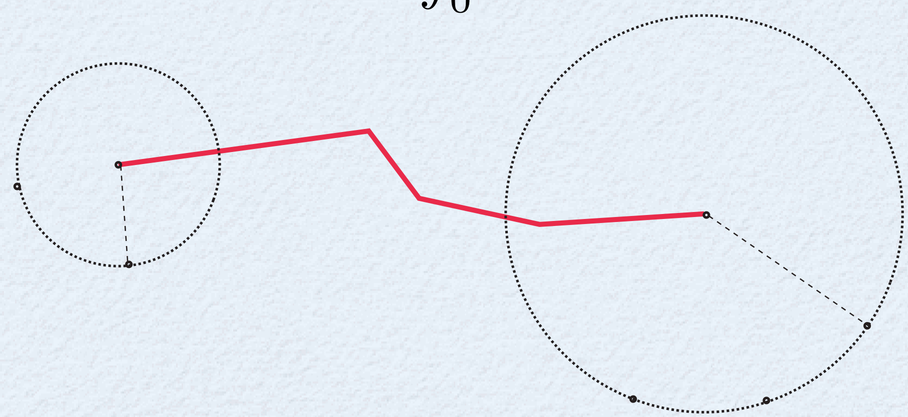


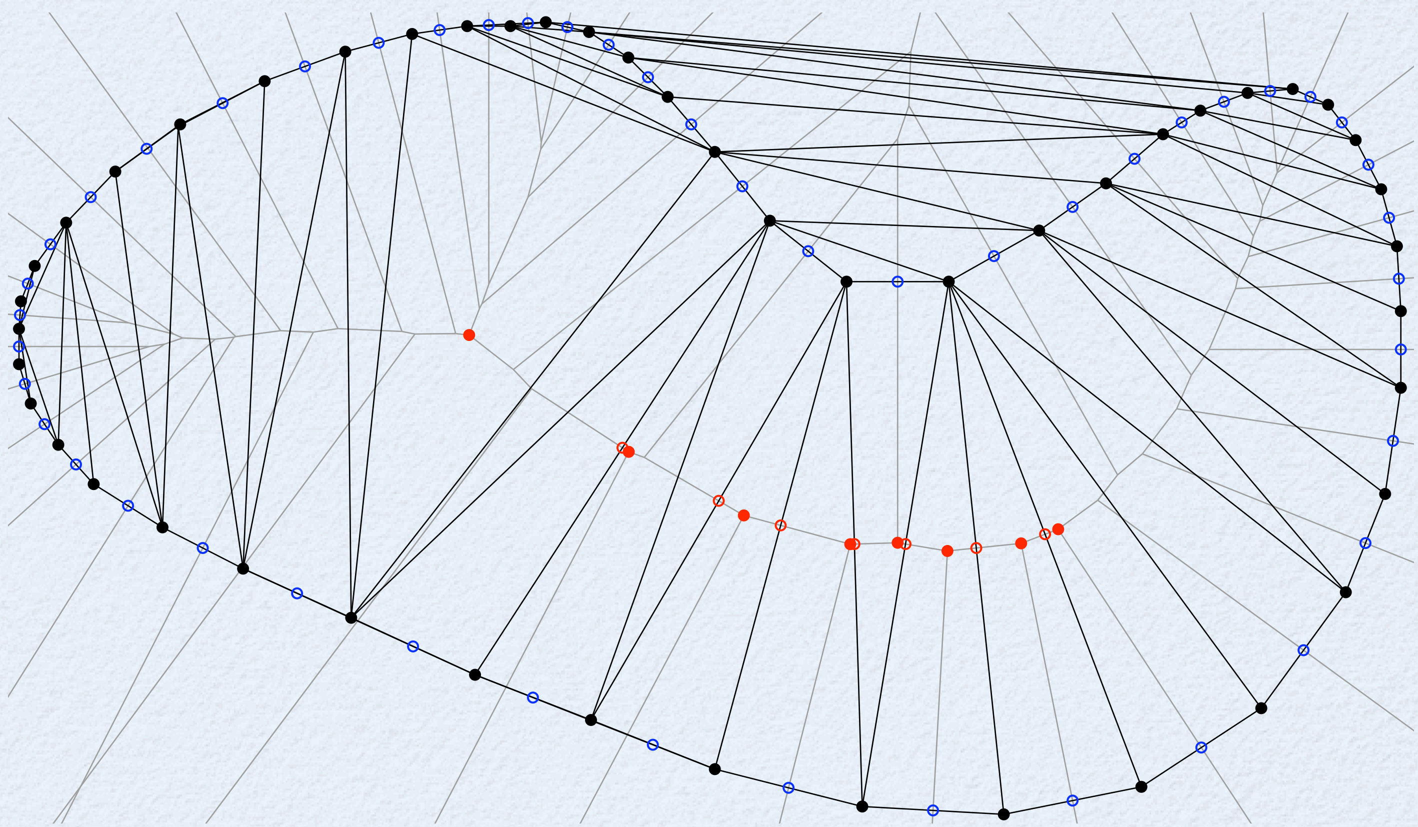
Theorem. For $y = \phi(t, x)$,

$$h(y) = h(x) + \int_0^t \|v(\phi(\tau, x))\|^2 d\tau.$$



Critical Points of Distance Function

A point c with $v(c) = 0$ is called critical.



The index of a critical point c is the dimension of $D(c)$.

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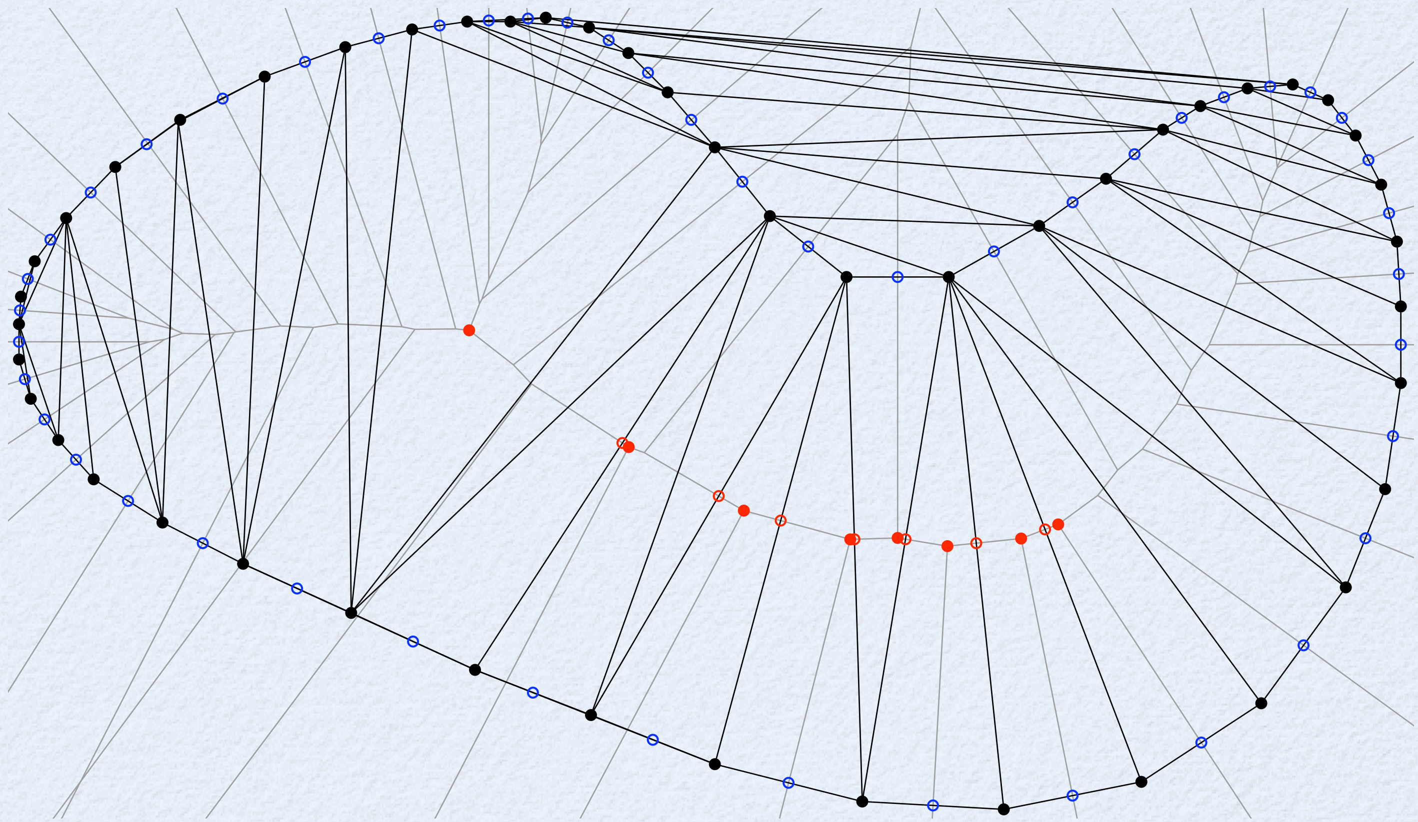

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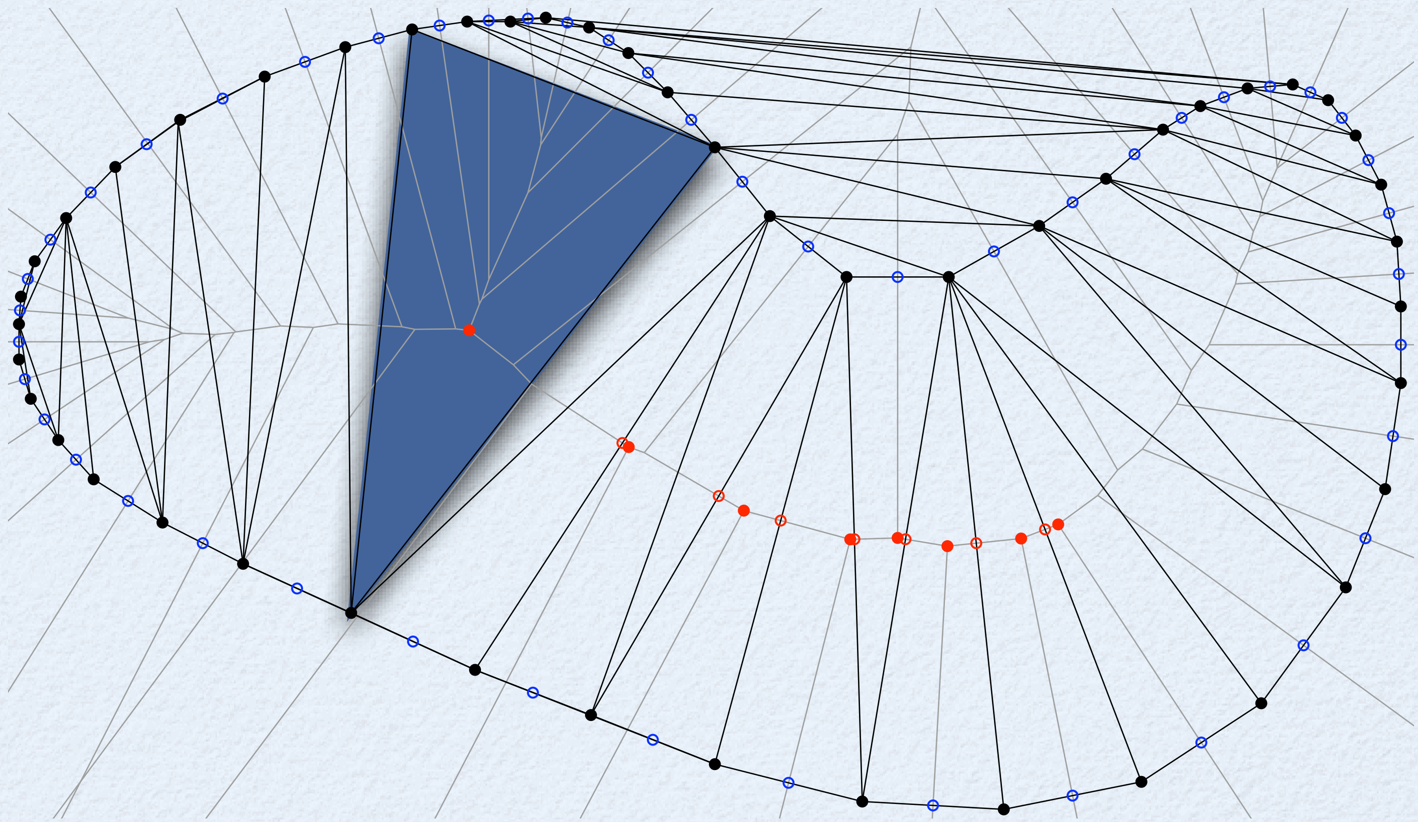

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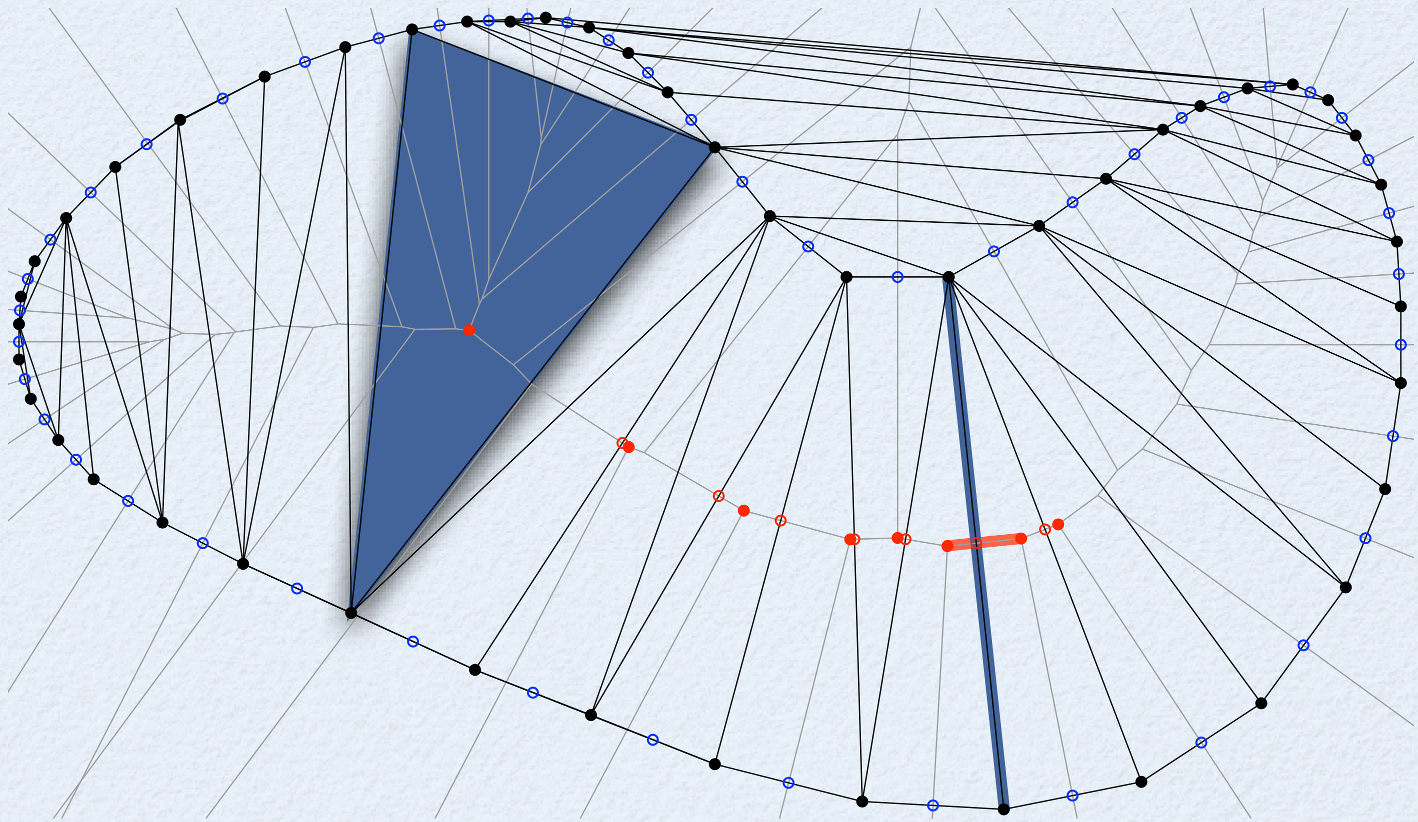

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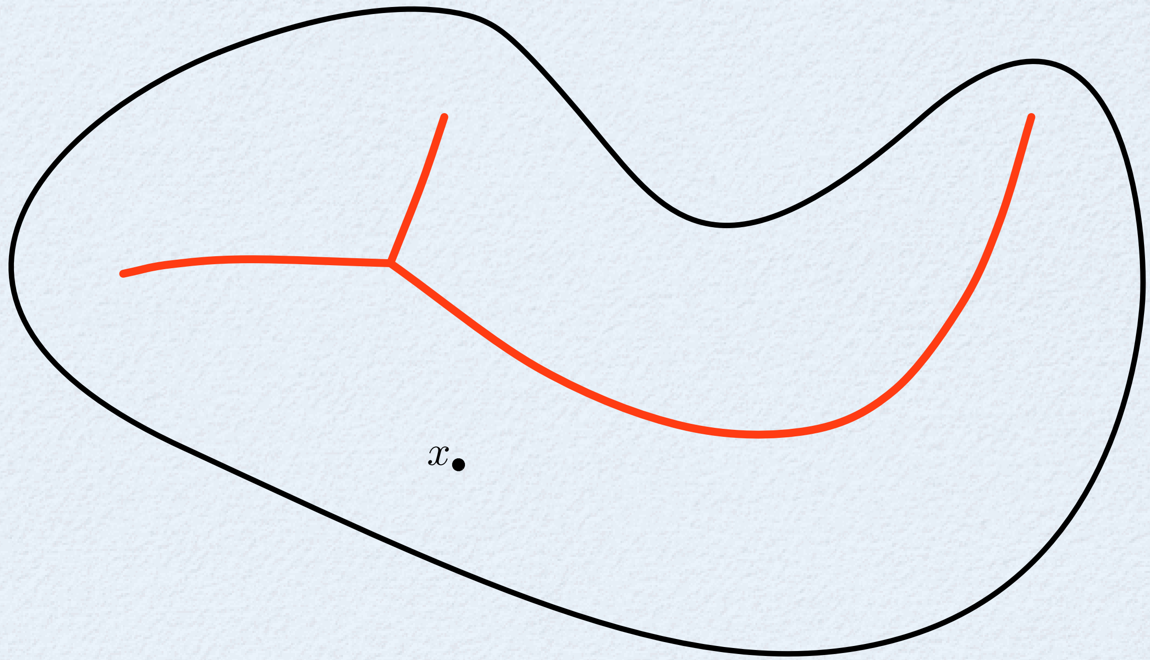

The δ-tubular neighborhoods of Σ and M:

$$\begin{aligned} \Sigma_\delta &= \{x \in \mathbb{R}^n \setminus M : \|x - \hat{x}\| < \delta f(\hat{x})\} \\ M_\delta &= \{x \in \mathbb{R}^n \setminus \Sigma : \|x - \check{x}\| < \delta f(\hat{x})\} \end{aligned}$$

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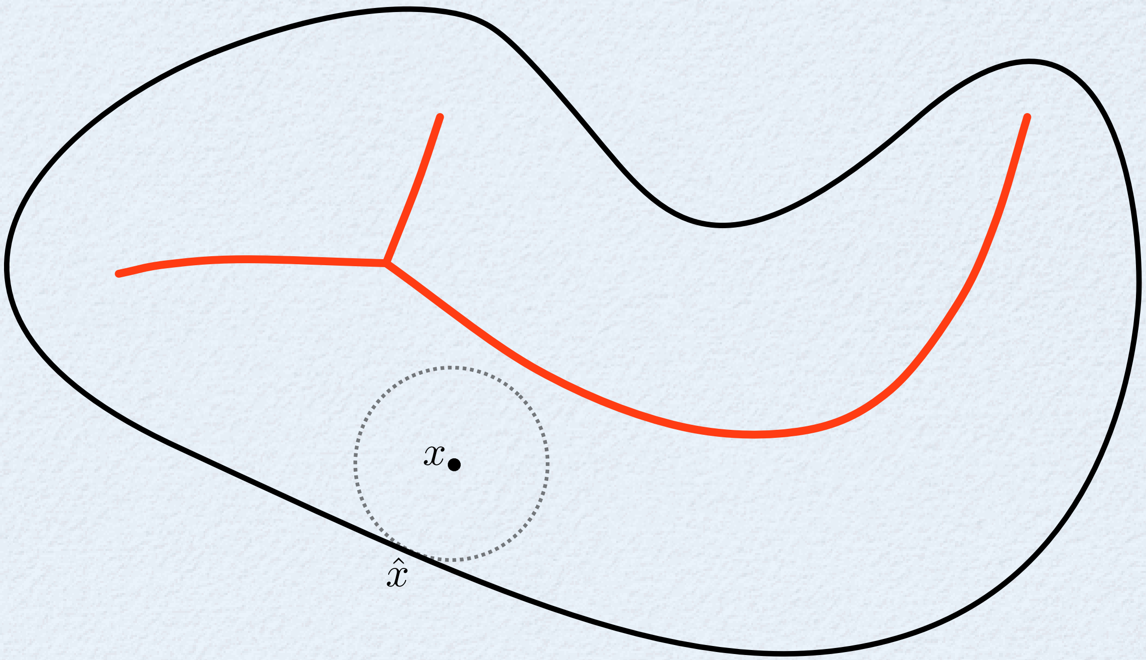

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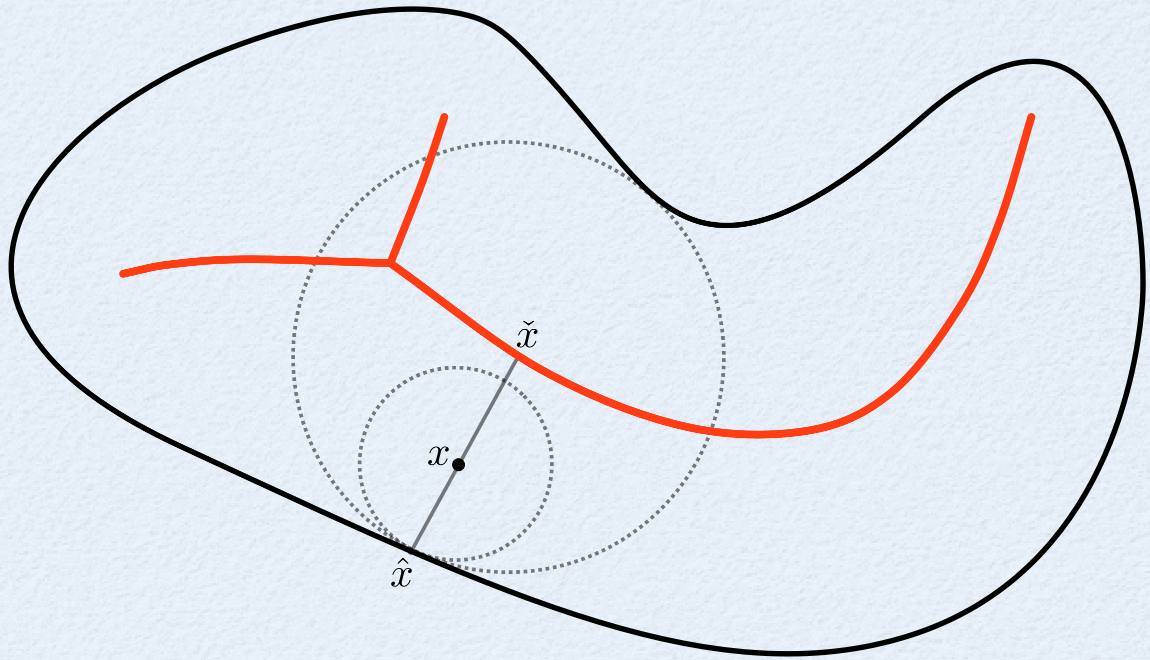

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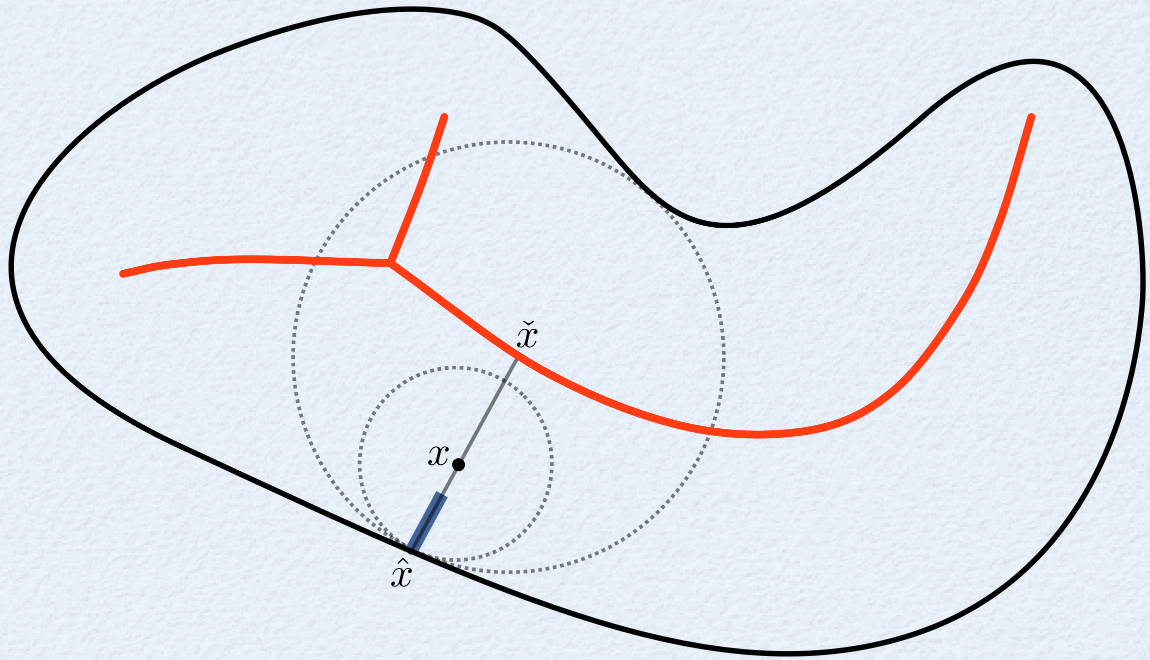

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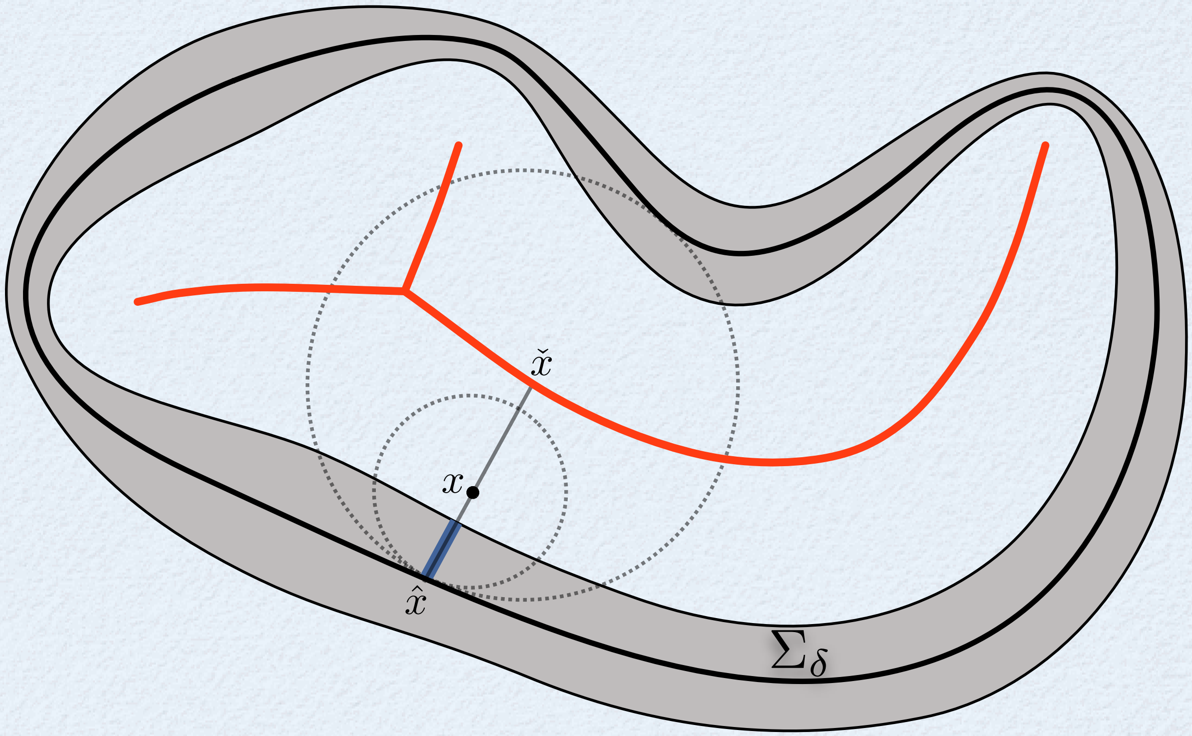

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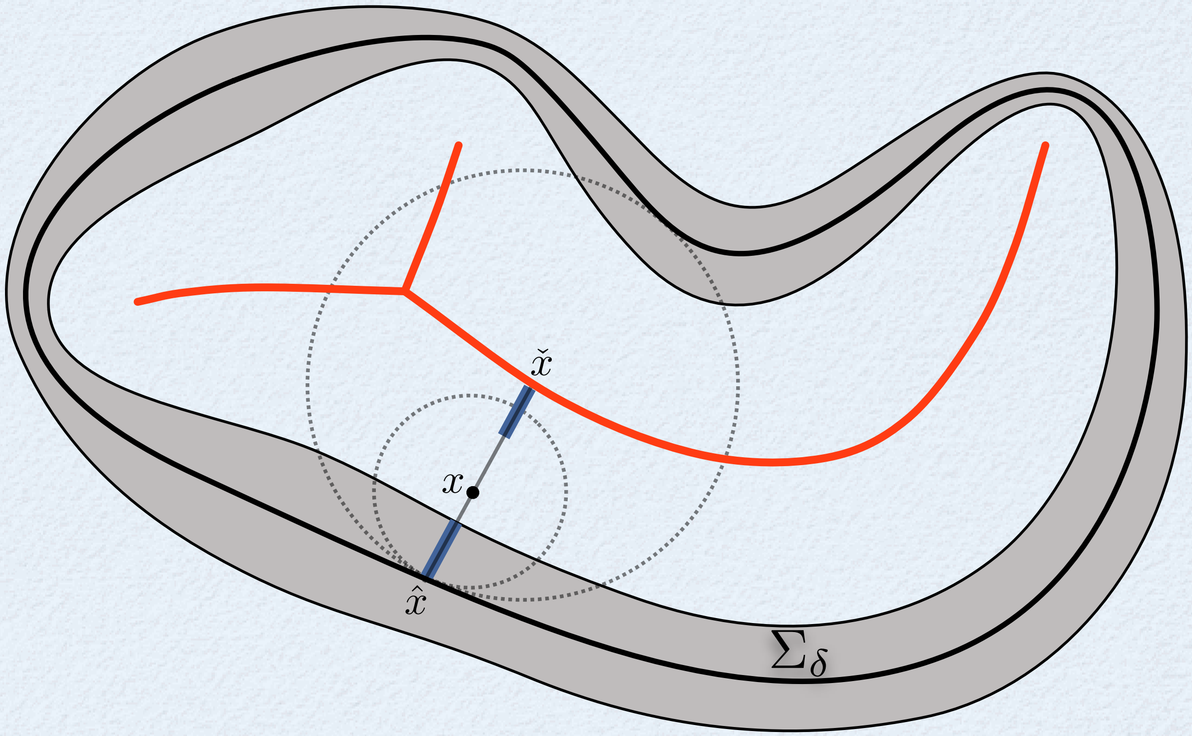

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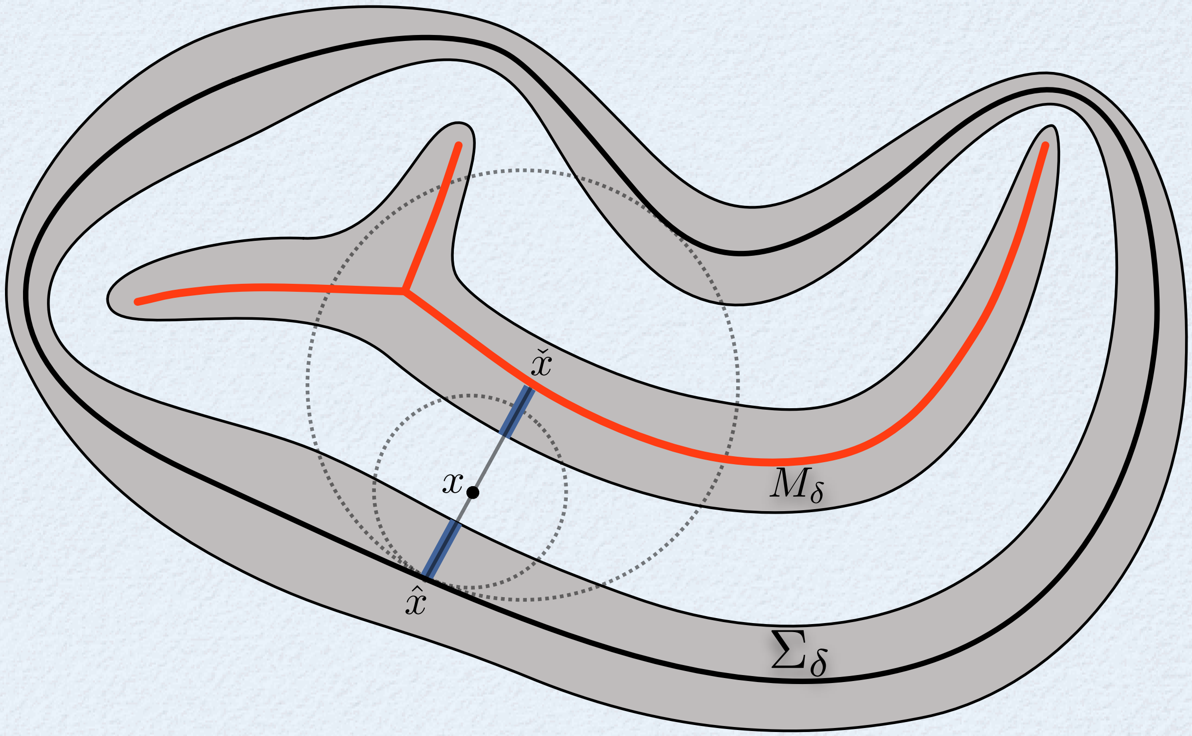

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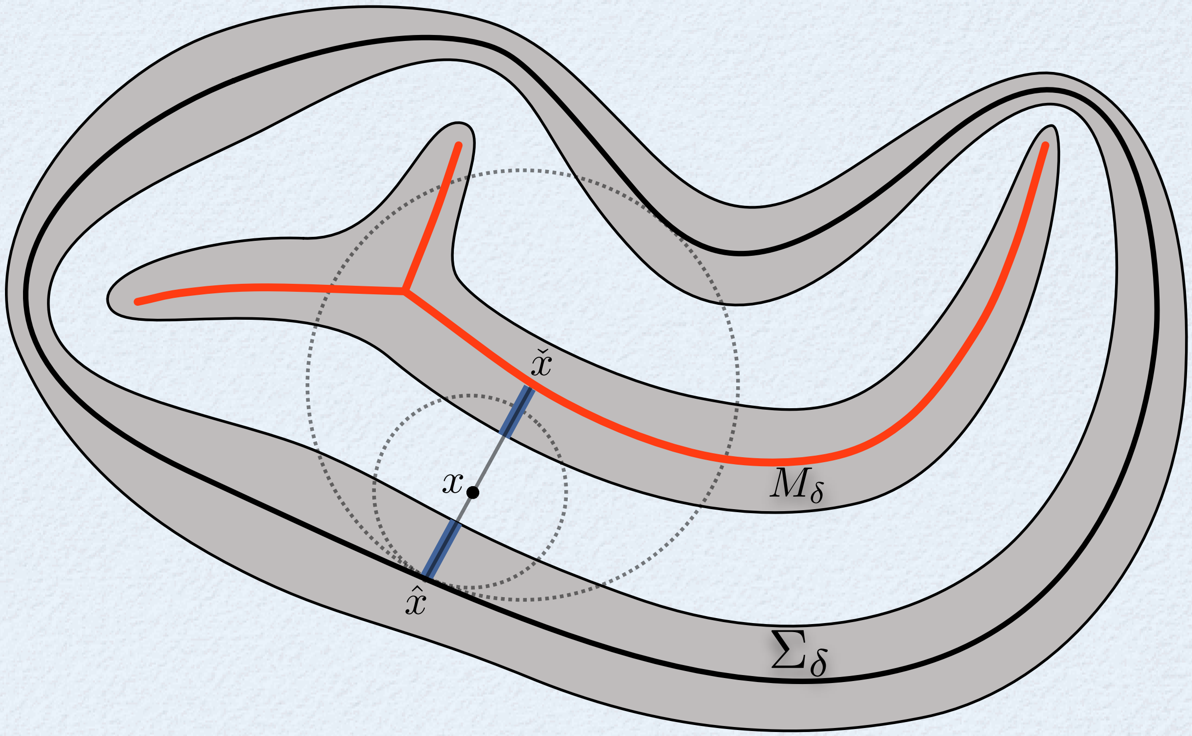
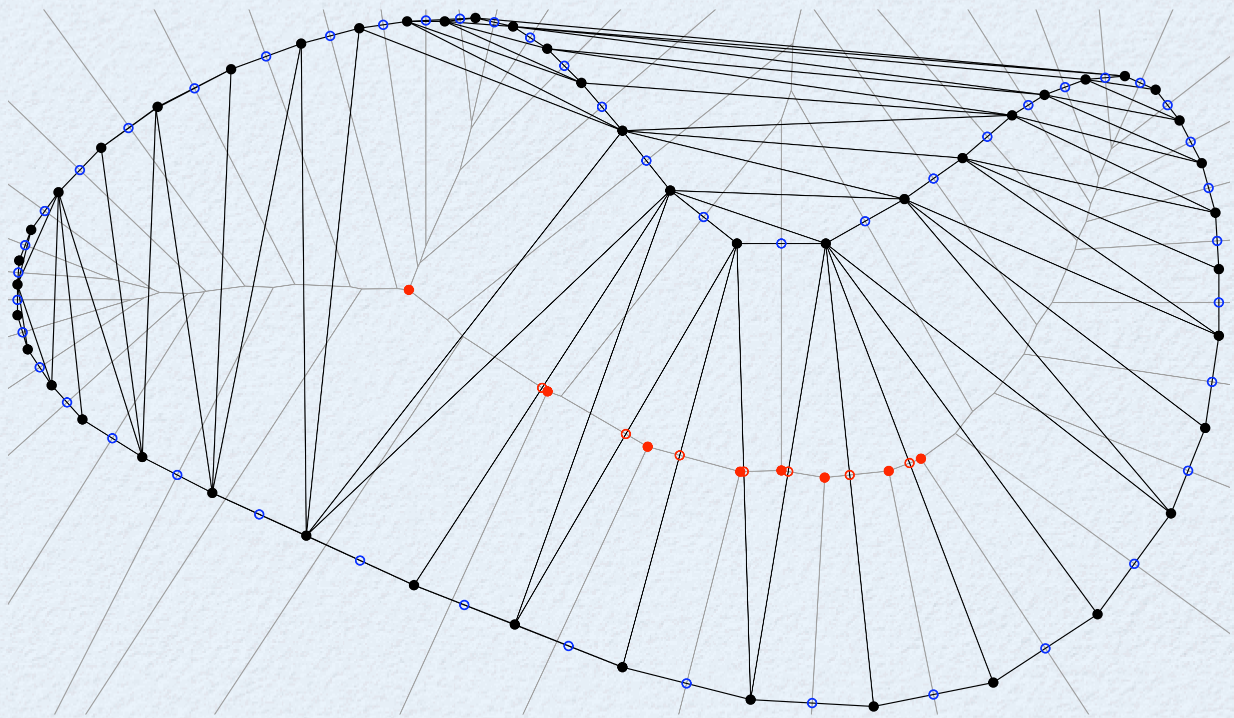


Theorem [DGRS'05]
If h is induced by an ε-sample of Σ with $\varepsilon < 1/\sqrt{3}$, the all critical points of h are contained in either Σ_{ε^2} or $M_{2\varepsilon^2}$.

Stable Manifold of a Critical Point

Stable manifold of a critical point c is everything that flows into c.

$$\mathsf{Sm}(c) = \{x : \phi(\infty, x) = c\}.$$



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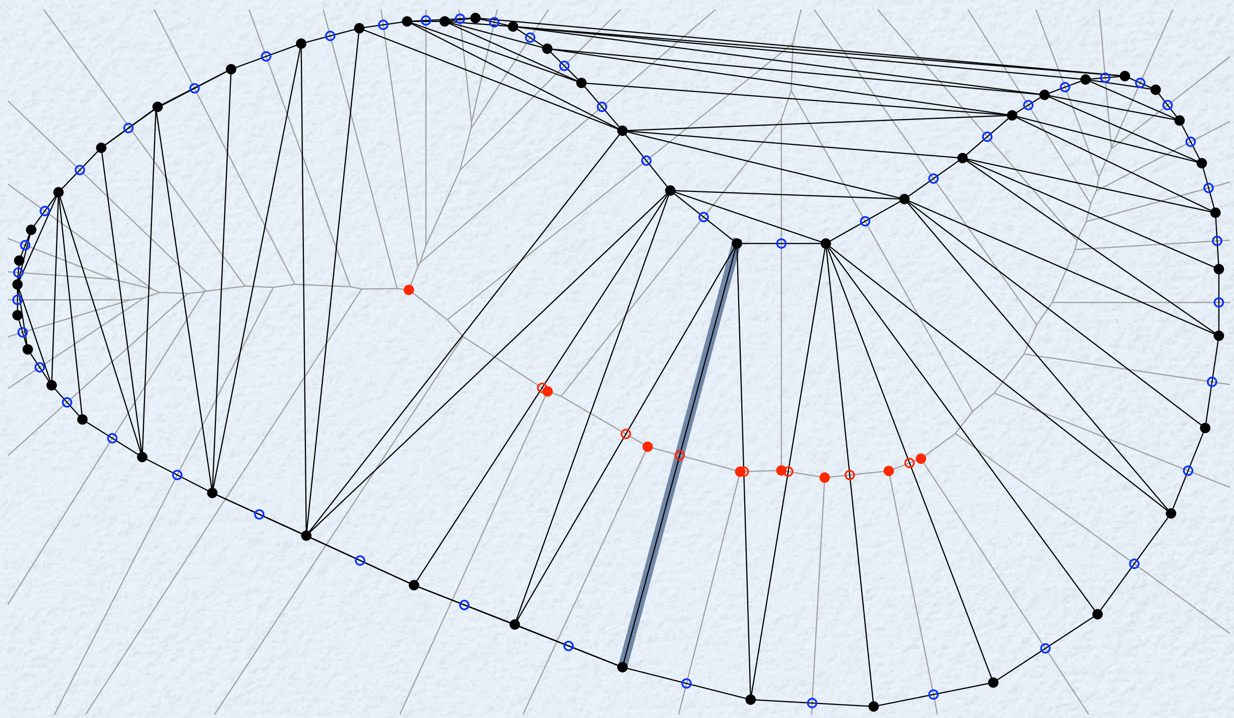

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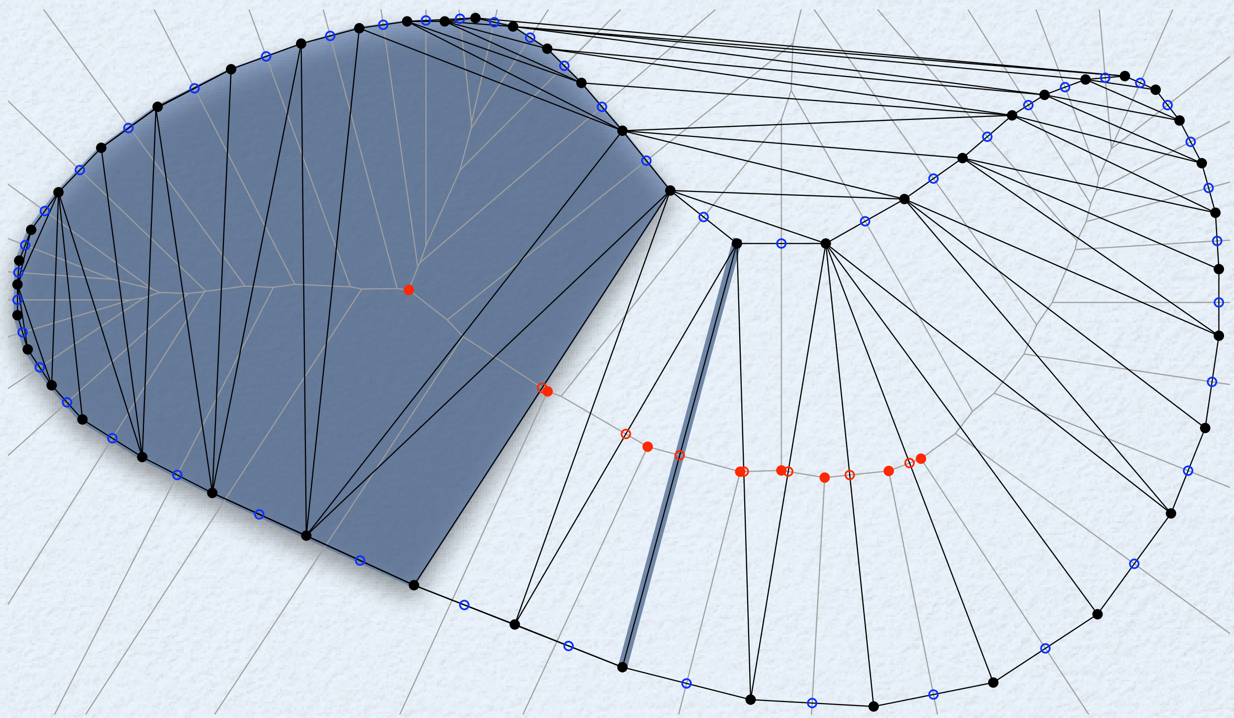

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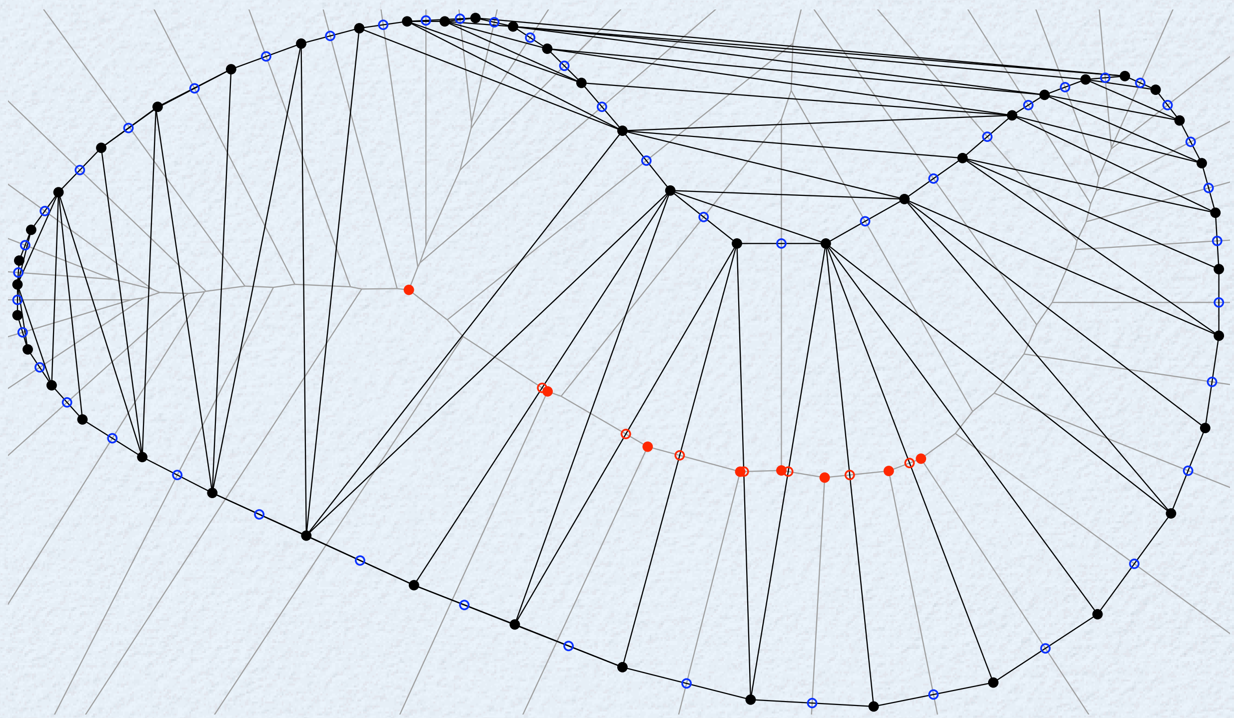

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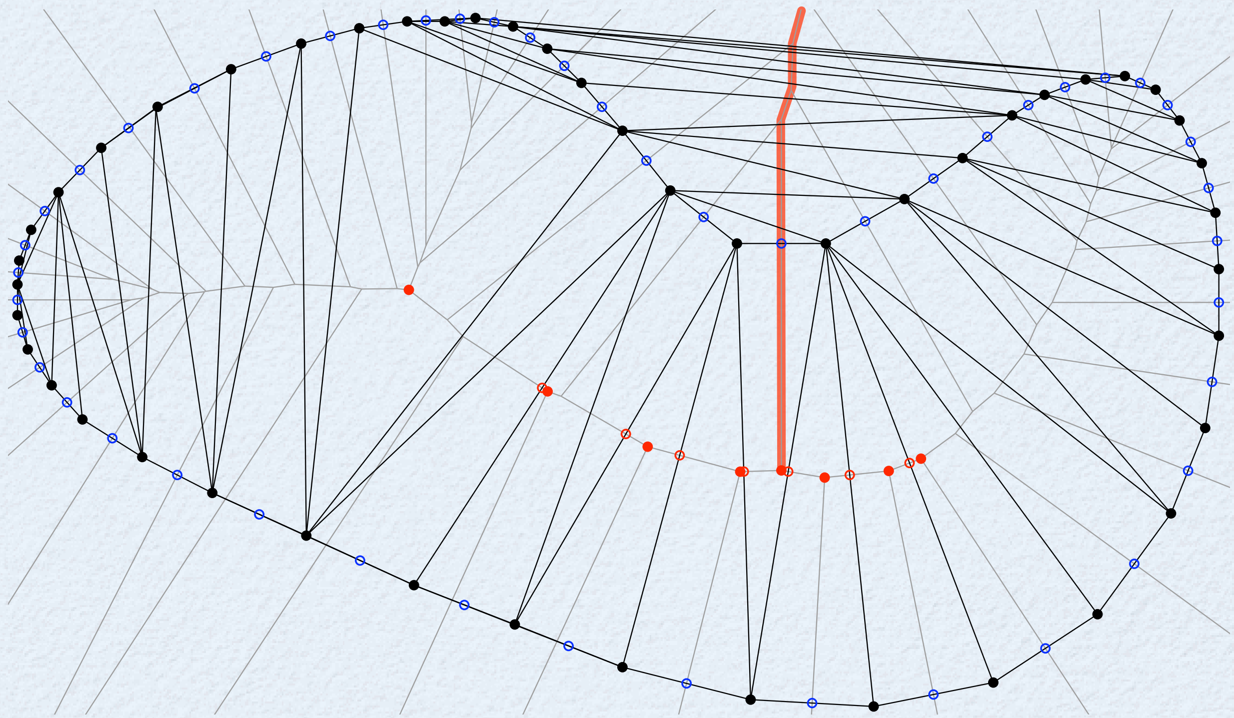

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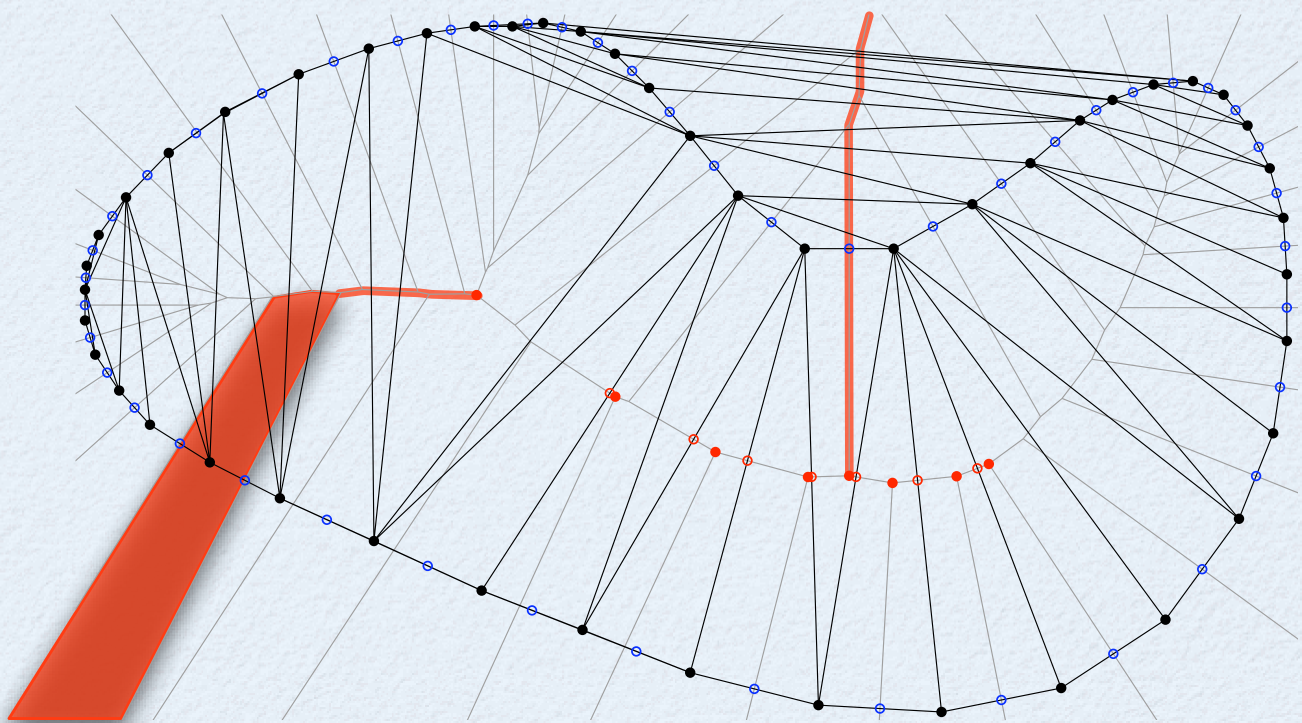


A Criterion for Homotopy Equivalence

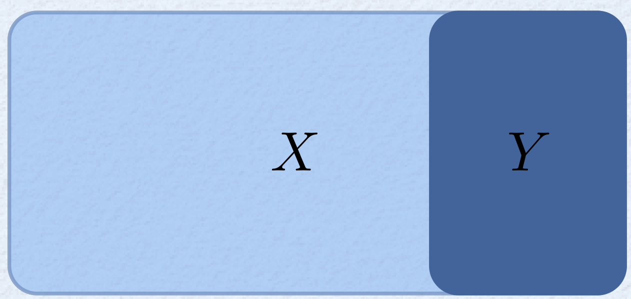
Proposition. Let X and $Y \subseteq X$ be arbitrary sets and

$$H : [0,1] \times X \to X$$

be a continuous function (on both variables) satisfying

1. $\forall x \in X : H(0,x) = x$
2. $\forall y \in Y, \forall t \in [0,1] : H(t,y) \in Y$
3. $\forall x \in X : H(1,x) \in Y$

Then X and Y have the same homotopy type.



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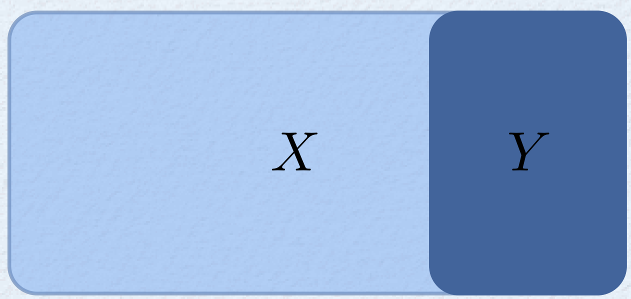

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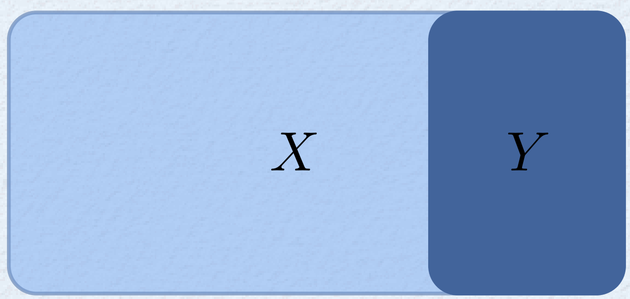

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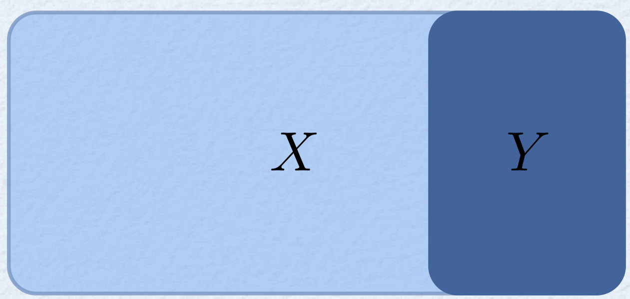

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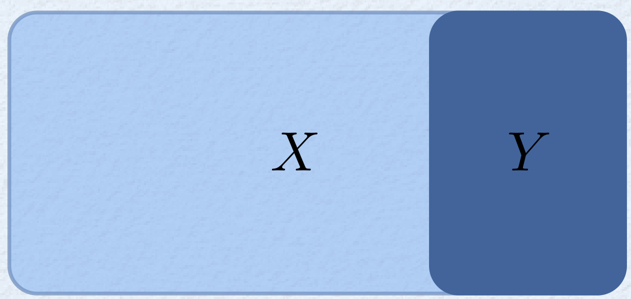

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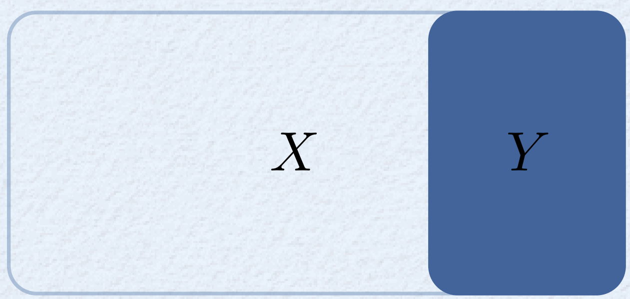

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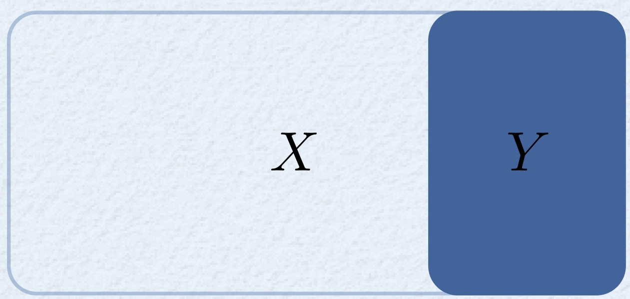

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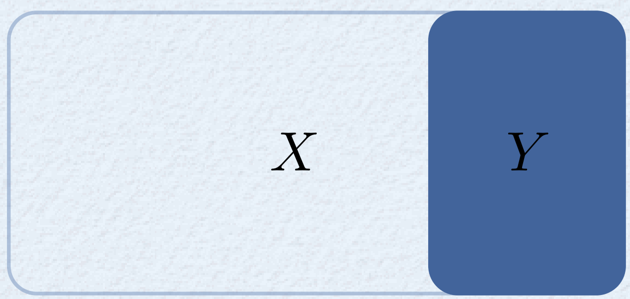

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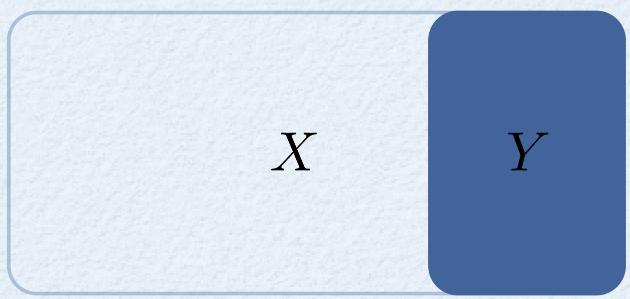

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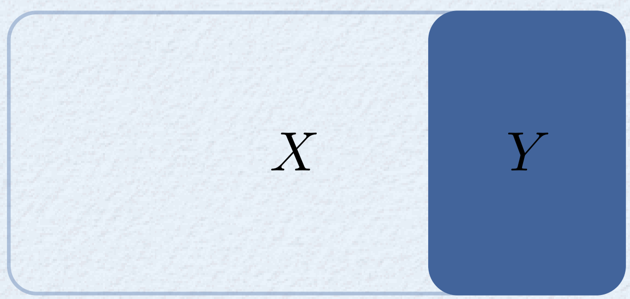

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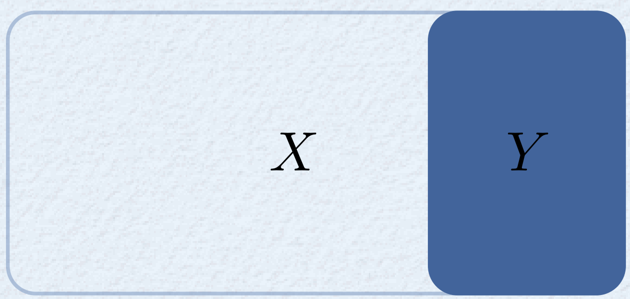

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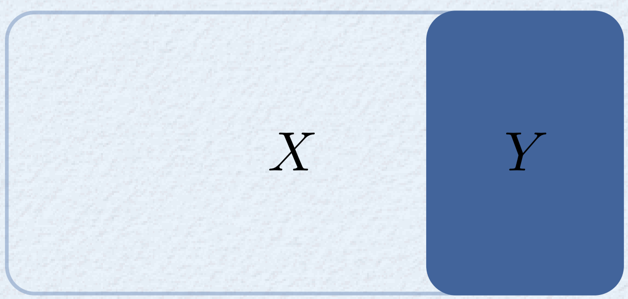

This is the idea Lieutier used in [Lieuteir'04] to show $M(S) \simeq S$.

In Other Words ...

Key Theorem. If $Y \subset X$ are bounded and

1. $\phi(X) = X$ and $\phi(Y) = Y$, and
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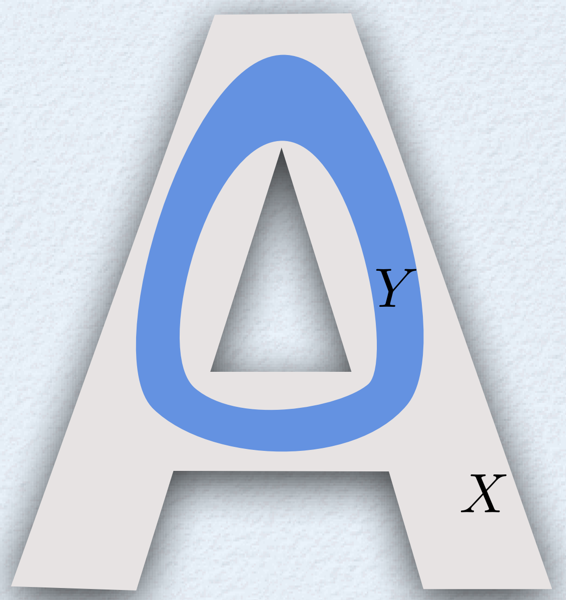

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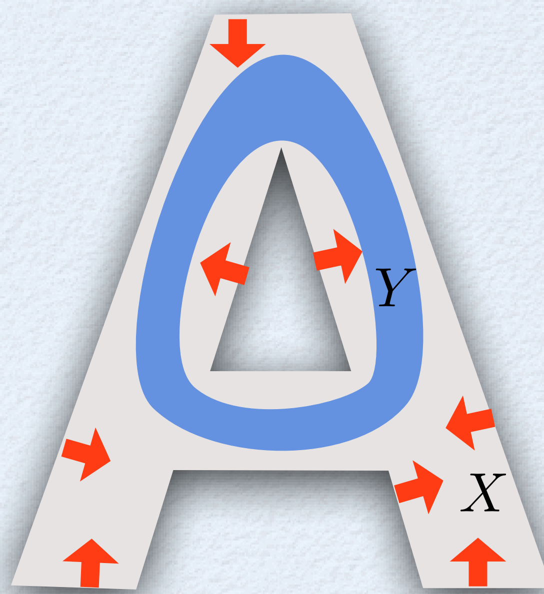

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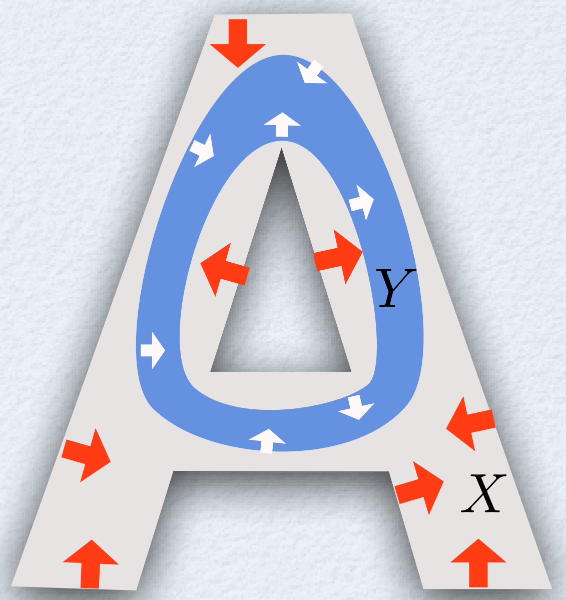

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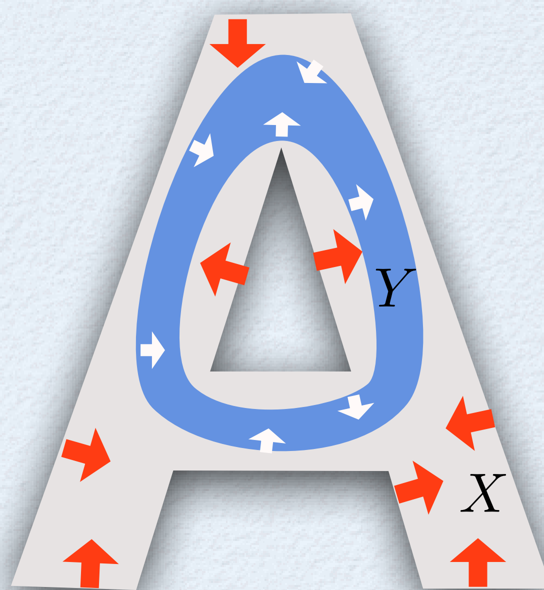


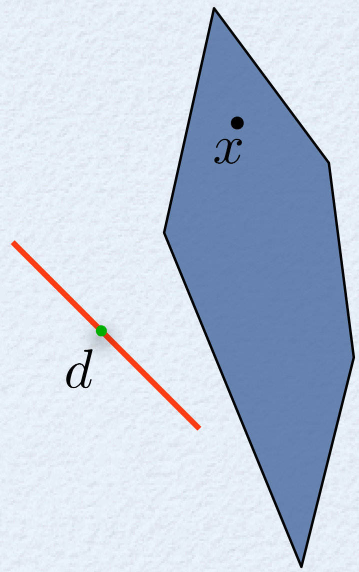
Proof. If $\phi(t,x) \notin Y$, then

$$\begin{aligned} h(\phi(t,x)) &= h(x) + \int_0^t \|v(\phi(\tau,x))\|^2 d\tau \\ &\geq h(x) + \int_0^t c^2 d\tau \\ &= h(x) + tc^2 \\ &< d_H(X,P)^2. \end{aligned}$$

A Handy Lower Bound for Speed

If $V(x) \cap D(x) = \emptyset$ then

$$\begin{aligned} \|v(x)\| &= 2 \cdot \|x - d(x)\| \\ &\geq 2 \cdot \mathsf{dist}(V(x), D(x)). \end{aligned}$$



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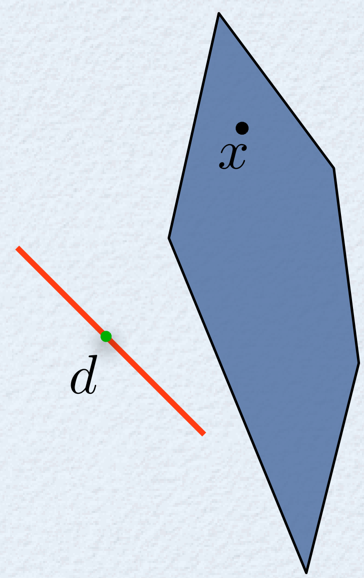
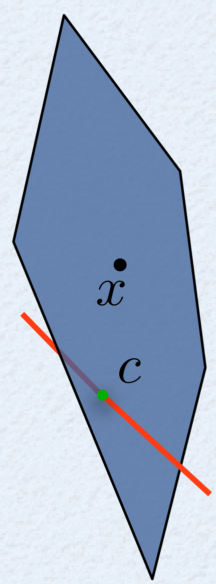


If $V(x) \cap D(x) = \{c\}$ then $x \in \mathsf{Um}(c)$.



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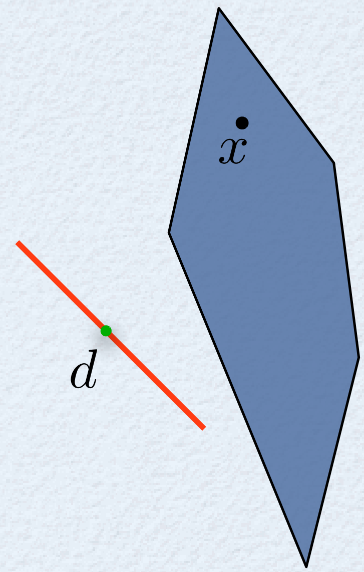

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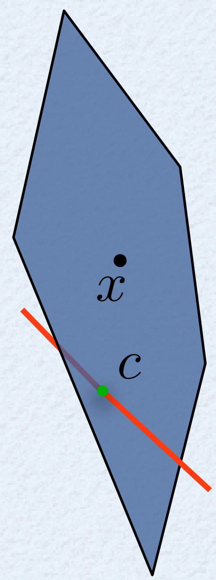


So, if $\text{Um}(c) \subset Y$ we are fine!

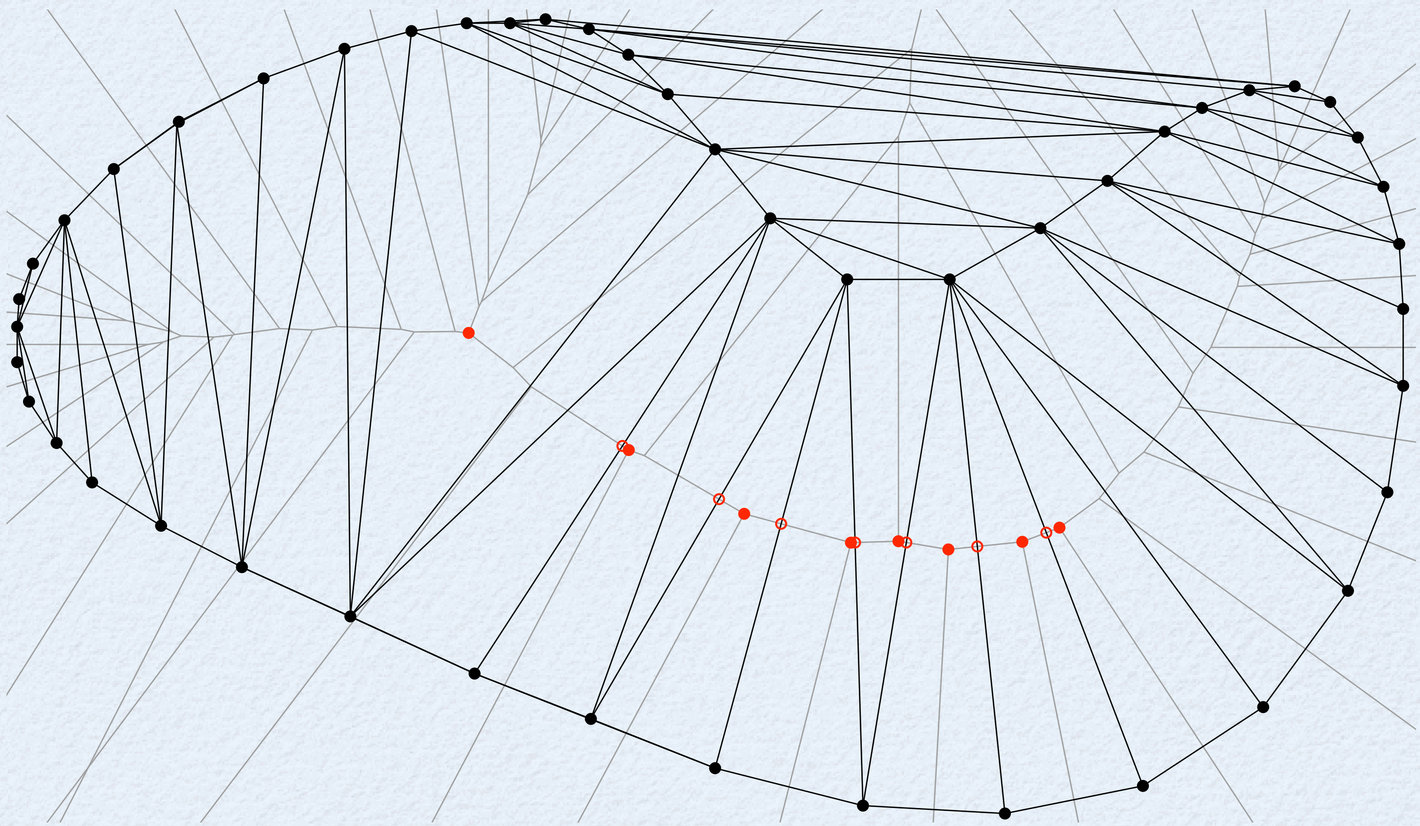
2

Medial Axis Approximation

The Inner CORE

Definition. Let N be the set of all inner medial axis critical points of h.

$$\text{CORE} = \bigcup_{c \in N} \text{Um}(c)$$



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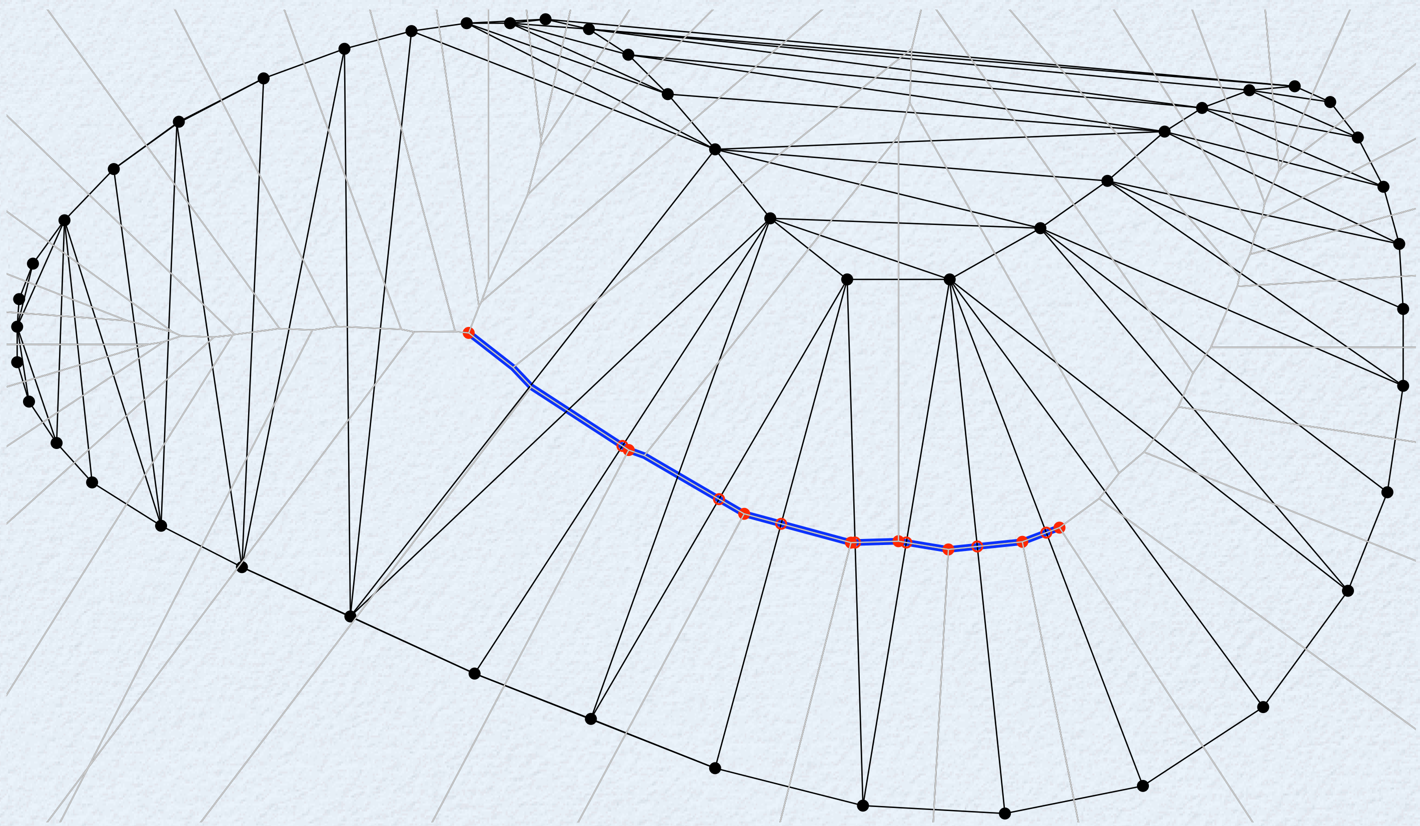
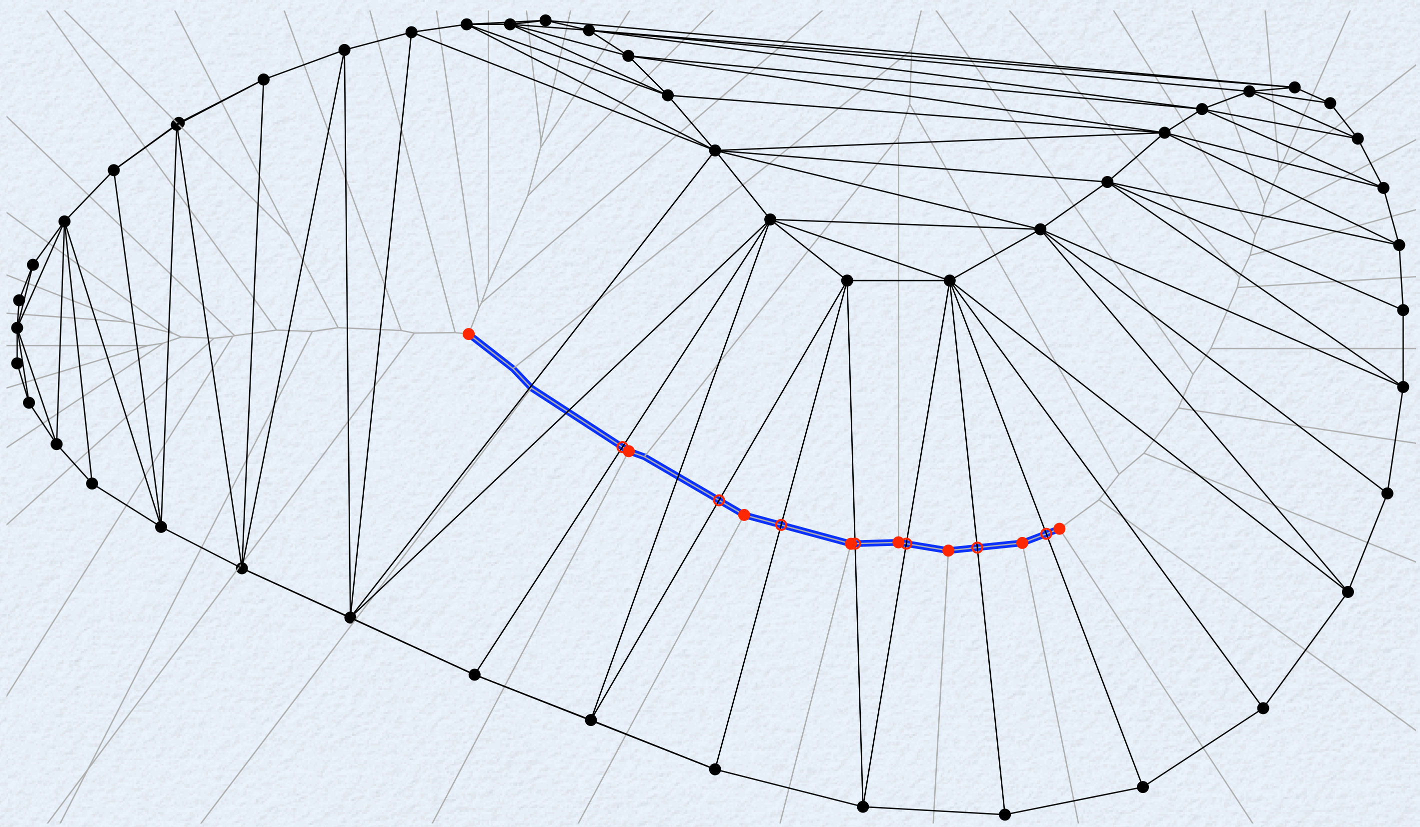


Homotopy Type of CORE

Theorem. CORE and shape are homotopy equivalent (for small ε).



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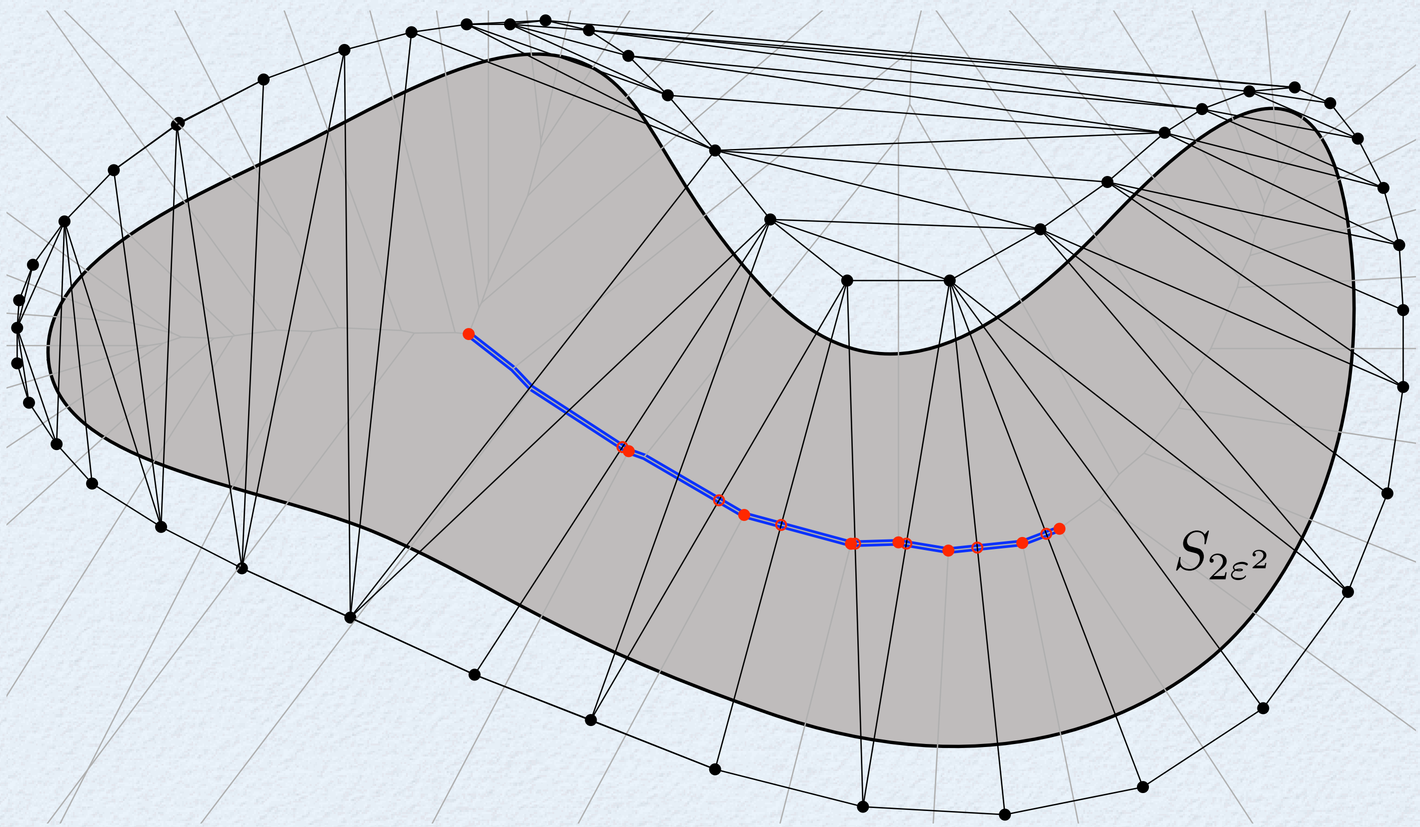

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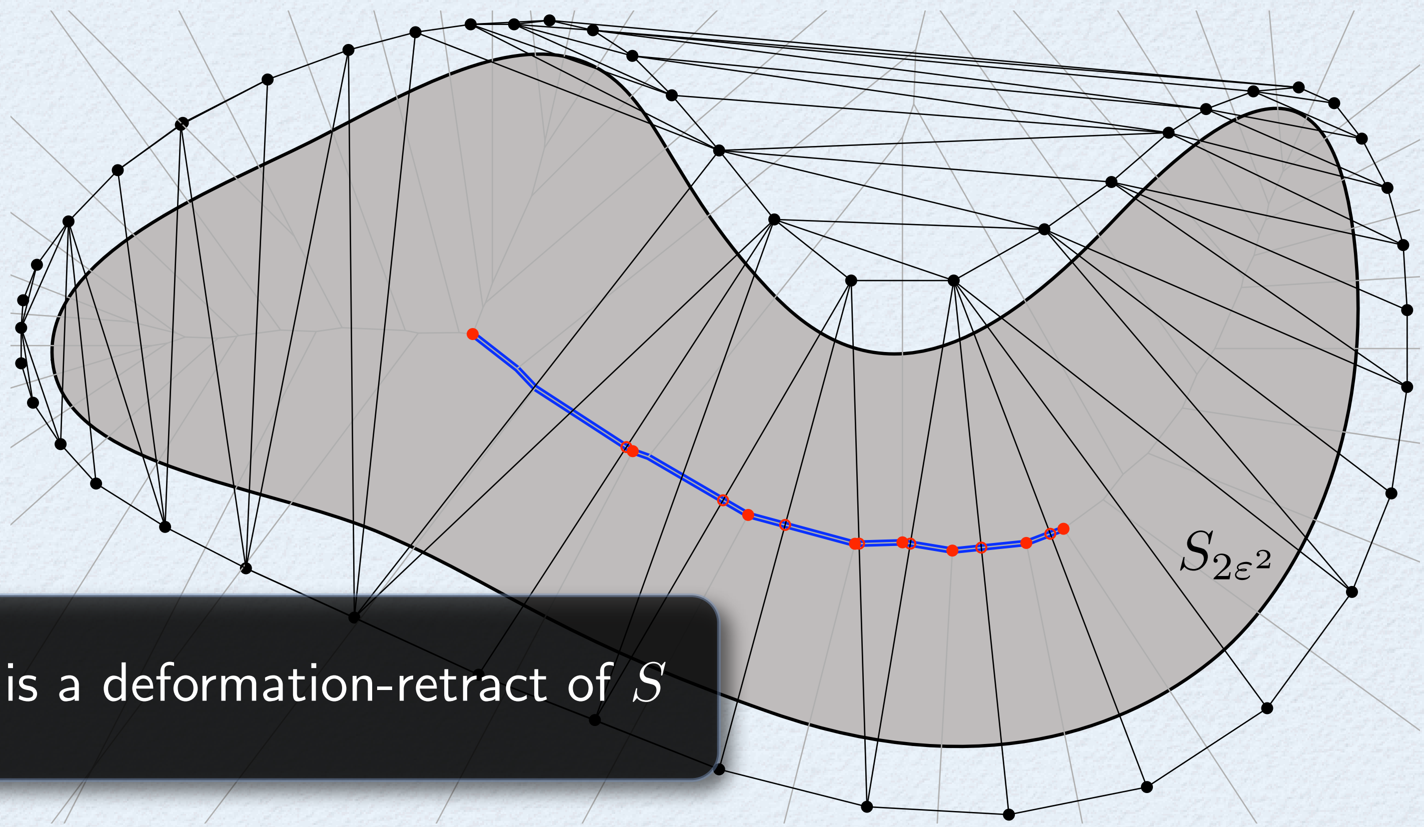

$S_{2\varepsilon^2}$ is a deformation-retract of S

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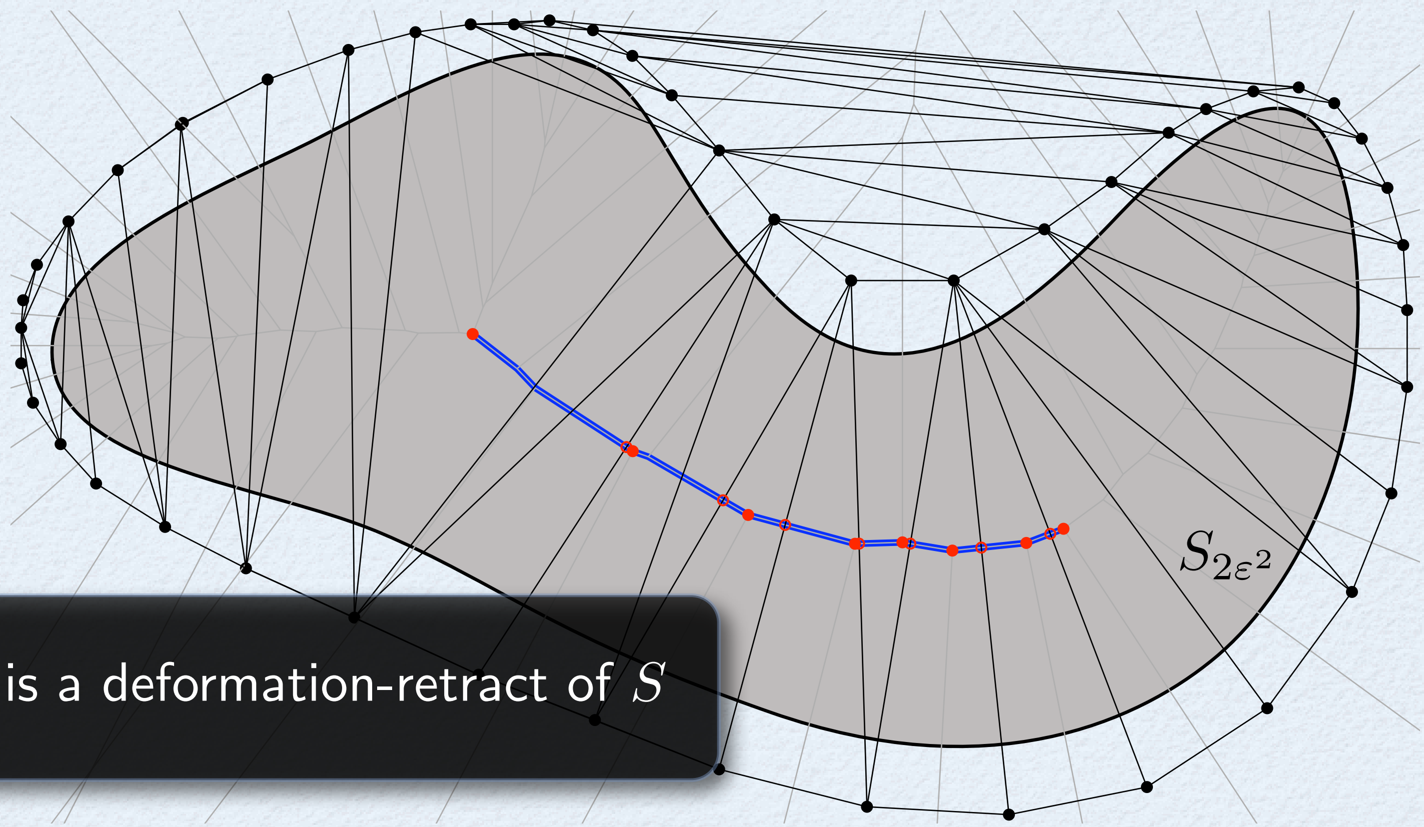

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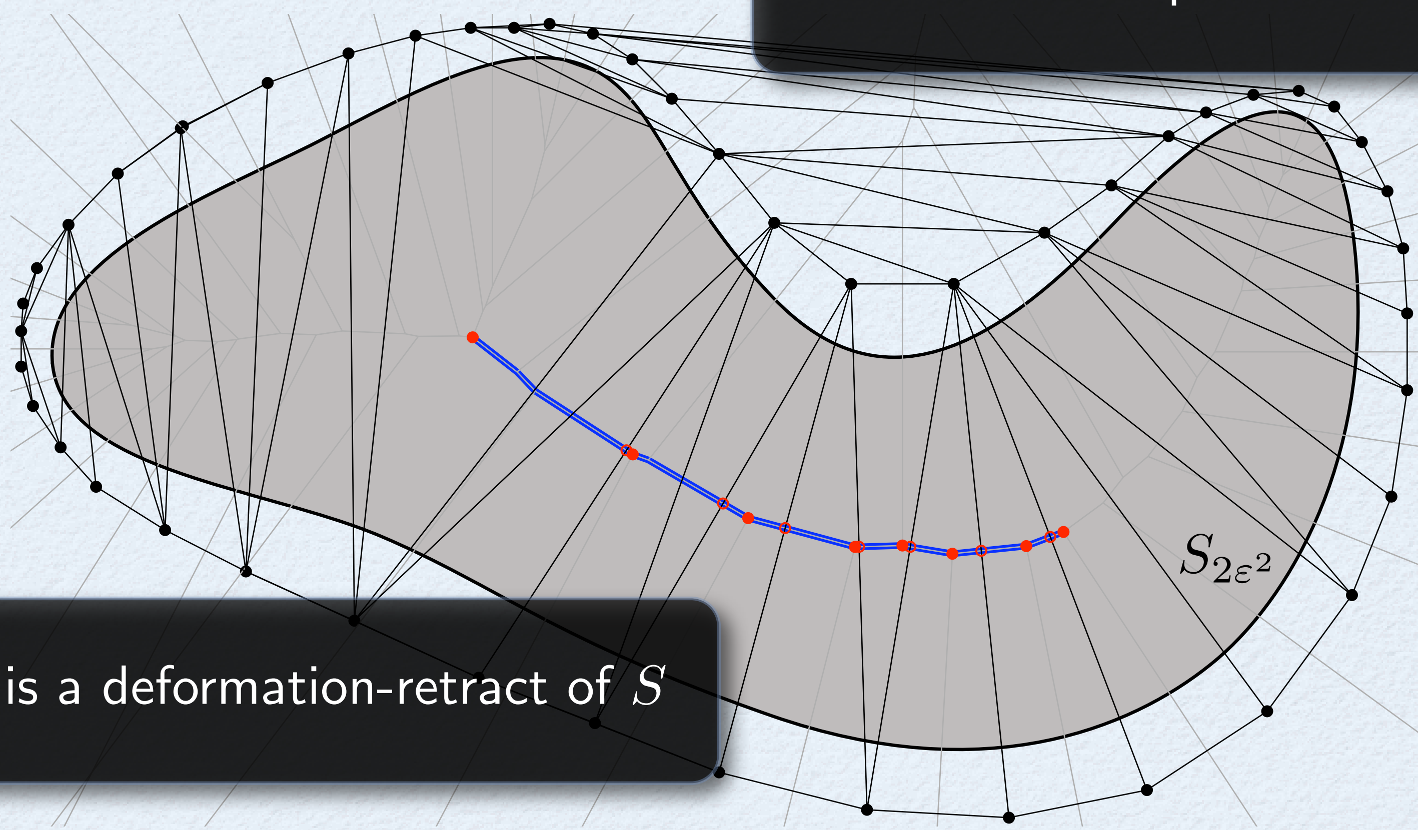
Surface critical points are in Σ_{ε^2}



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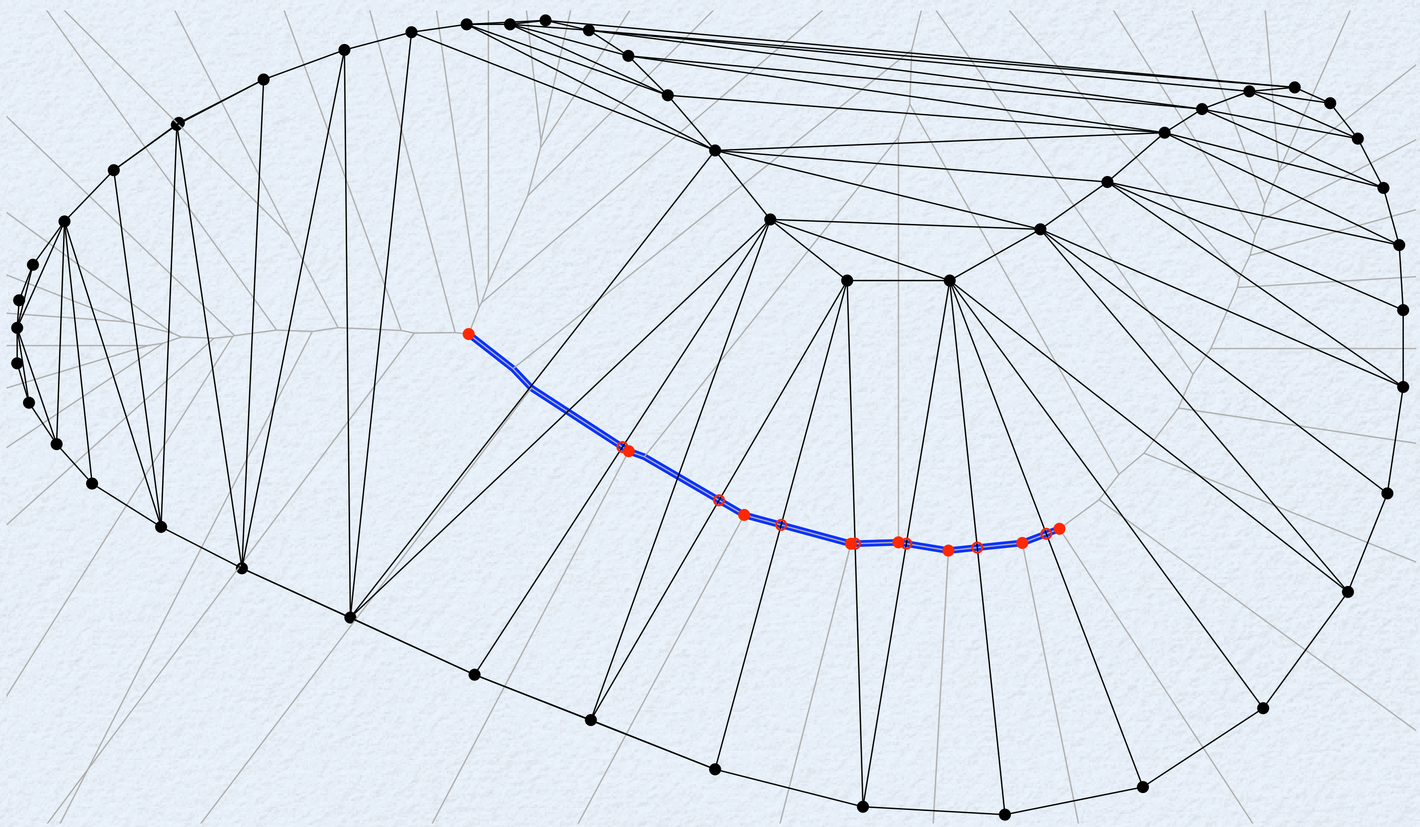
Extending the CORE

Theorem. If $U \subset S_{2\varepsilon^2}$, then

$$\mathsf{CORE} \cup \phi(U) \simeq S.$$



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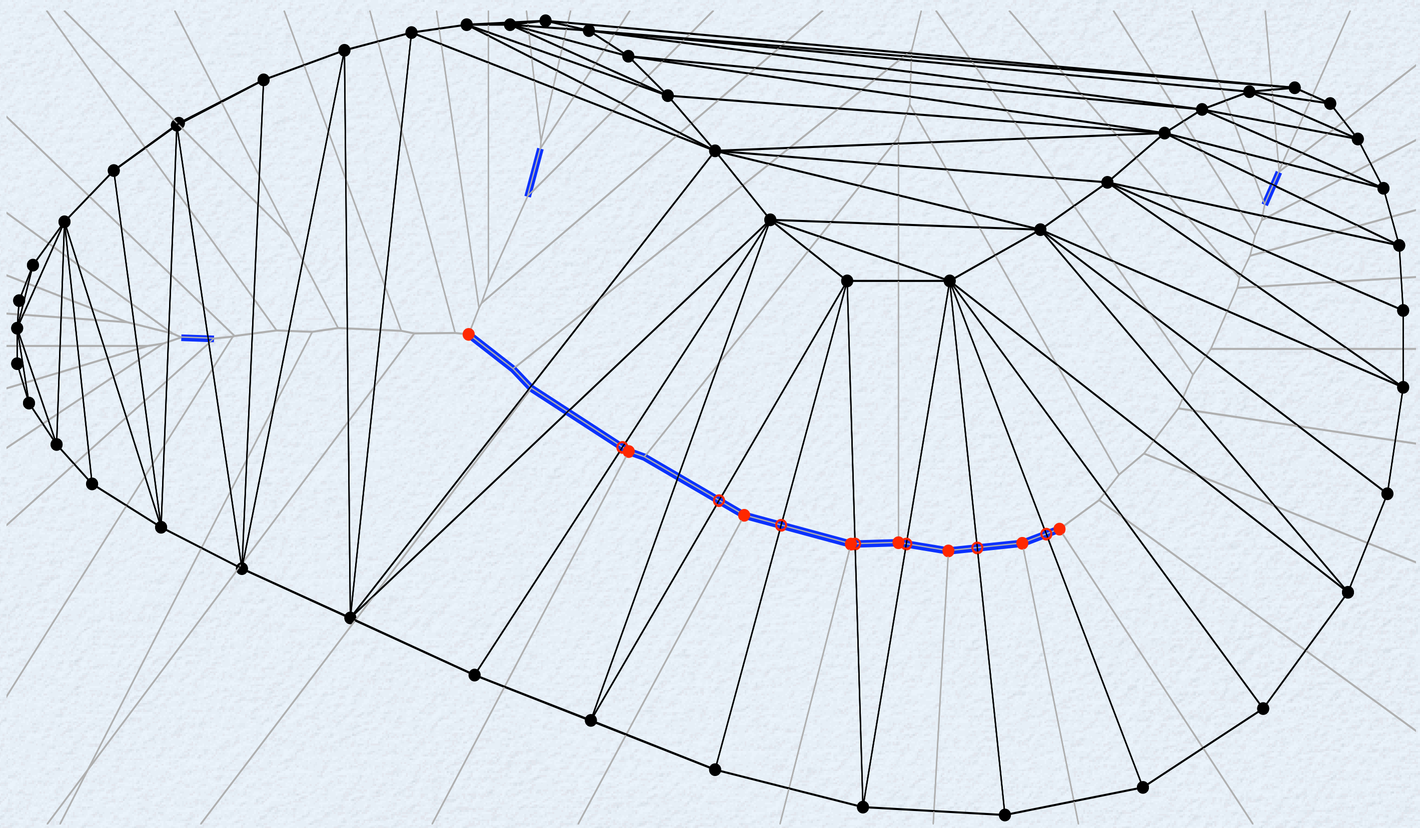

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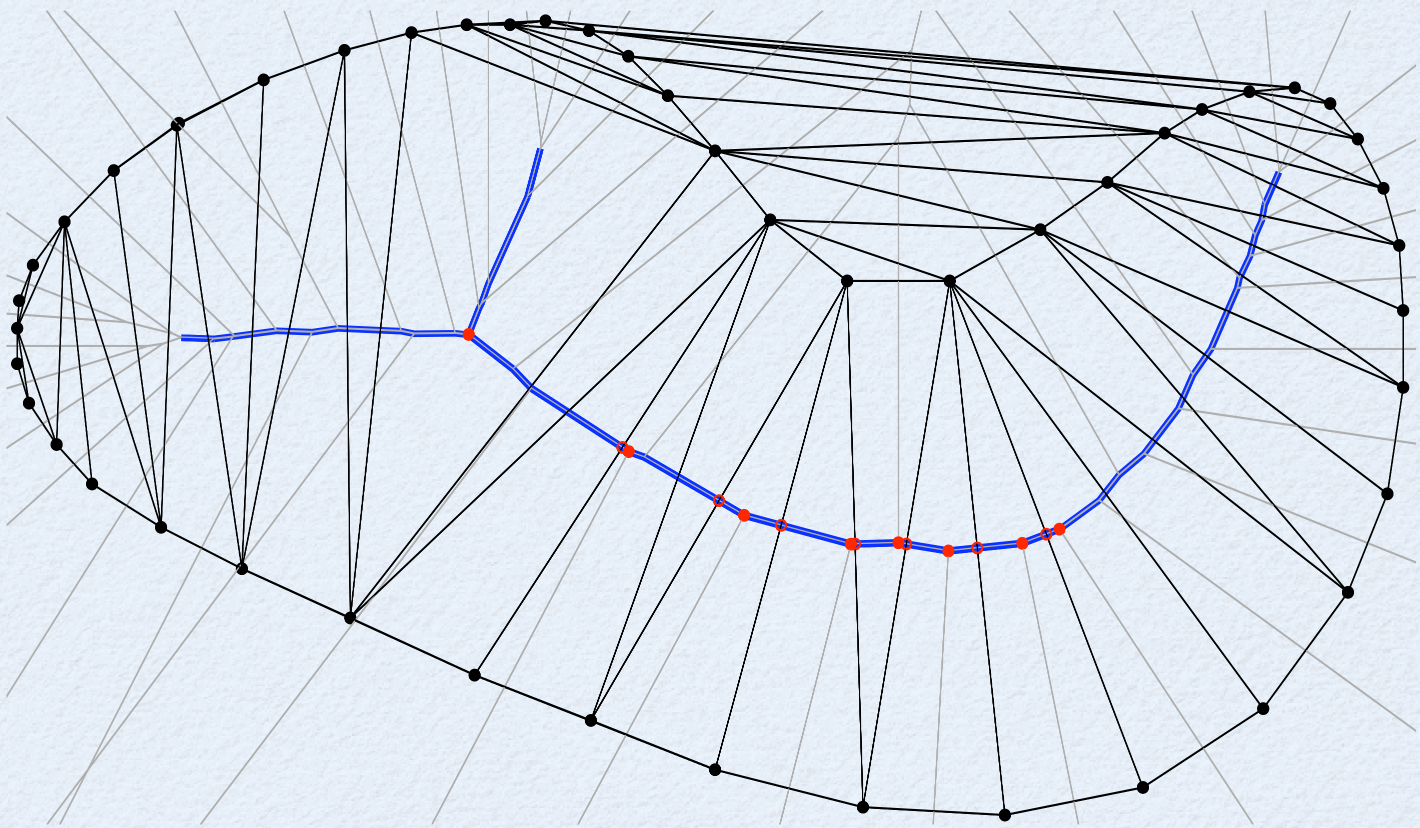

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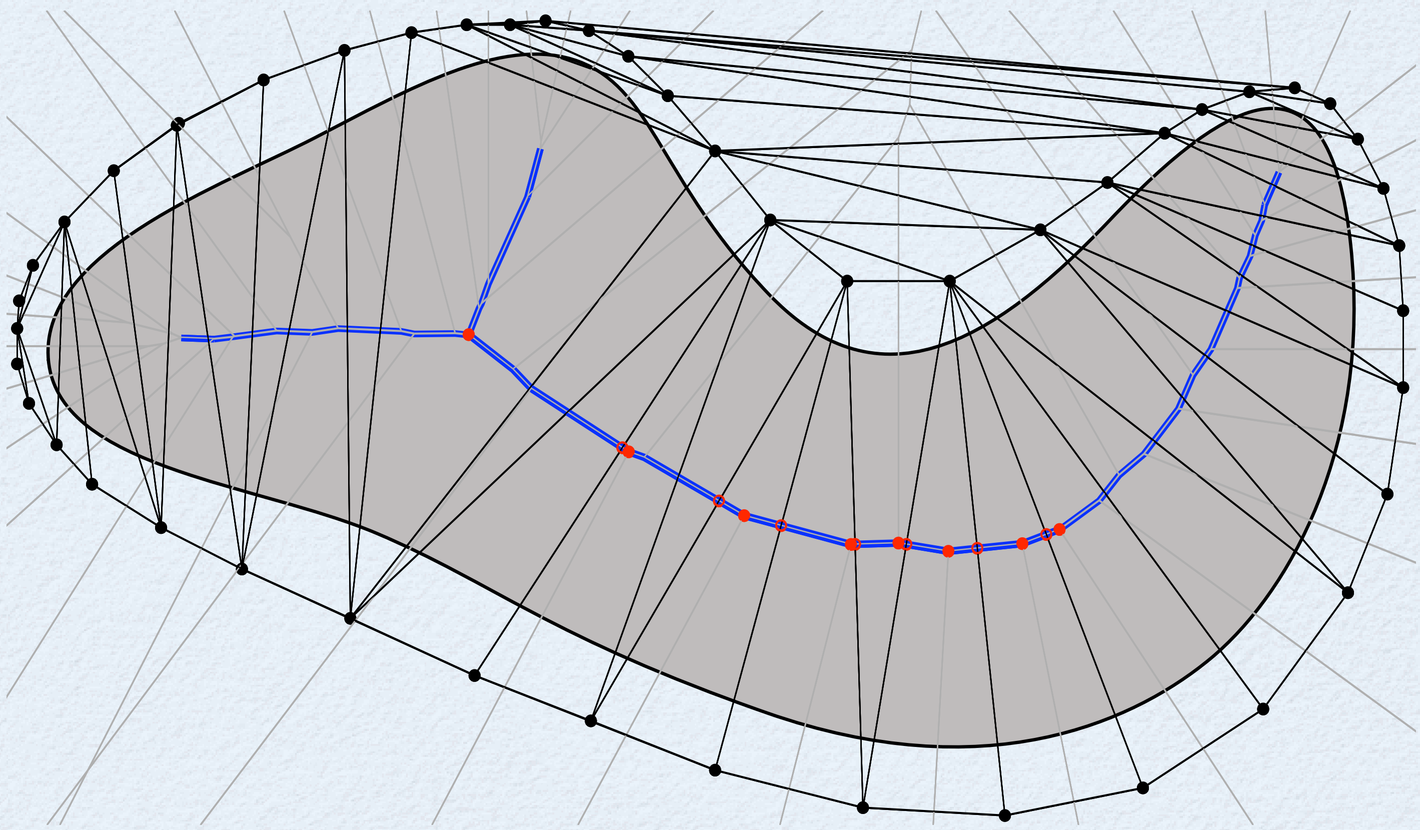
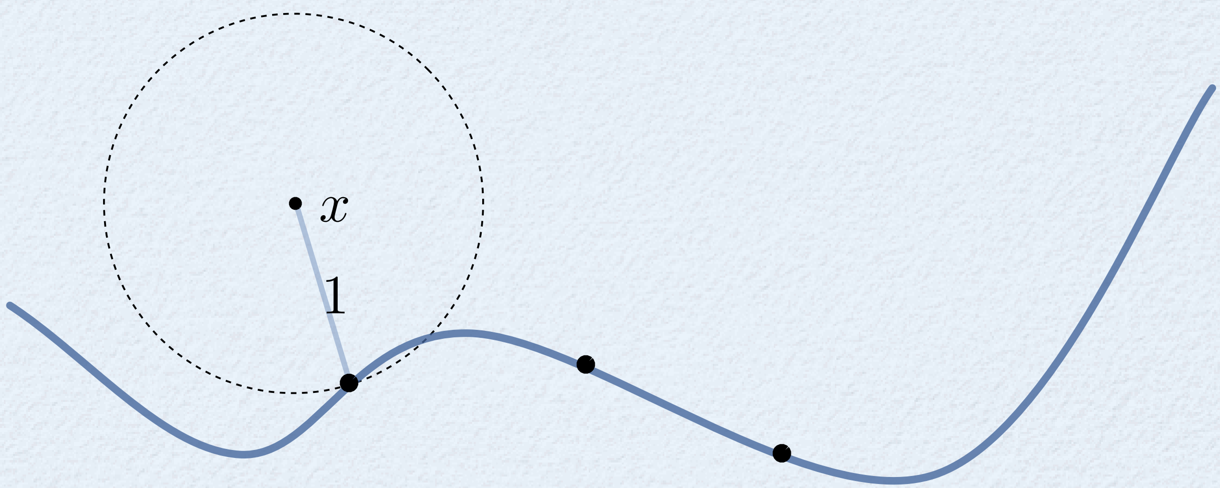


Geometric Quality and Flow Closure

Can we diverge too far away from MA when taking flow closure?

Theorem. If $\rho(x) = \sqrt{h(x)} = 1$ and x has a medial axis point within distance $O(\sqrt{\varepsilon})$, then for any $t \geq 0$, $y = \phi(t, x)$ has a medial axis point within distance

$$O(\sqrt{\varepsilon})\rho(y)^{1+O(\sqrt{\varepsilon})}.$$



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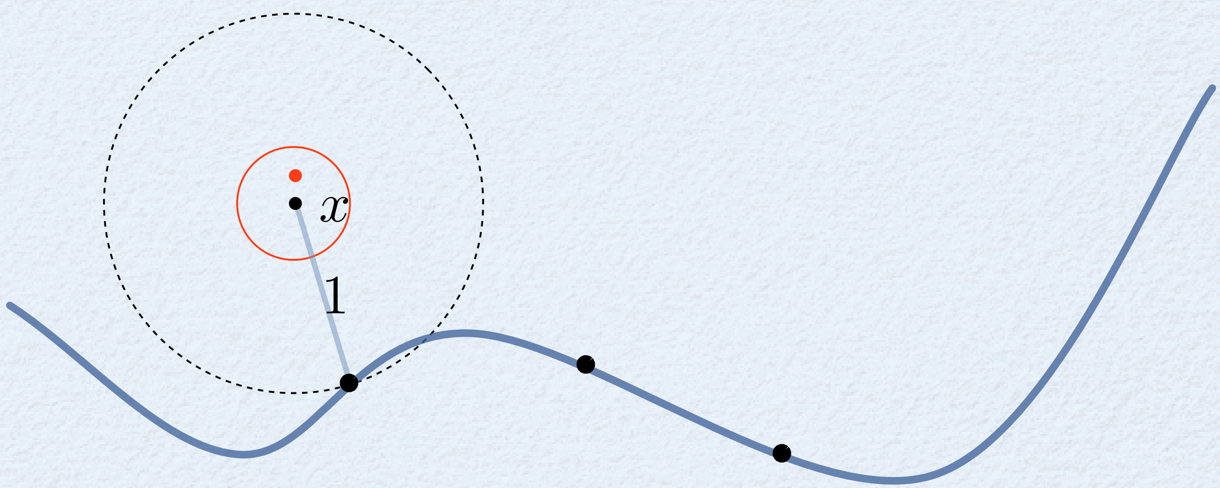

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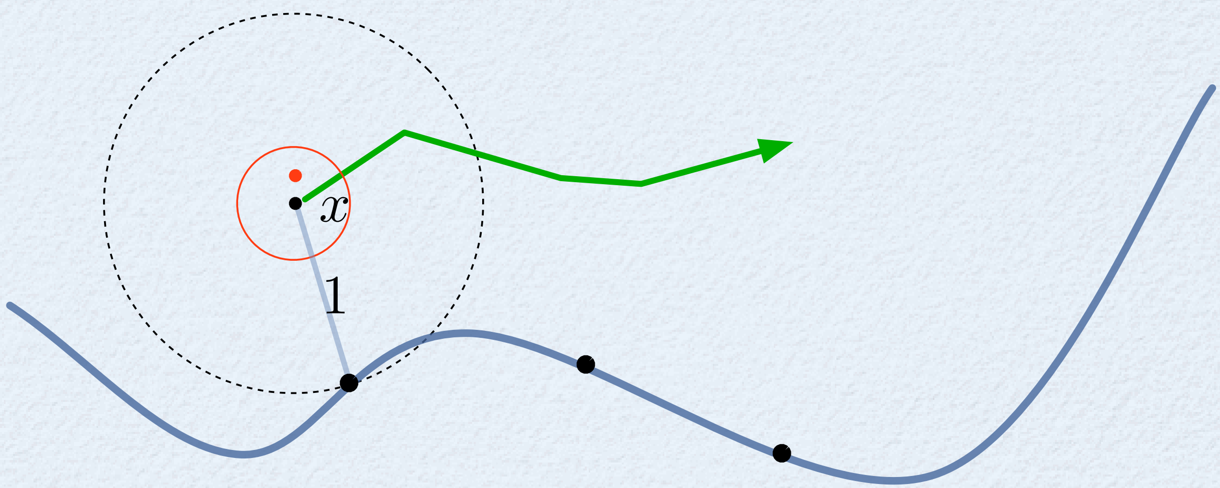

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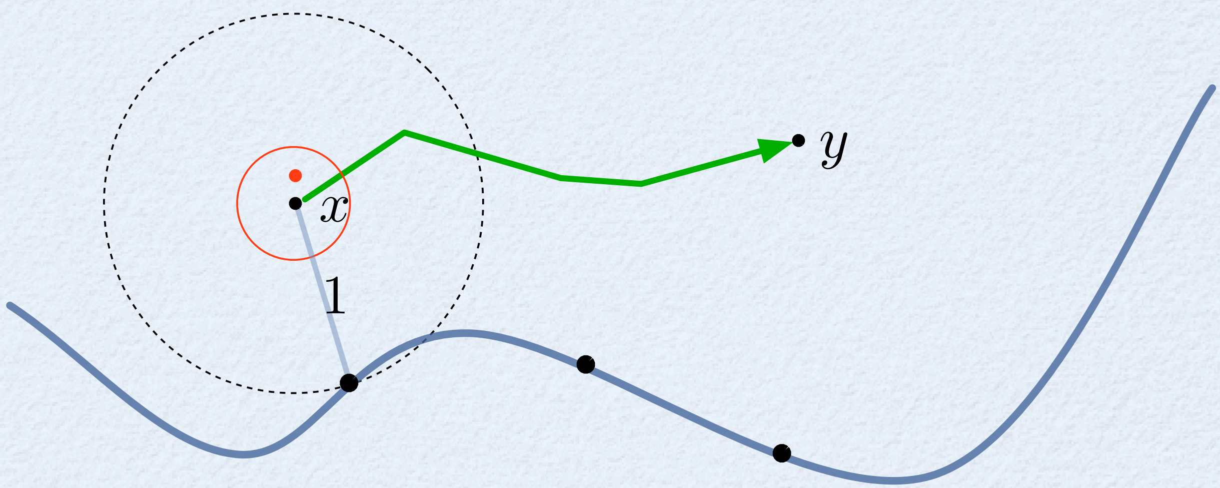

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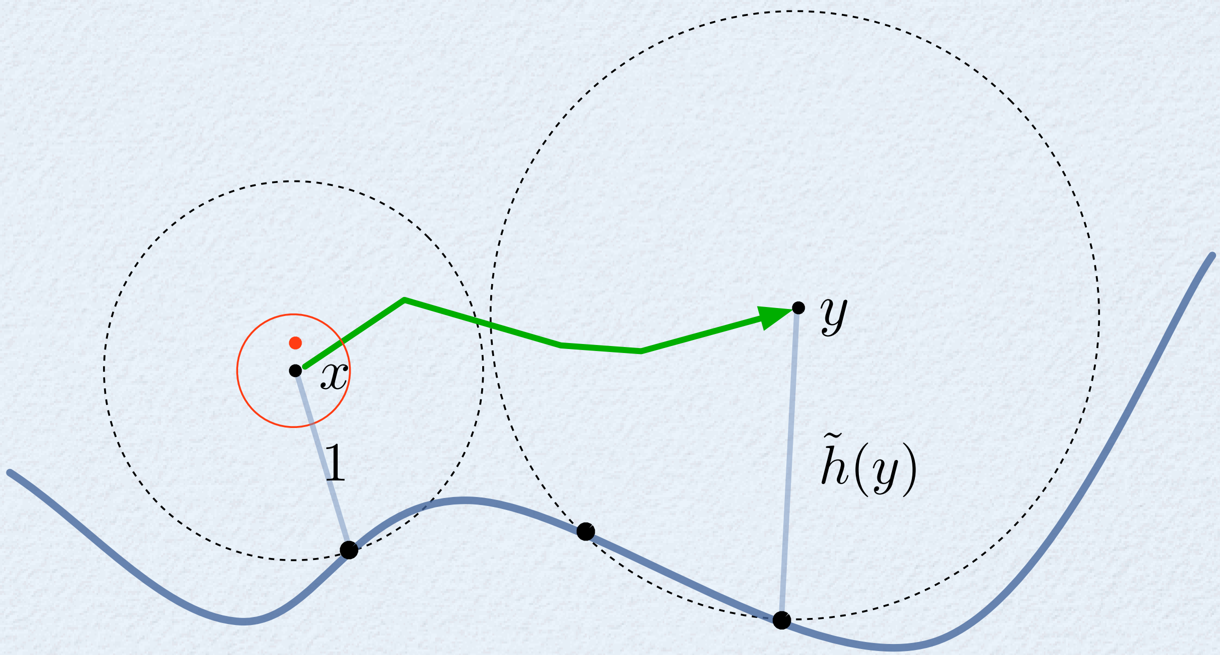

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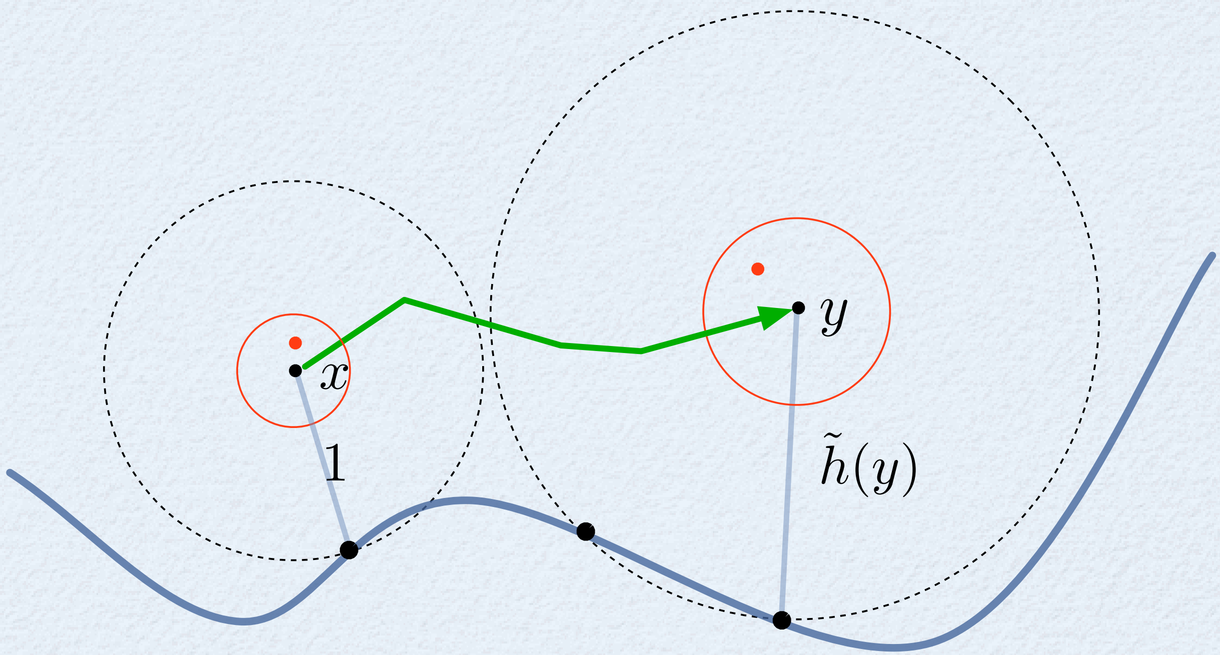

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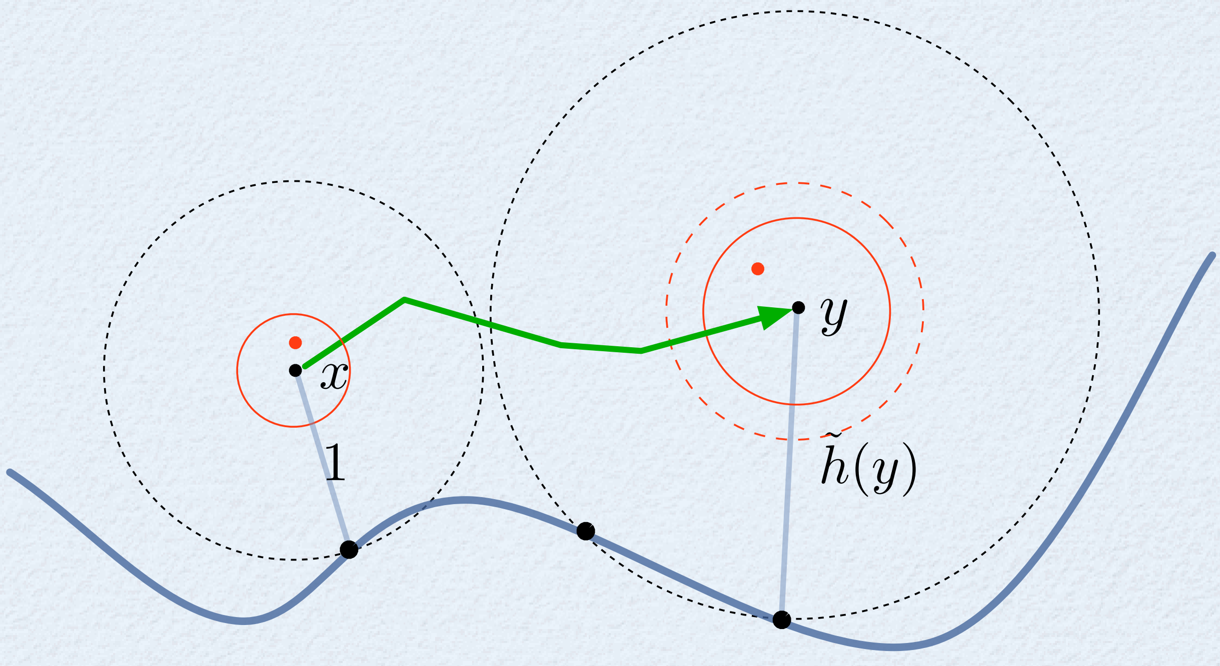


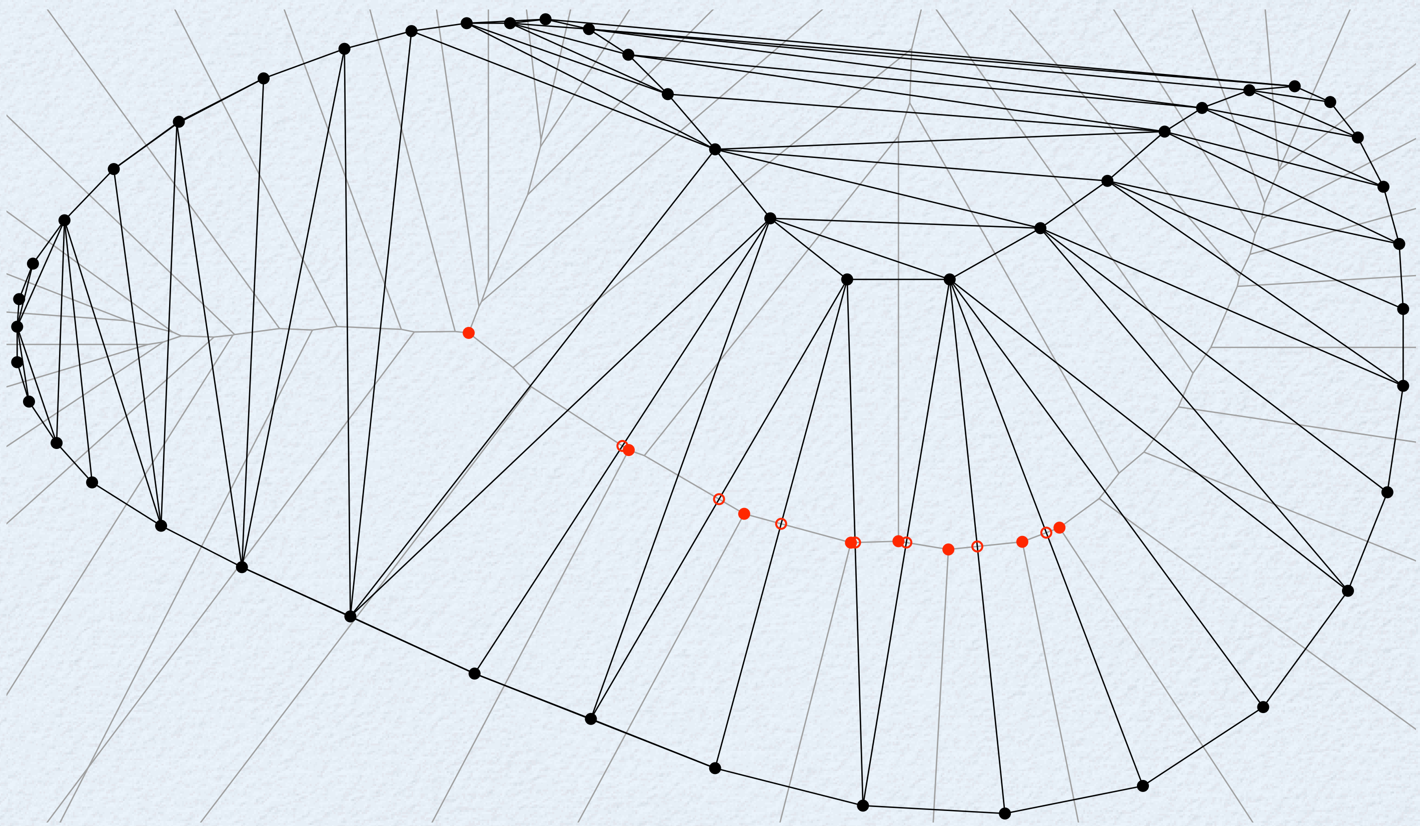
3

Surface (Shape) Reconstruction

Reconstruction as a Union of Stable Manifolds

Definition. Let N be the set of all inner medial axis critical points of h.

$$\textcolor{green}{\mathsf{REC}} = \bigcup_{c \in N} \mathsf{Sm}(c)$$



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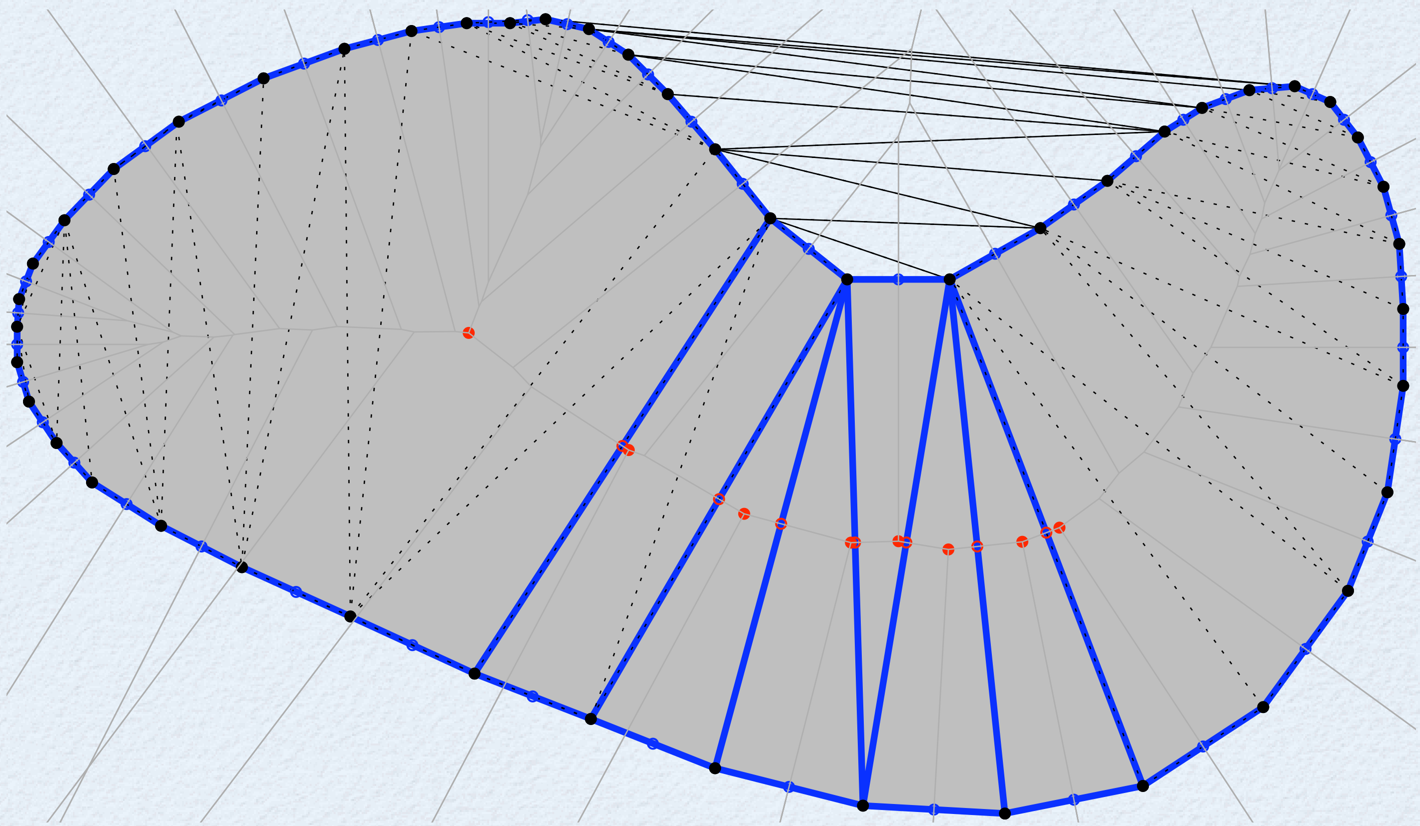
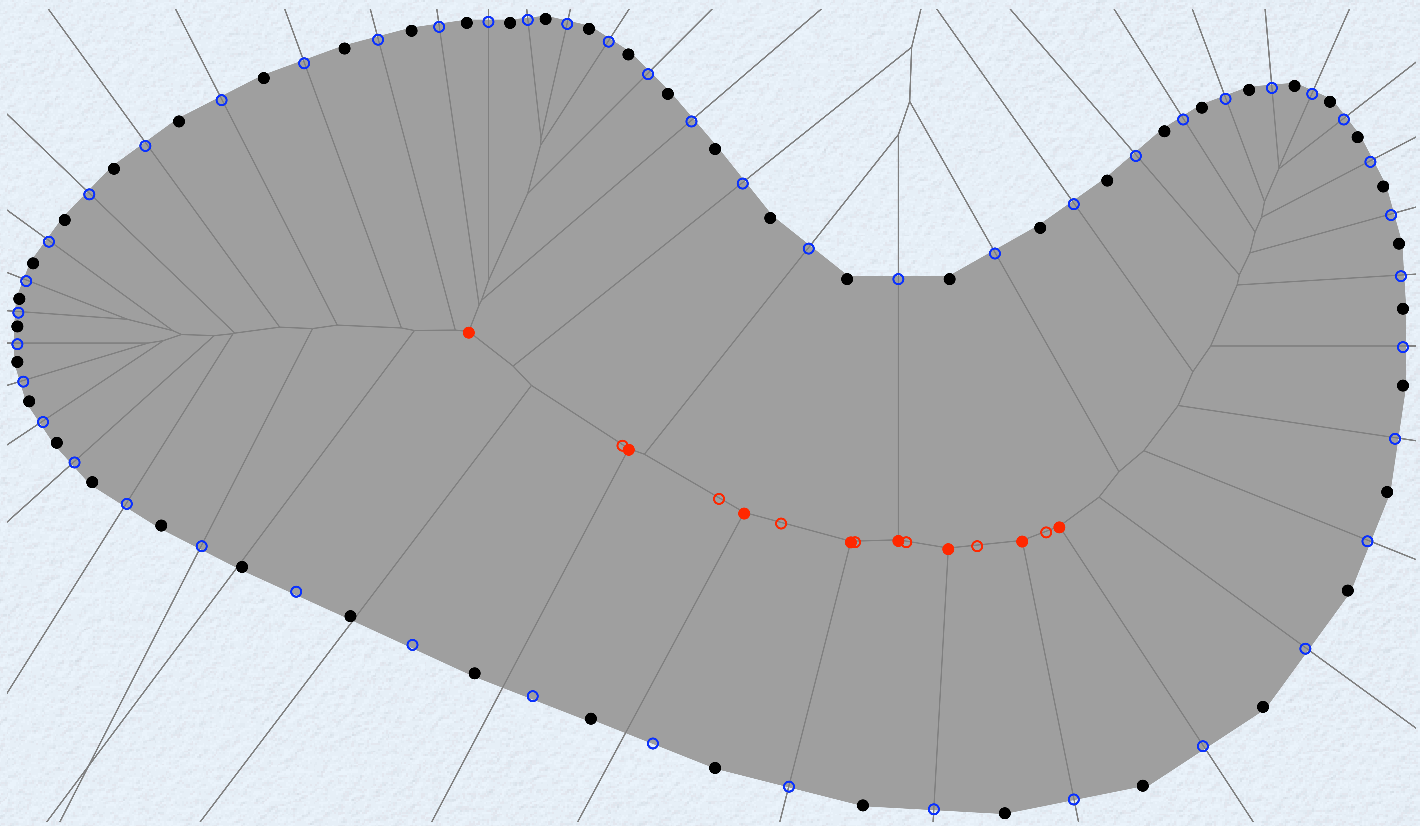


Homotopy Type of REC

Theorem. REC and shape are homotopy equivalent (for small ε).

Lemma. $\phi(S_{\varepsilon^2}) = S_{\varepsilon^2}$ (for small ε).



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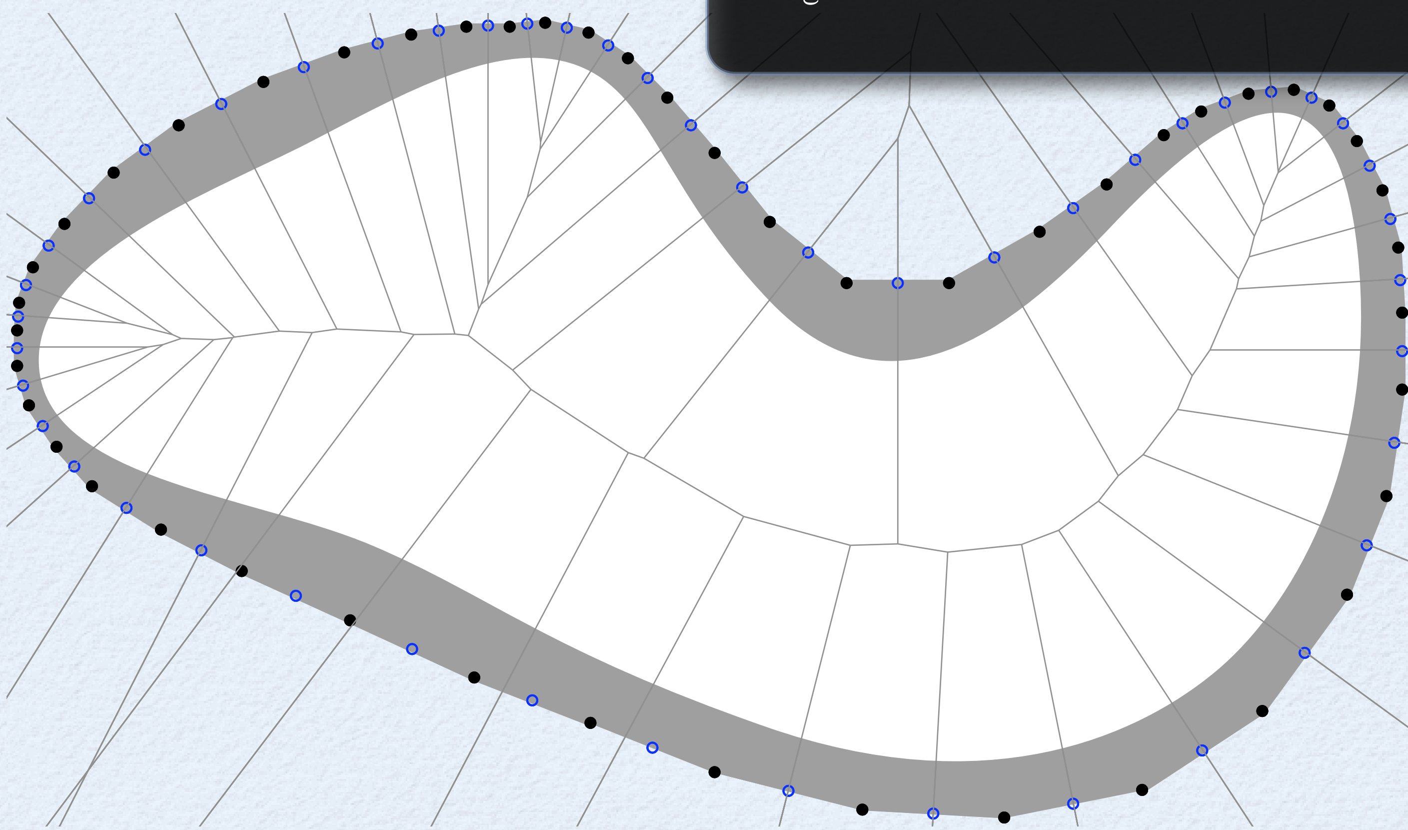

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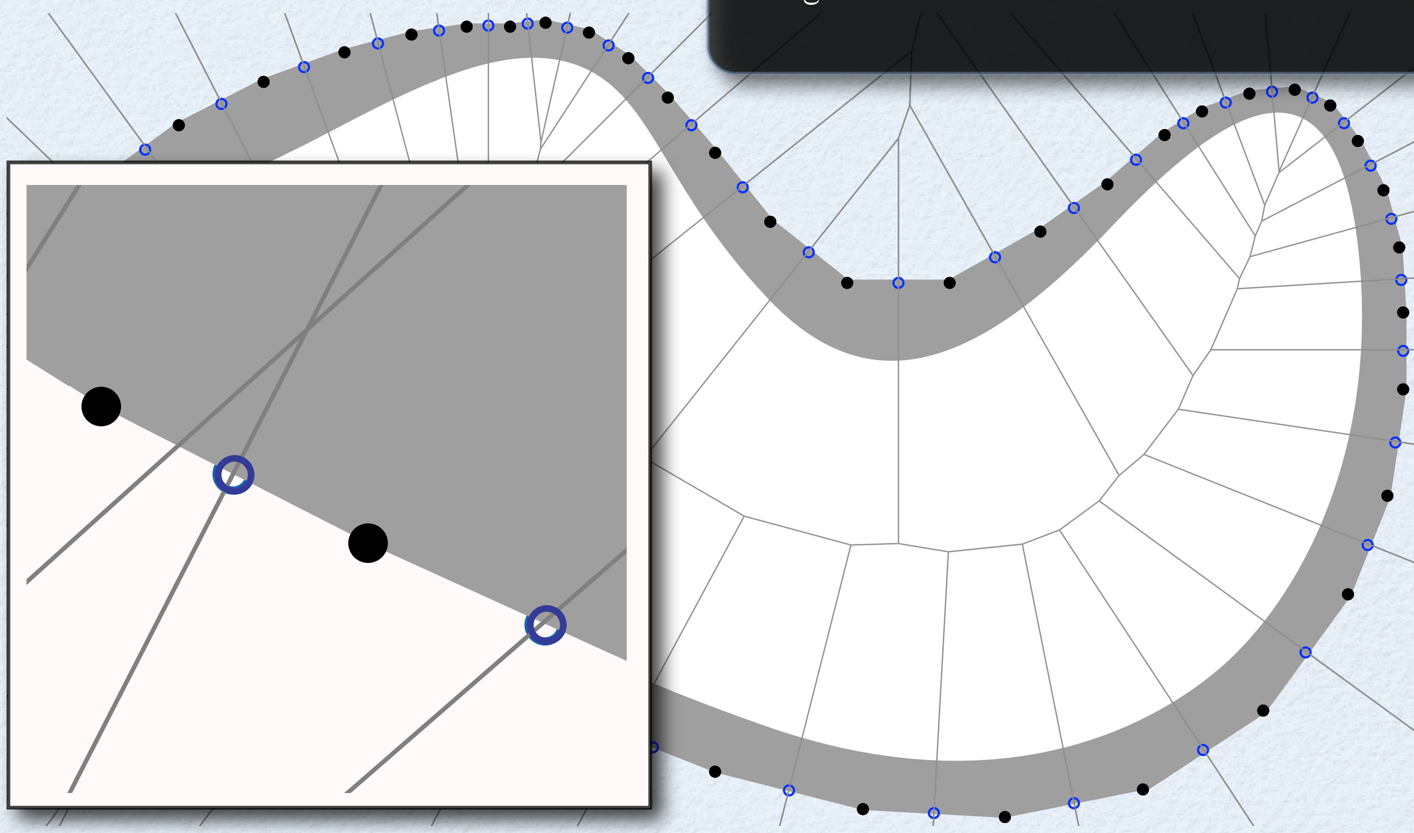

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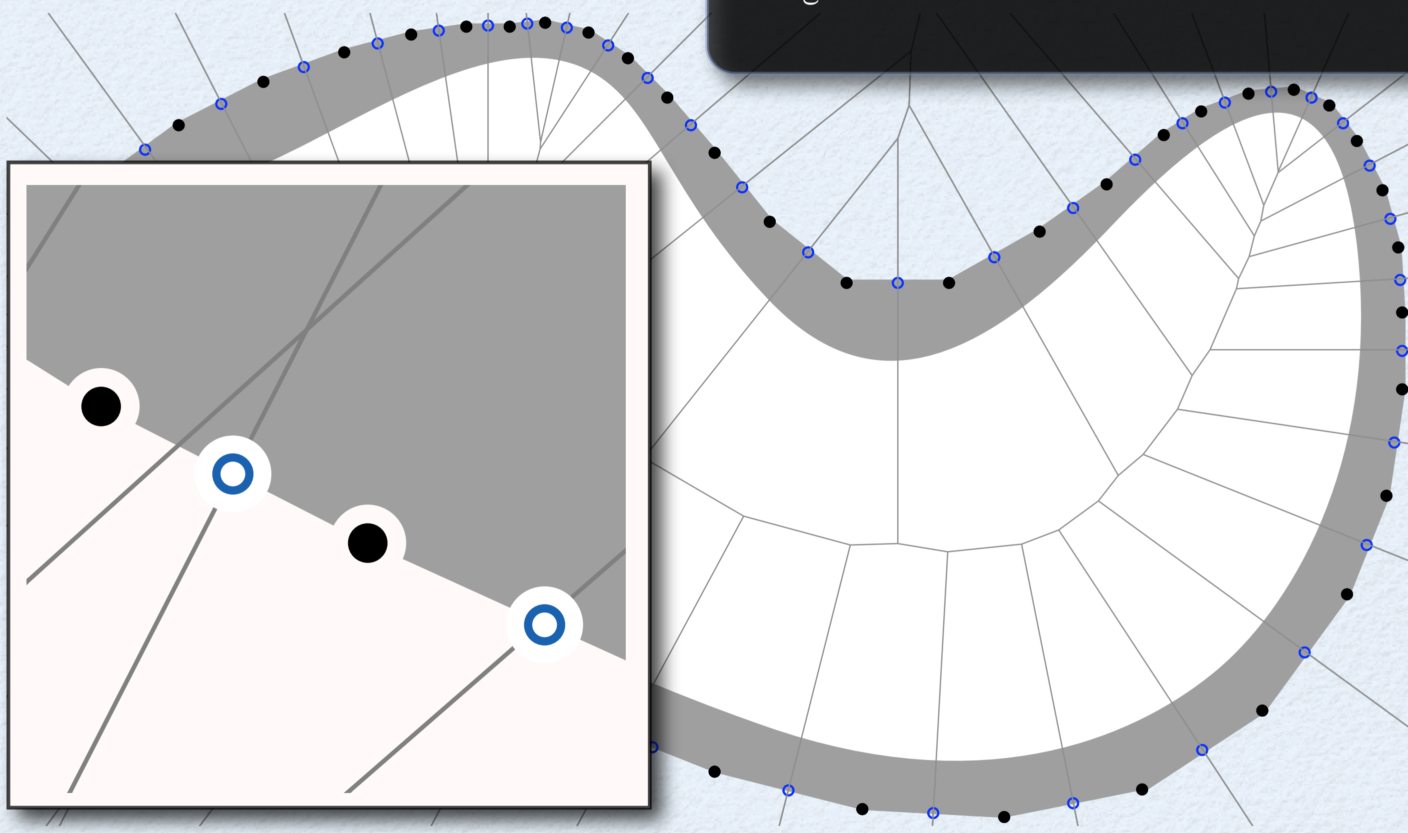

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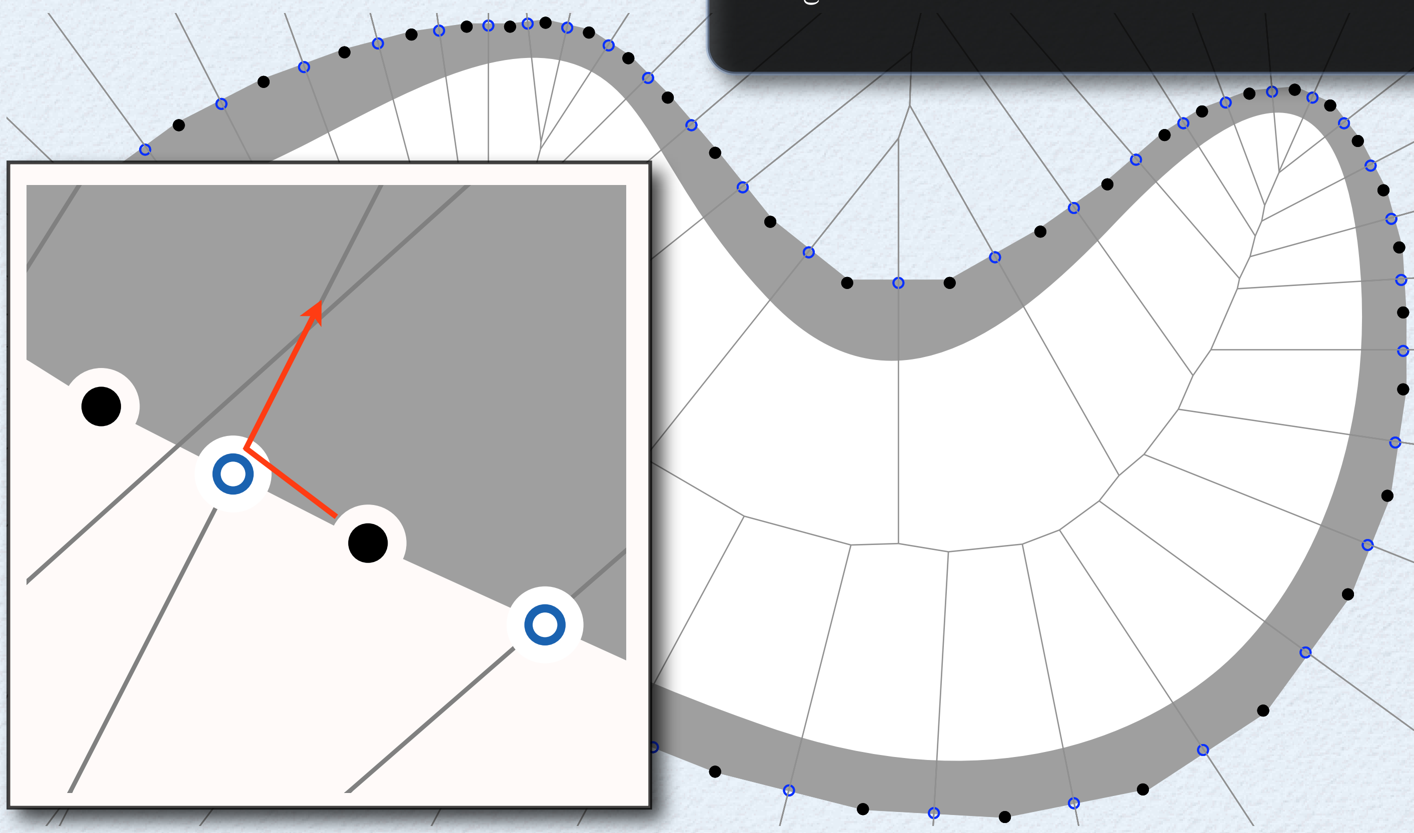

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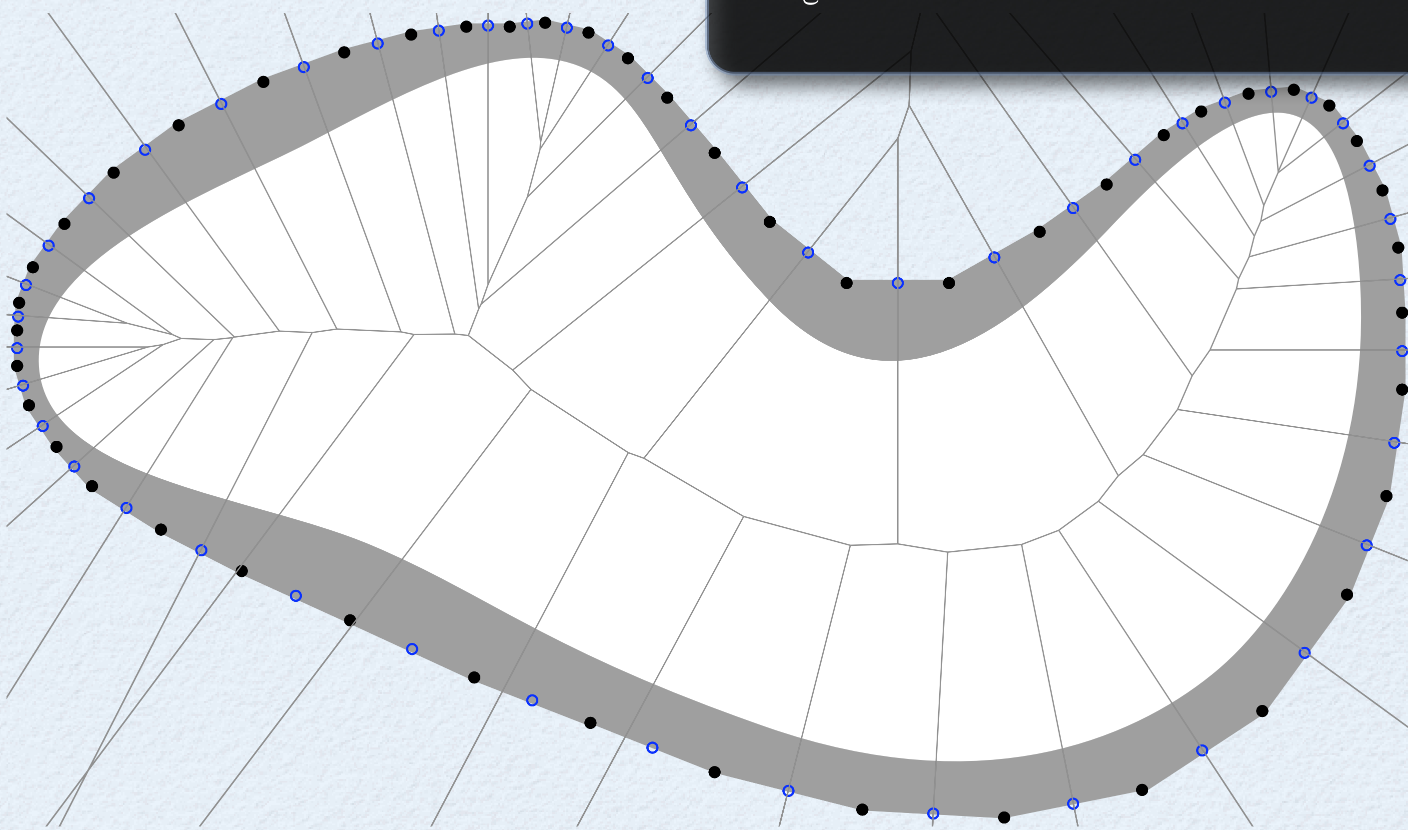

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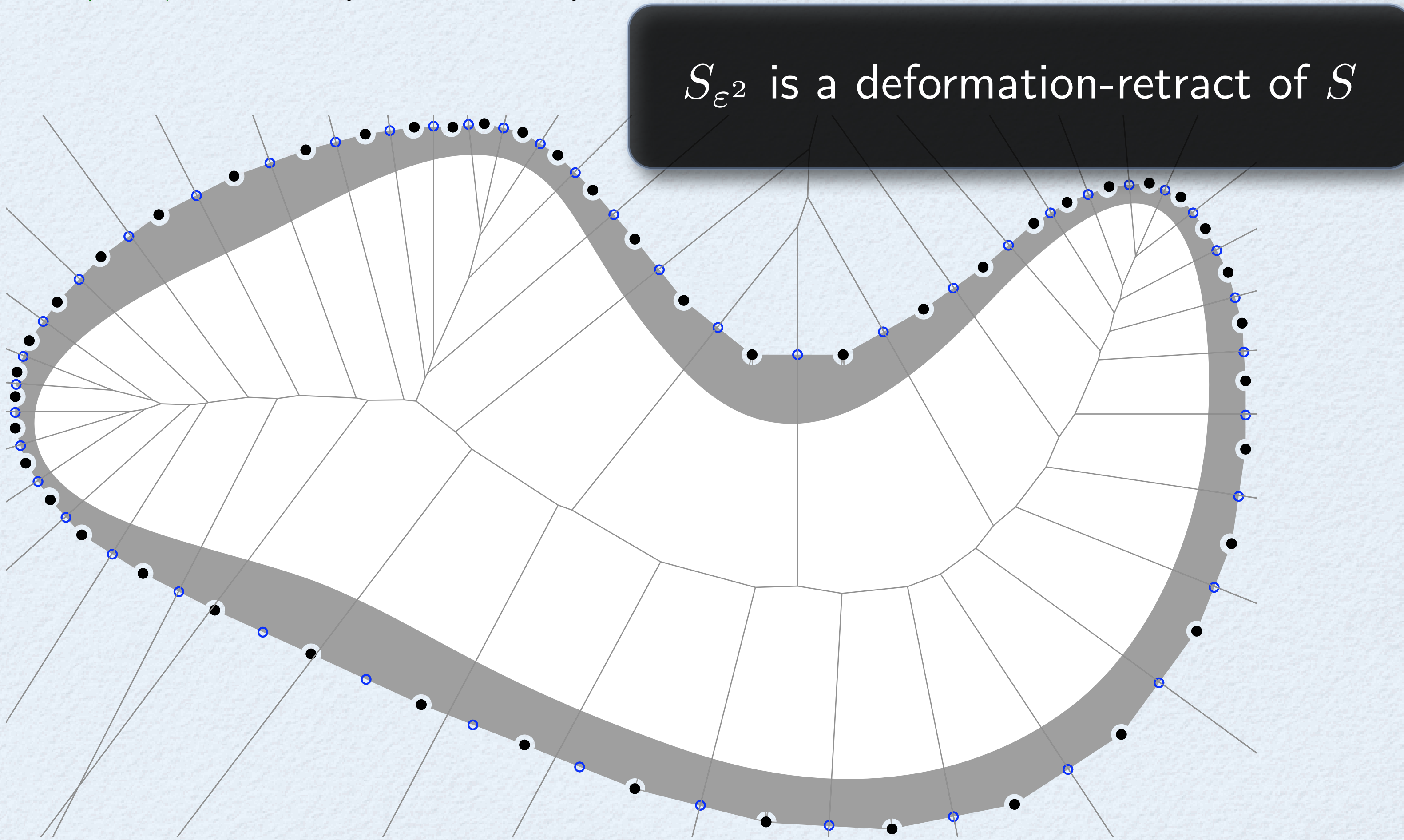

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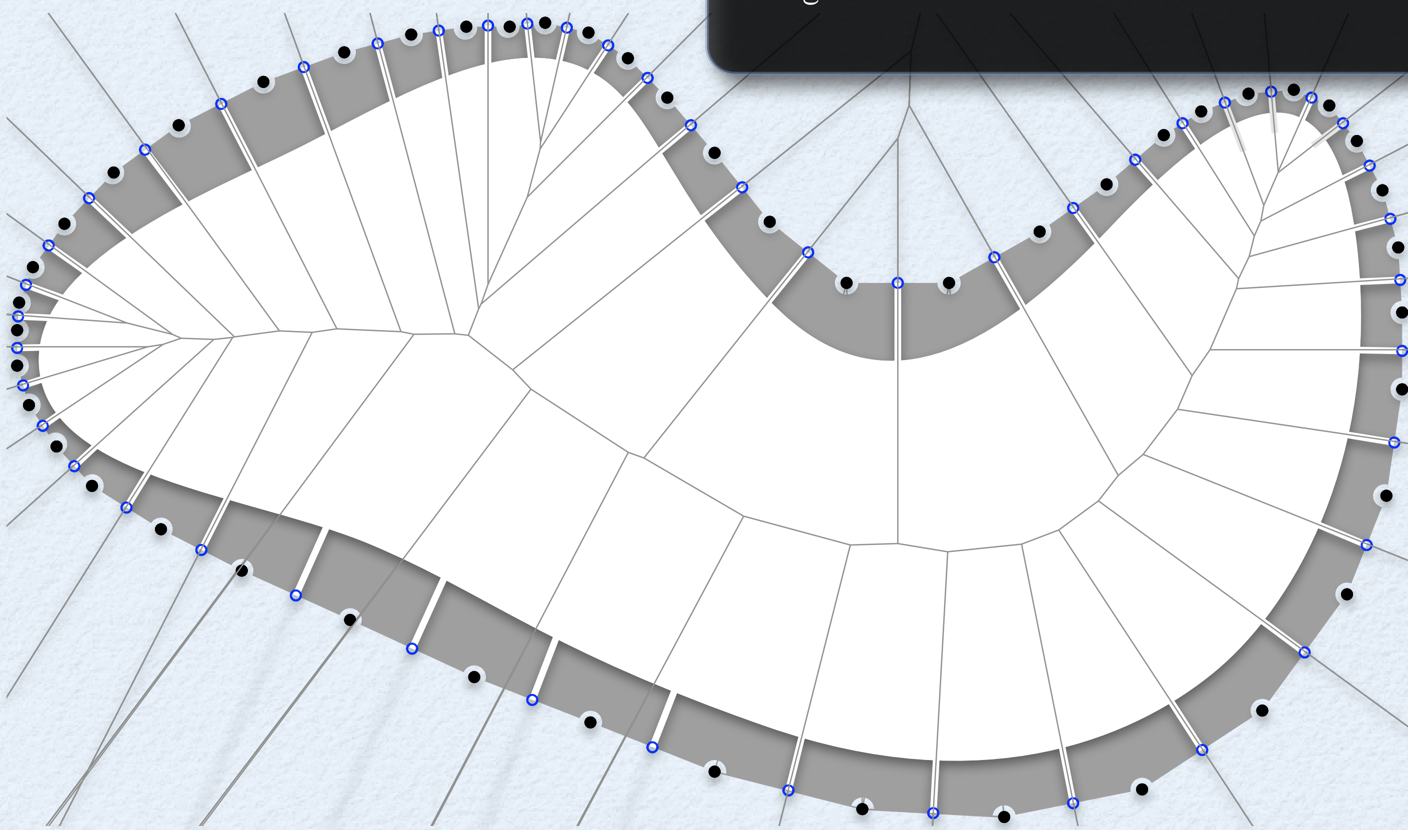

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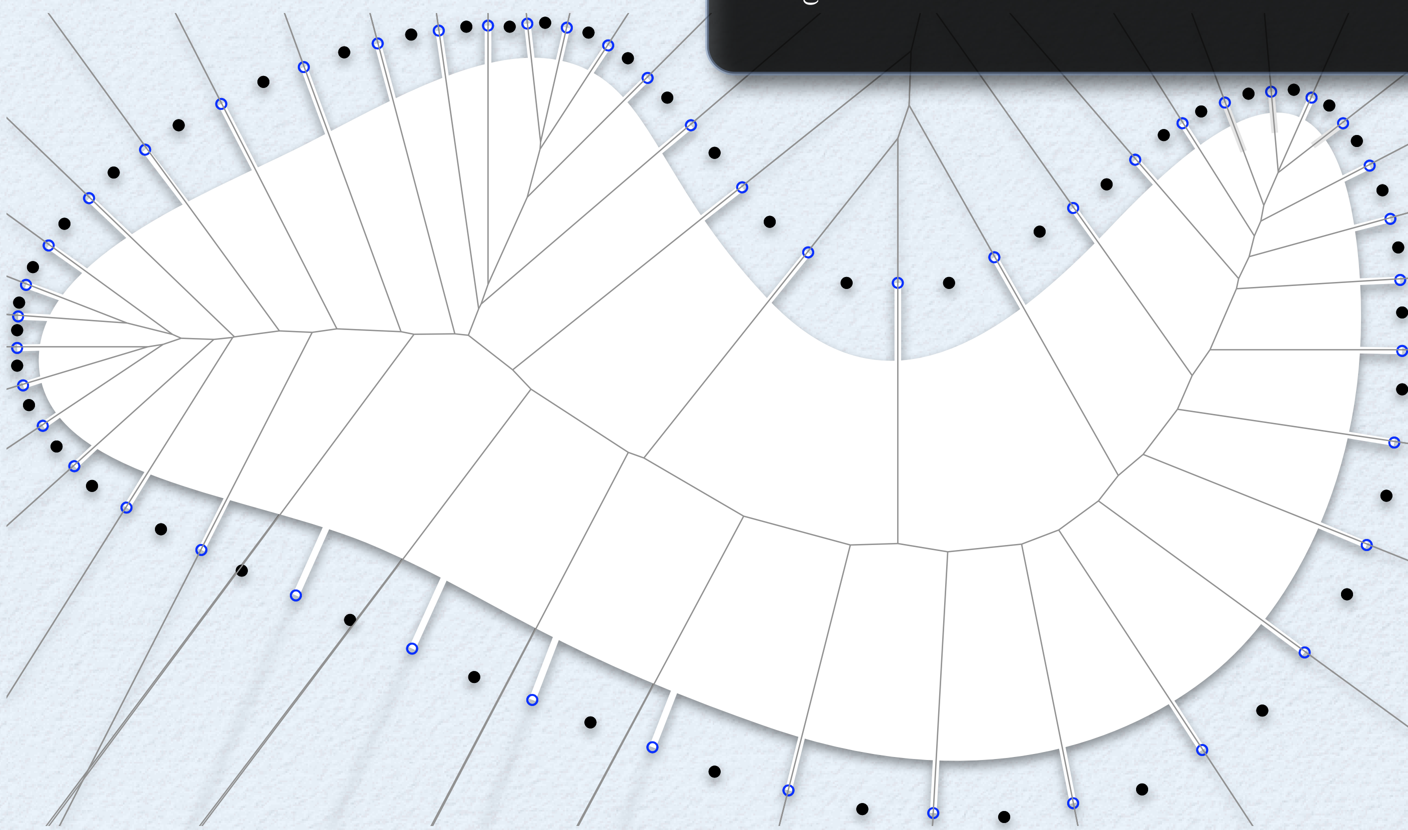

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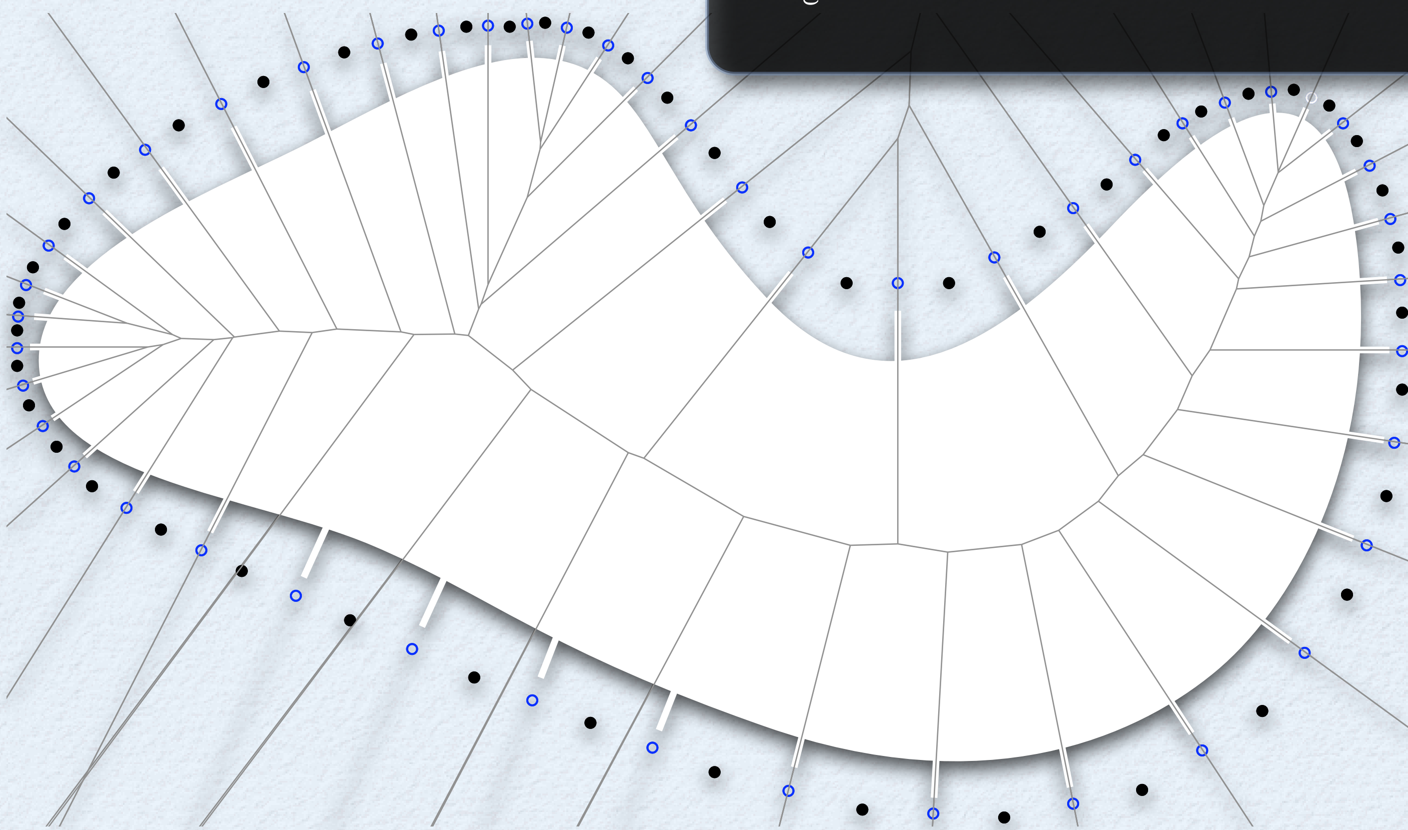

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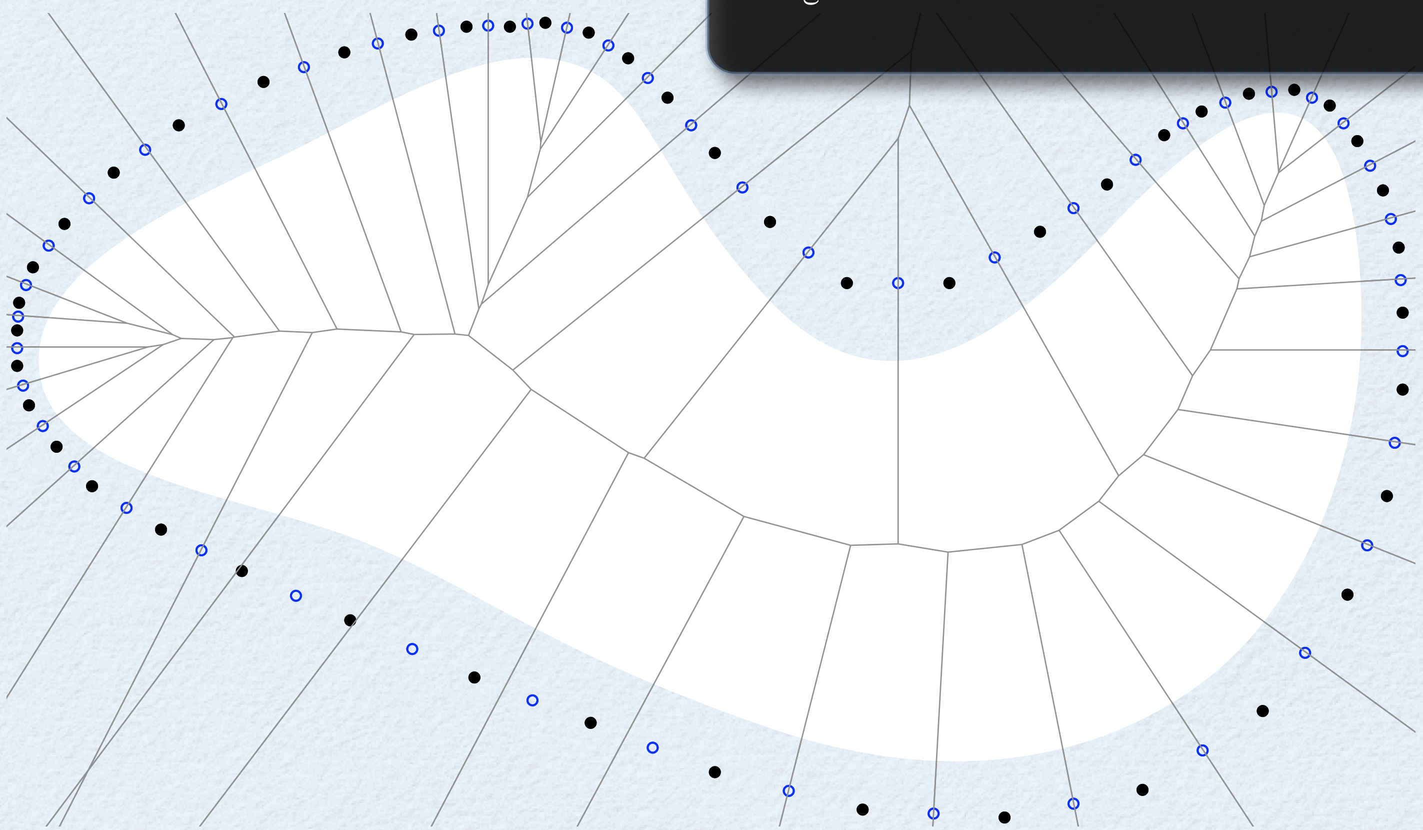
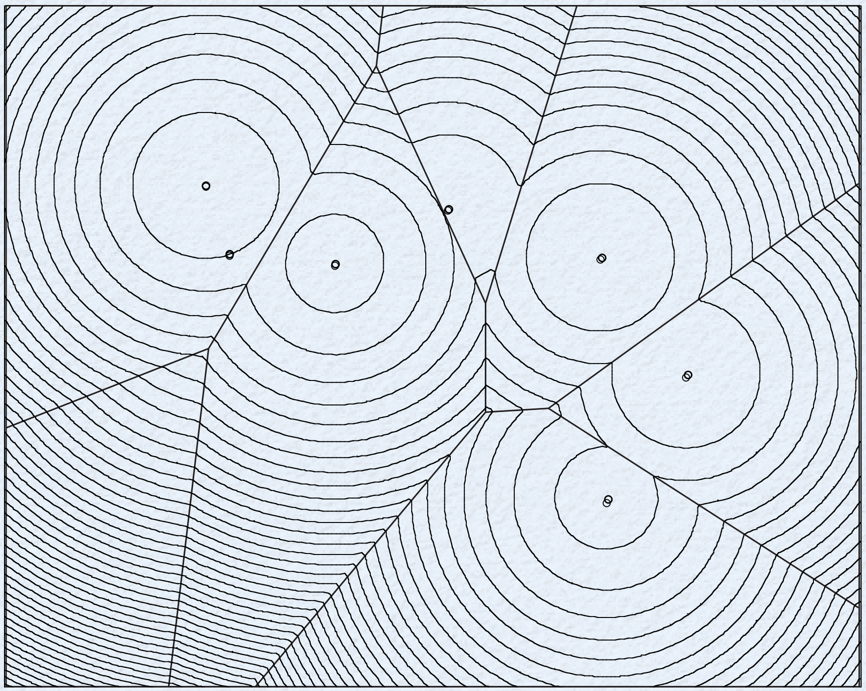


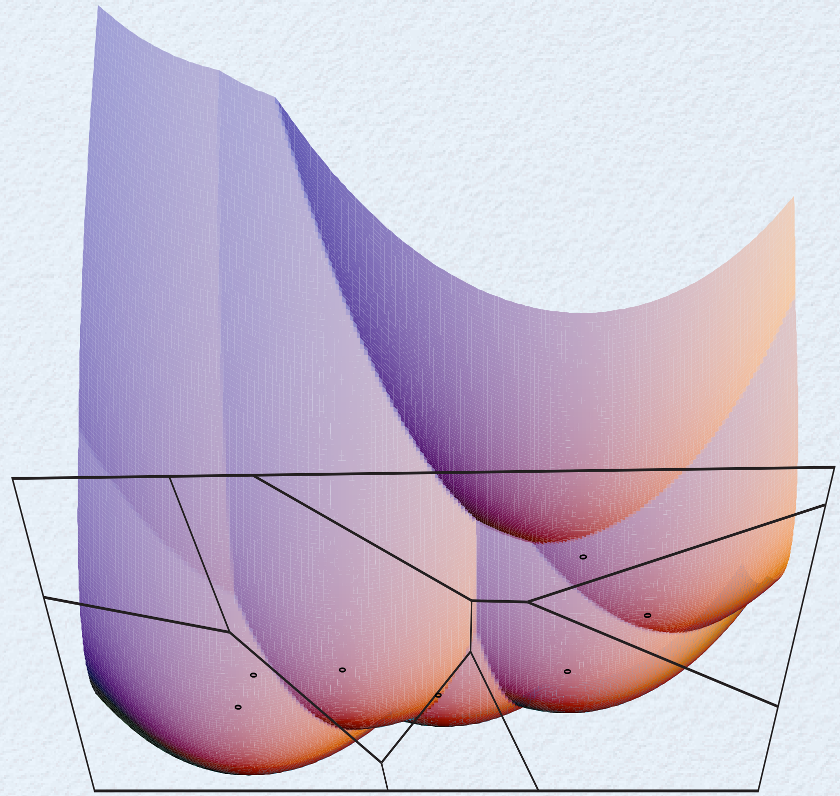
4

WRAP: Reconstruction Revisited

[Edelsbrunner'04] H. Edelsbrunner, Surface reconstruction by wrapping finite point-sets in space. Discrete & Computational Geometry, 2004.

Flow Induced by Weighted Points





Squared distance to p with weight w_p is $\|x - p\|^2 - w_p$.

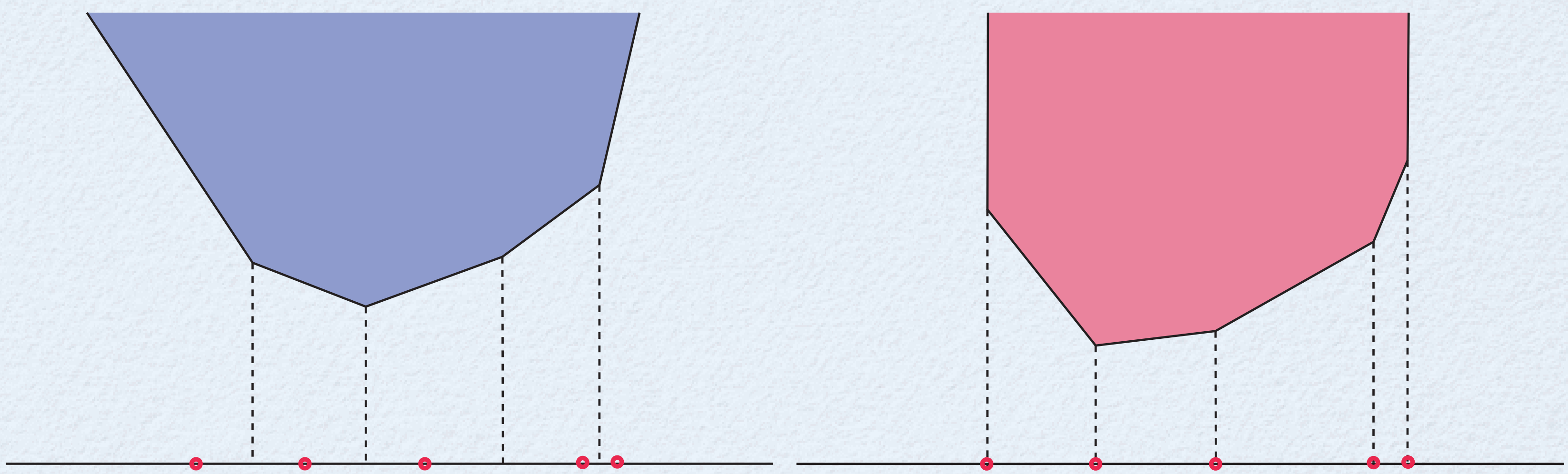
The squared distance to a set P of weighted points is

$$h(x) = \min_{p \in P} \|x - p\|^2 - w_p.$$

Polarity

For every set P of weighted points there is a set Q of weighted points such that

$$\text{Vor } P = \text{Del } Q \qquad \text{and} \qquad \text{Del } P = \text{Vor } Q$$



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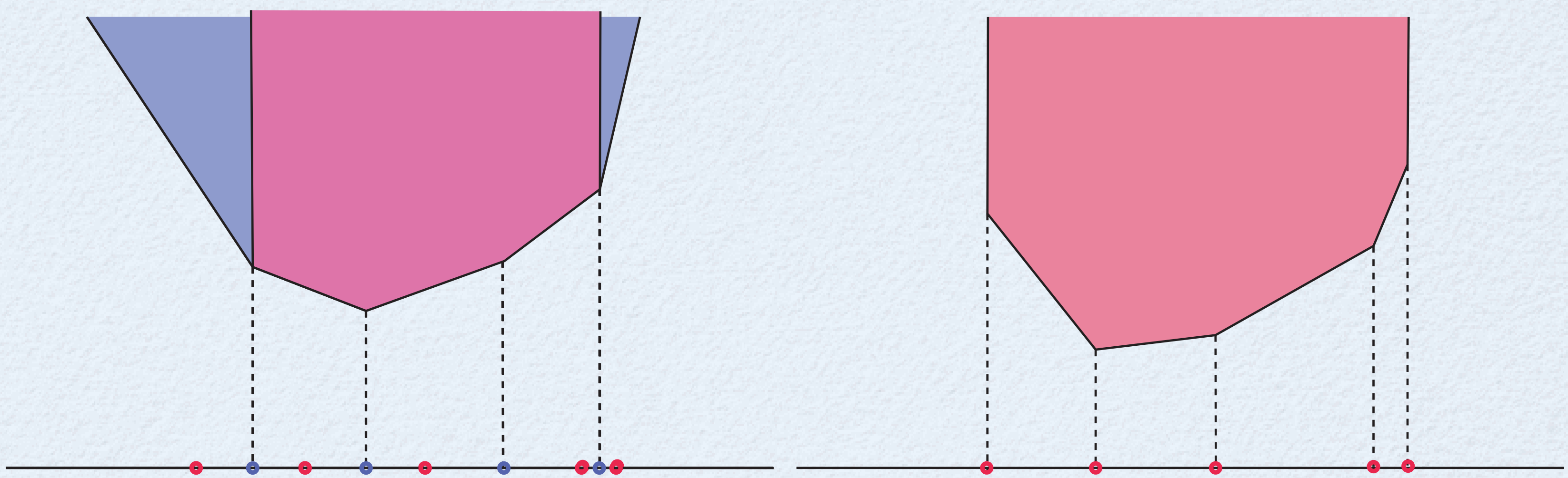
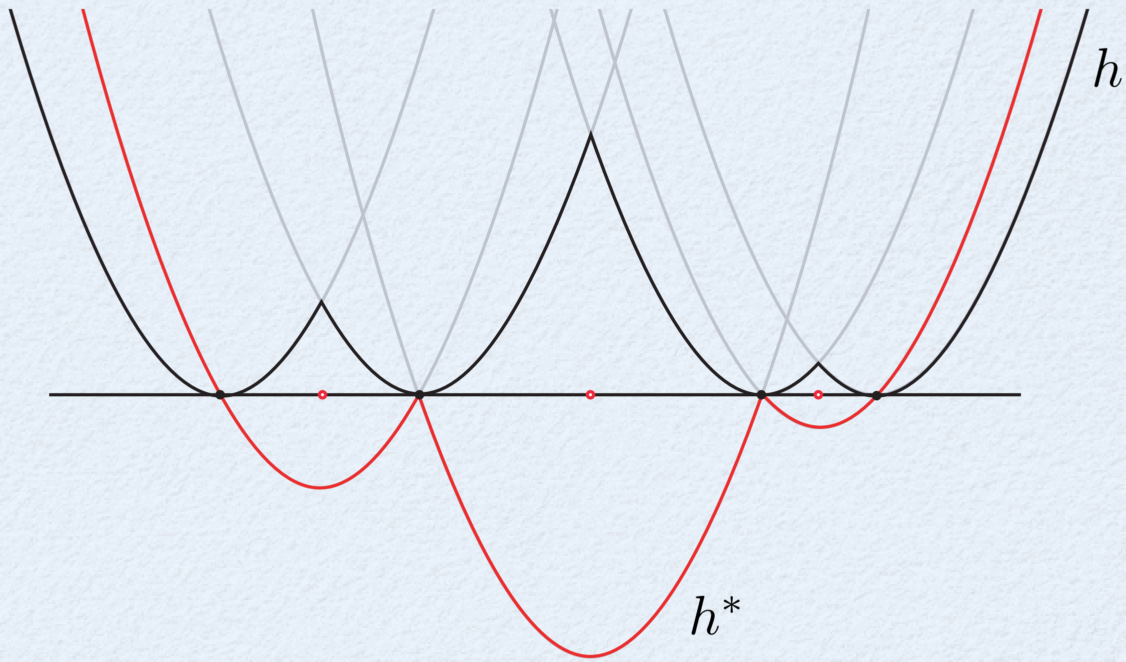


Voronoi Vertices as Weighted Points



For unweighted P, Q is the Voronoi vertices of P and for $q \in Q$:

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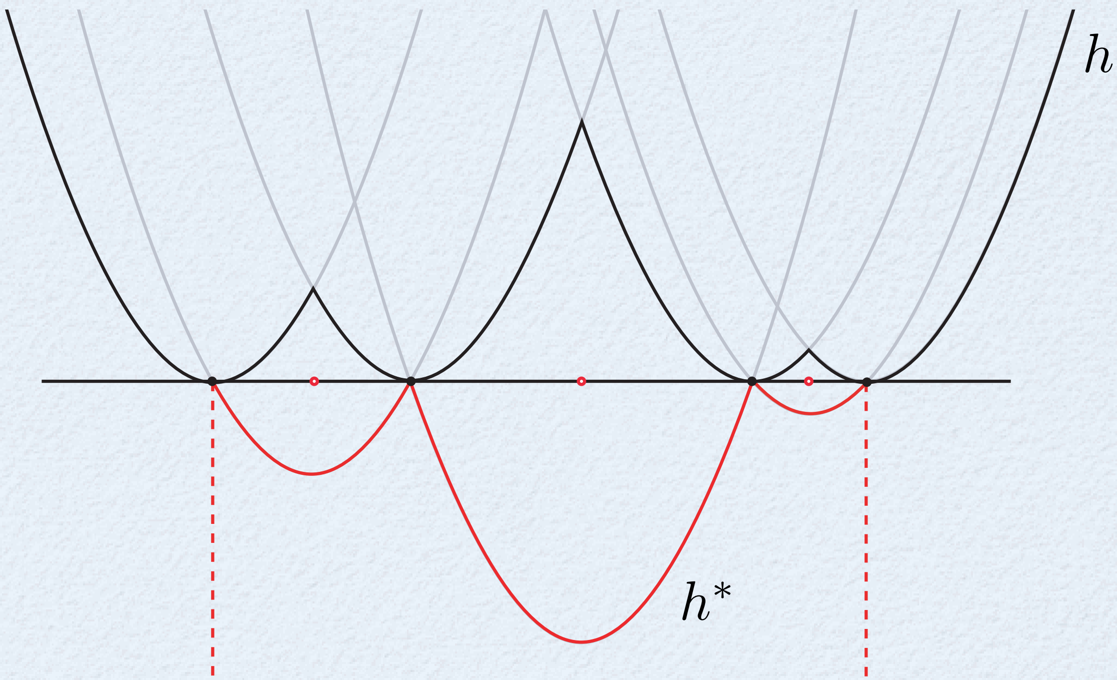

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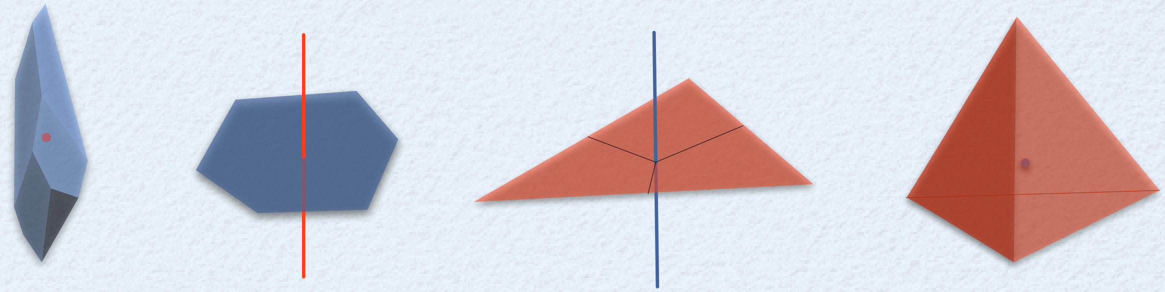
Critical Points of h^*

Observation. critical points of h^* and h are the same.



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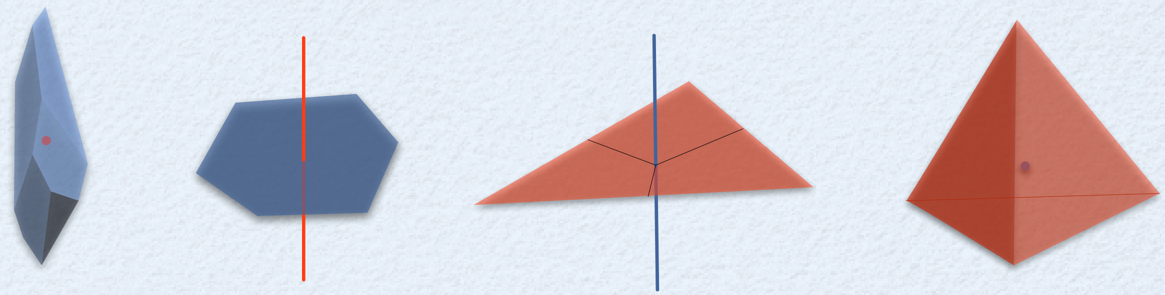

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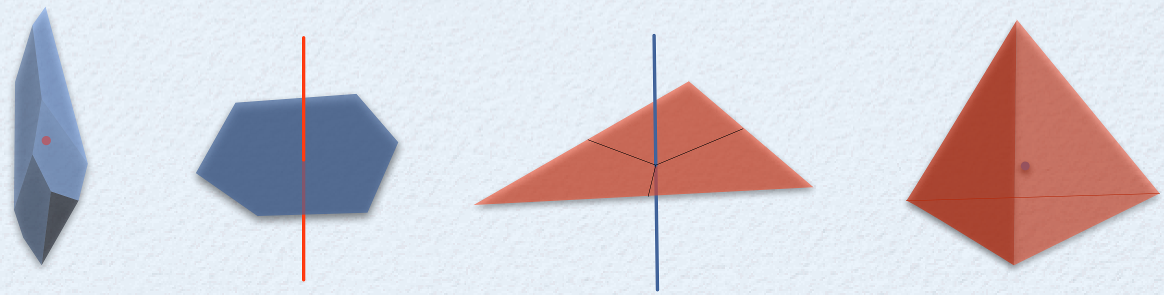

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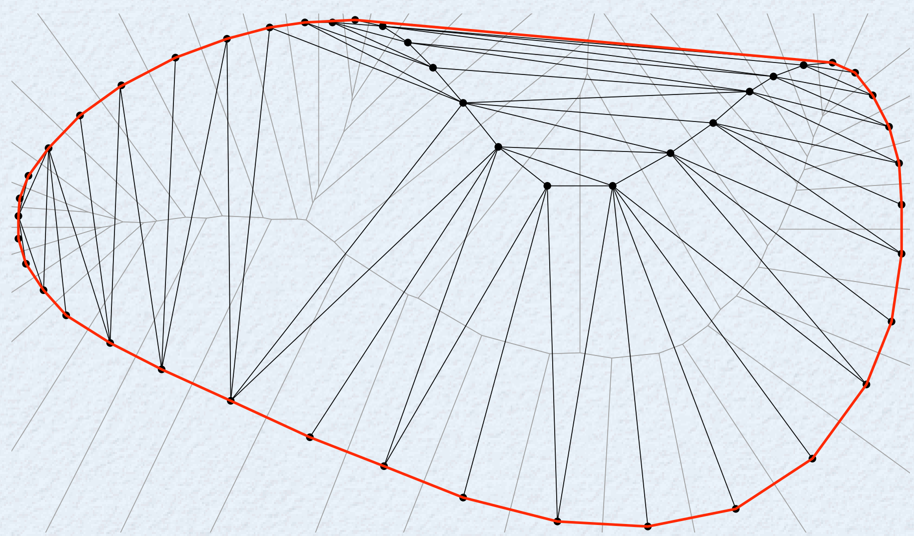
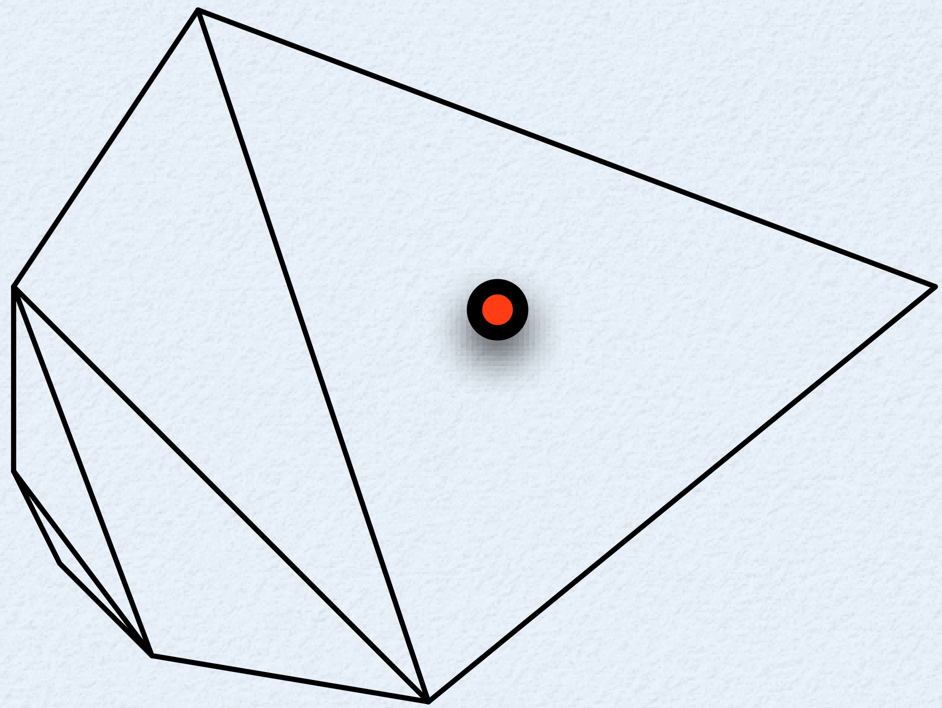
We treat $\mathbb{R}^n \setminus \mathrm{conv}\ P$ as an abstract critical simplex ω.



A Partial Order on Delaunay Simplices

$\tau \prec \sigma$: some flow line of ϕ^* visits relative interiors of σ and τ consecutively.



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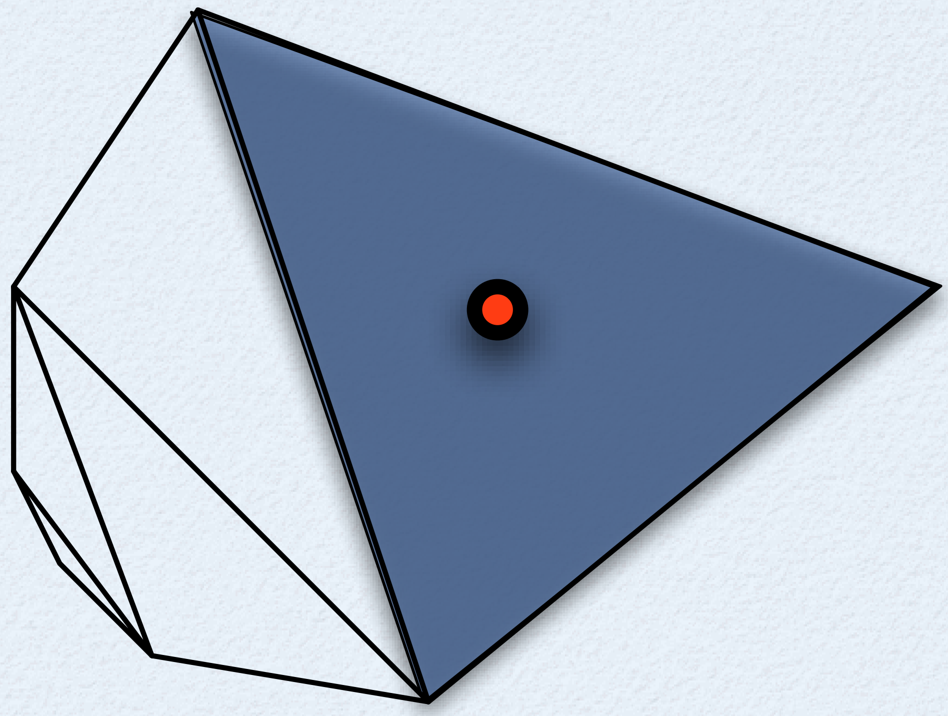

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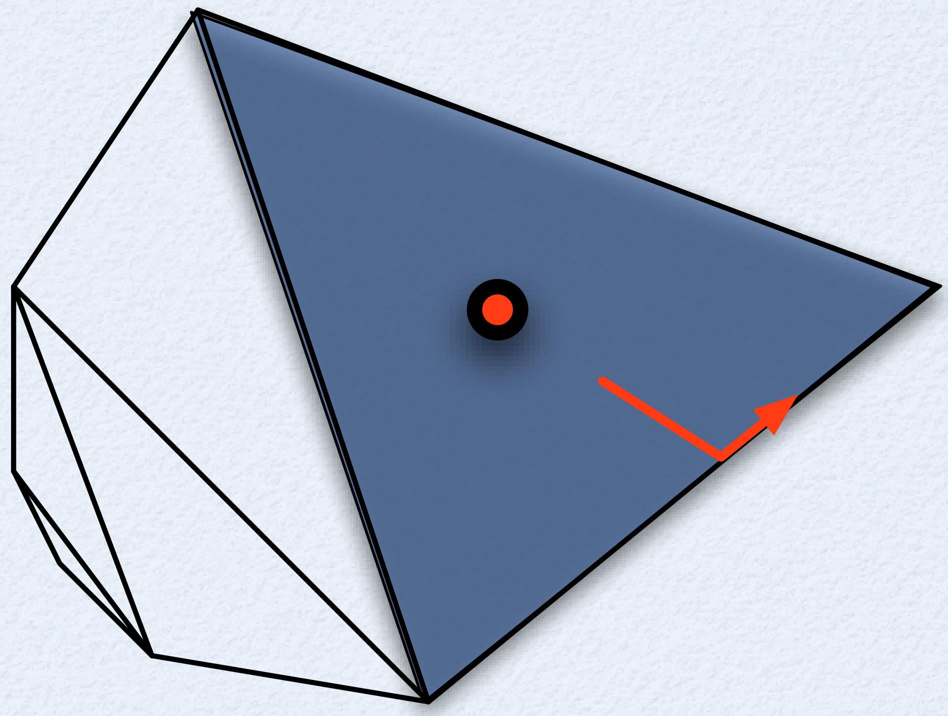

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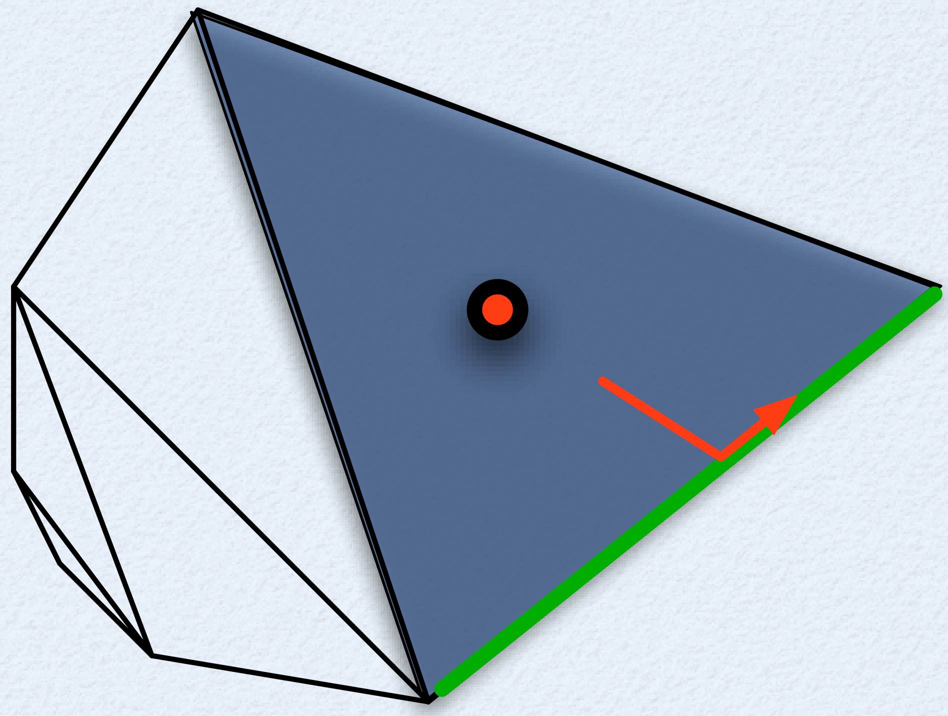

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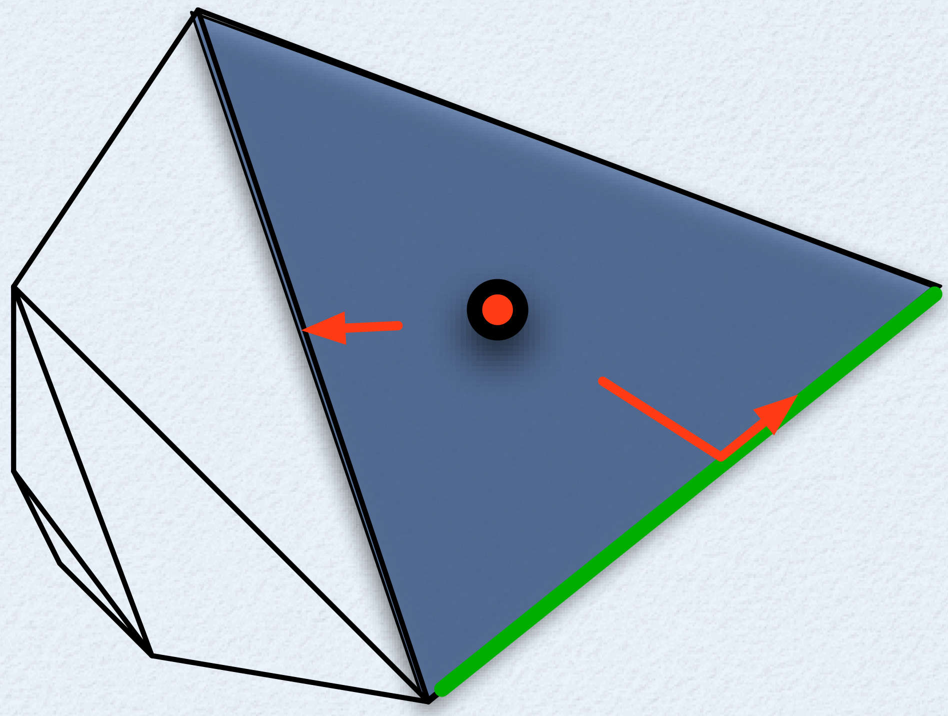

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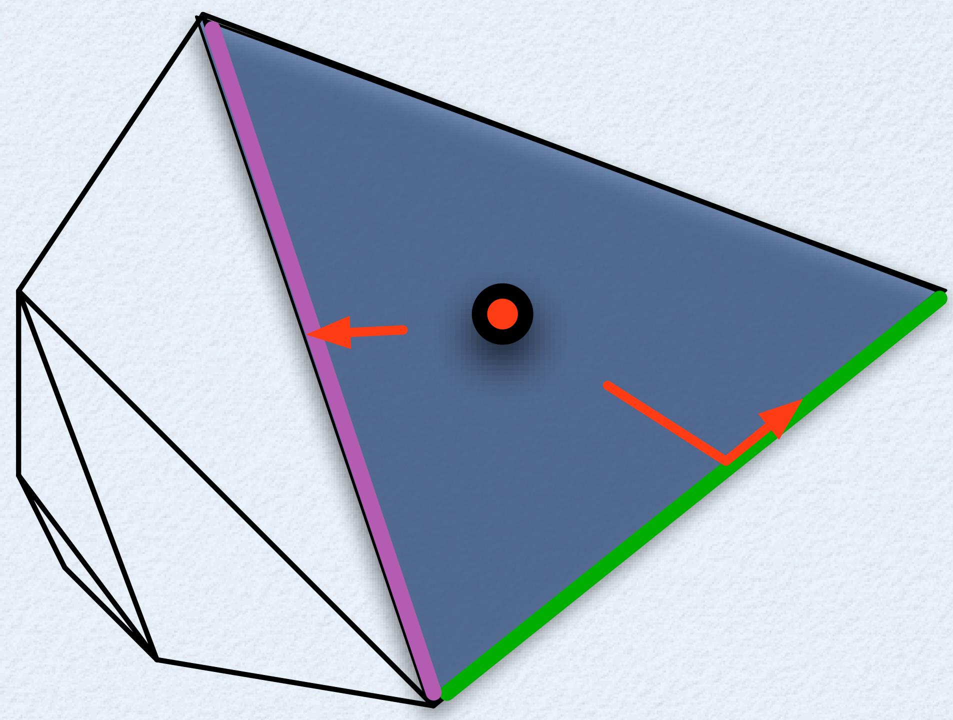

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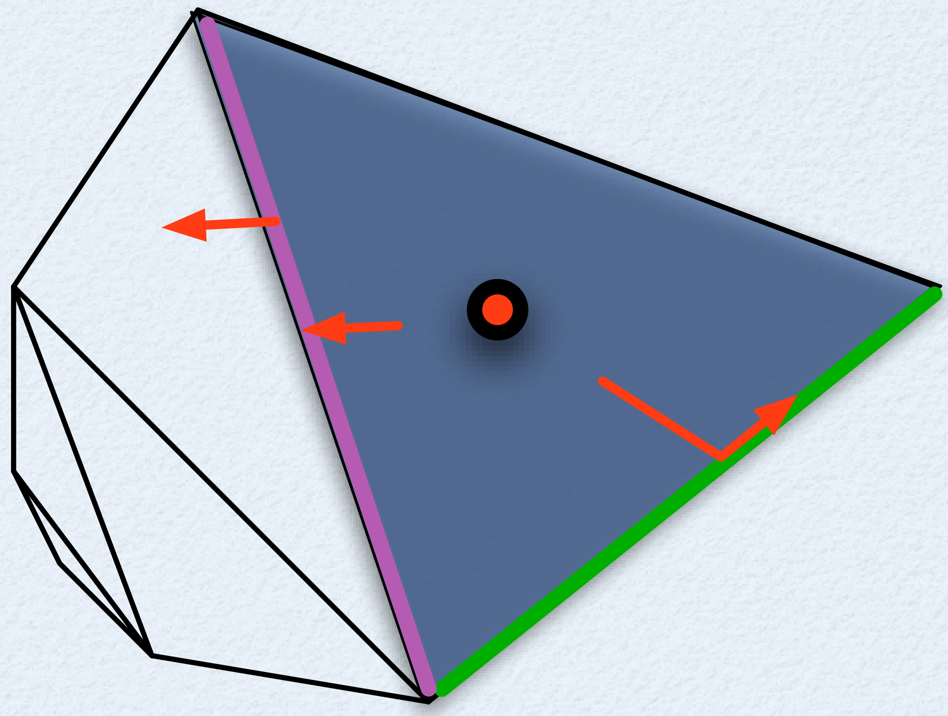

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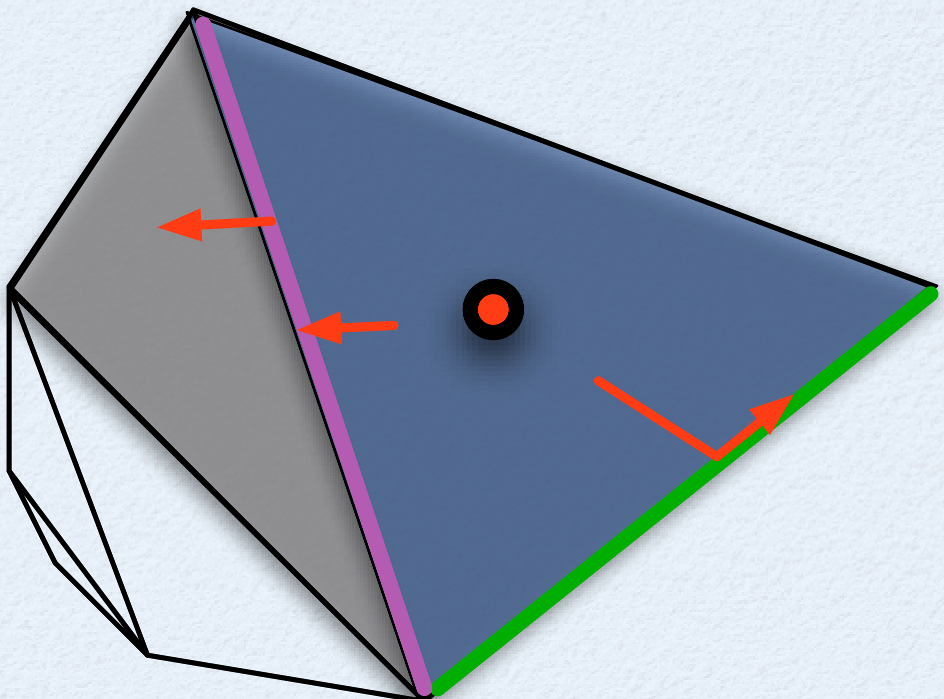

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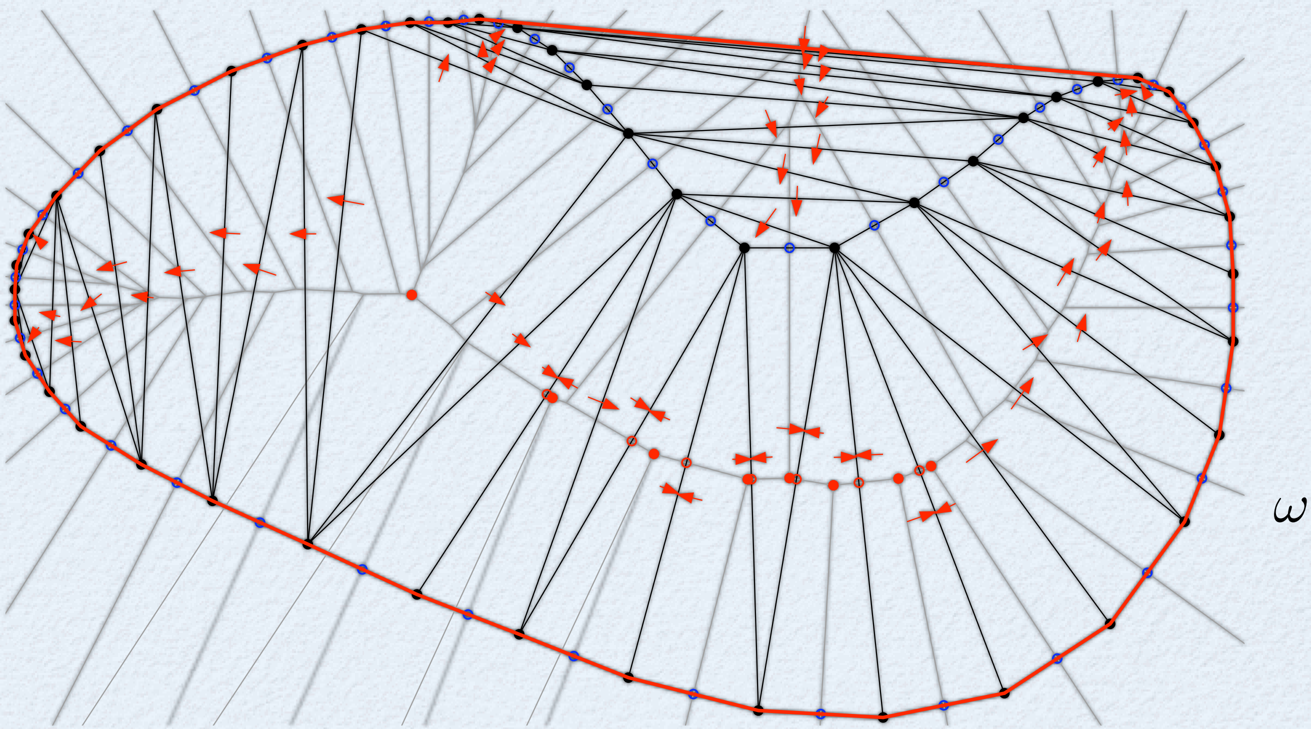
The WRAP Algorithm

The WRAP Algorithm [Edelsbrunner'04]

1. For every $\tau \in \mathrm{Del}\ P$, if the only critical simplex that precedes τ is the abstract critical simplex ω, then remove τ.

2. Return what is left as WRAP.



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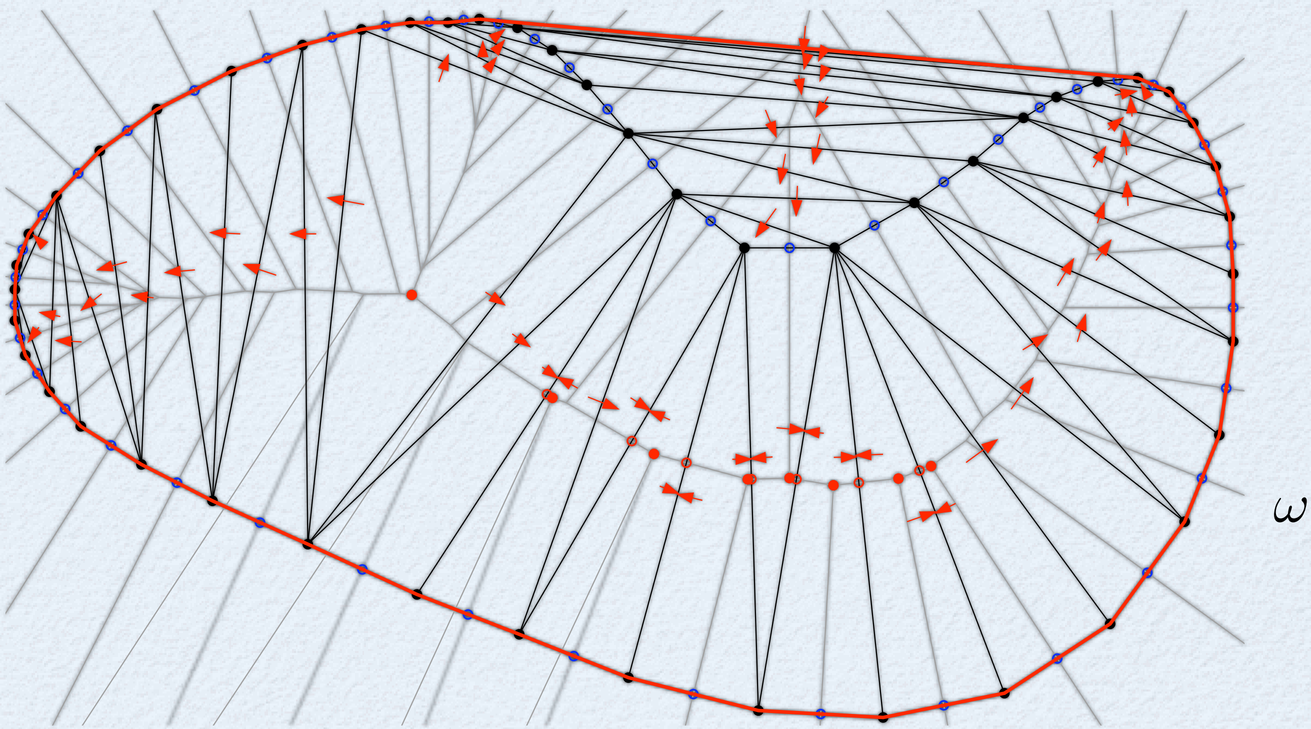

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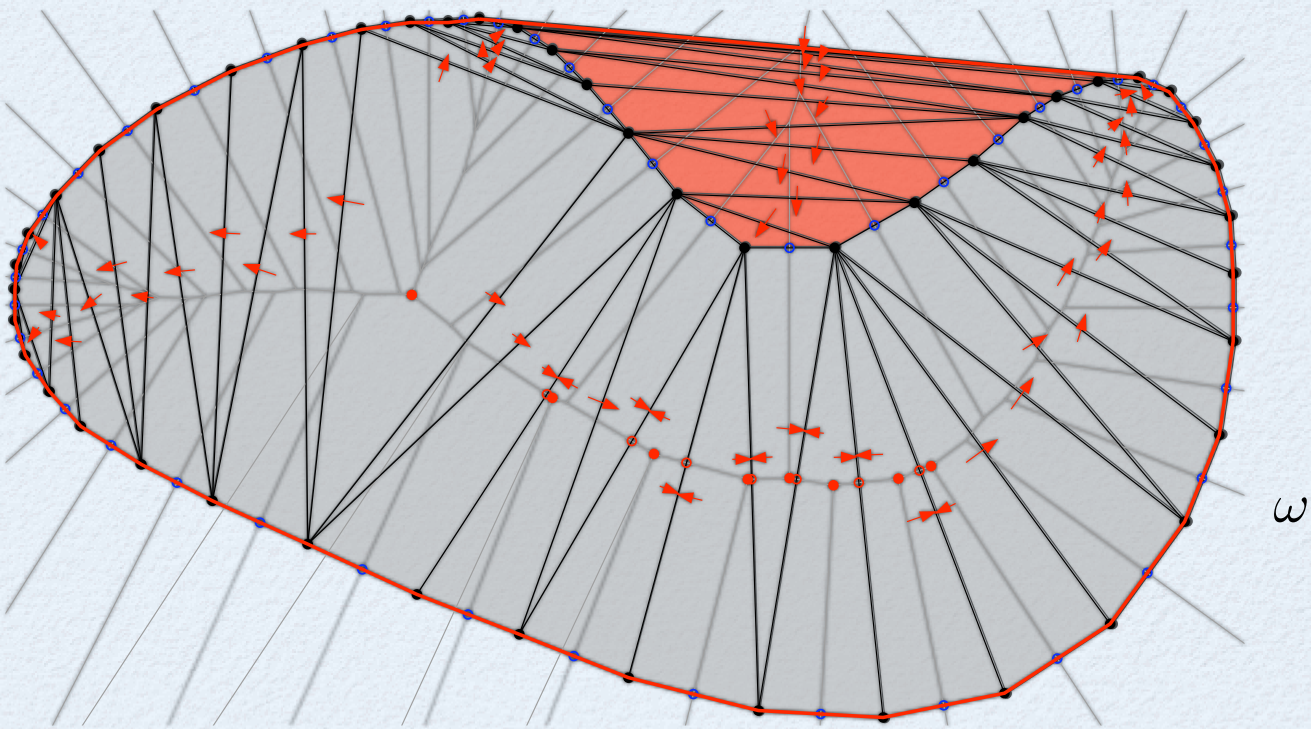

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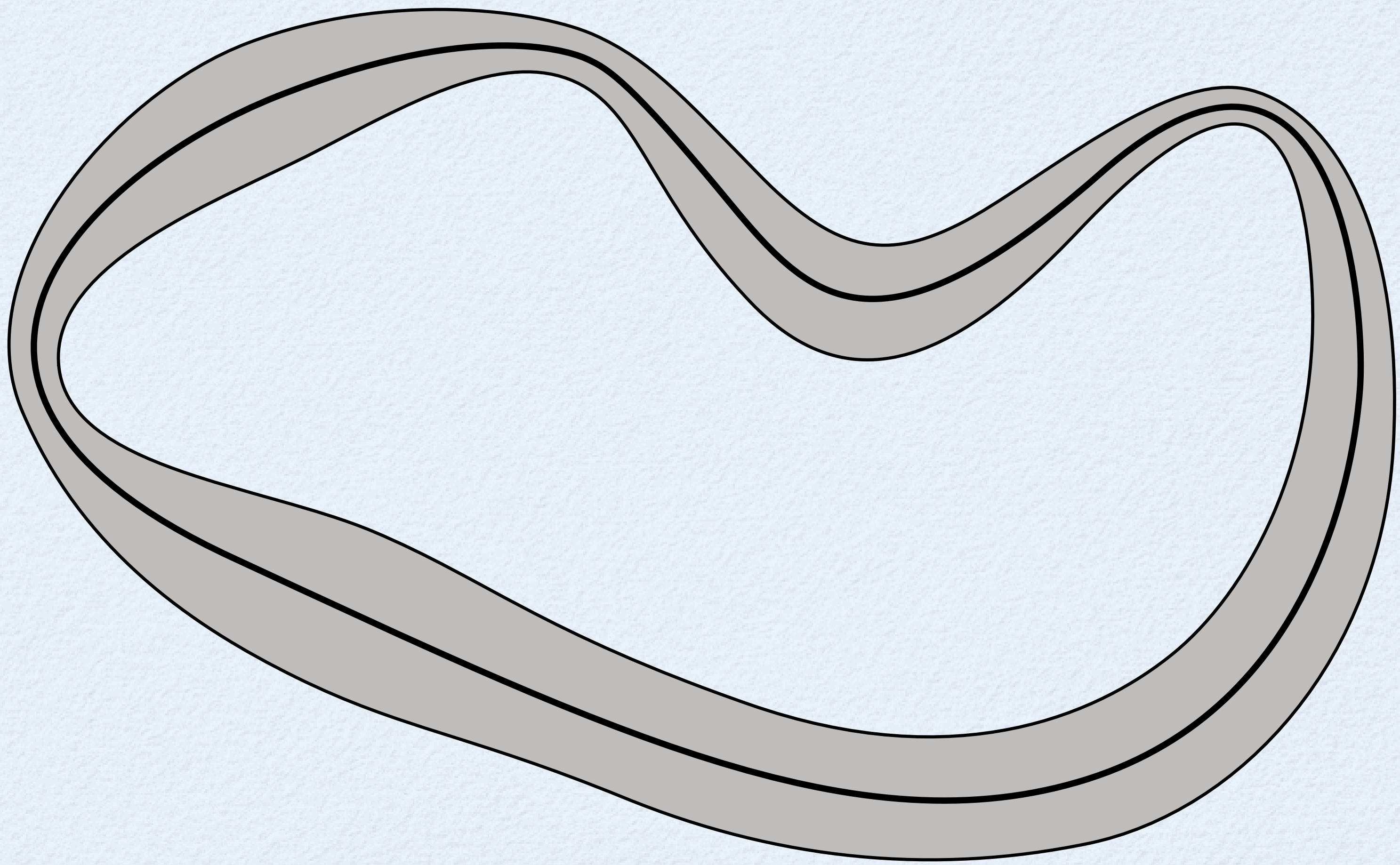

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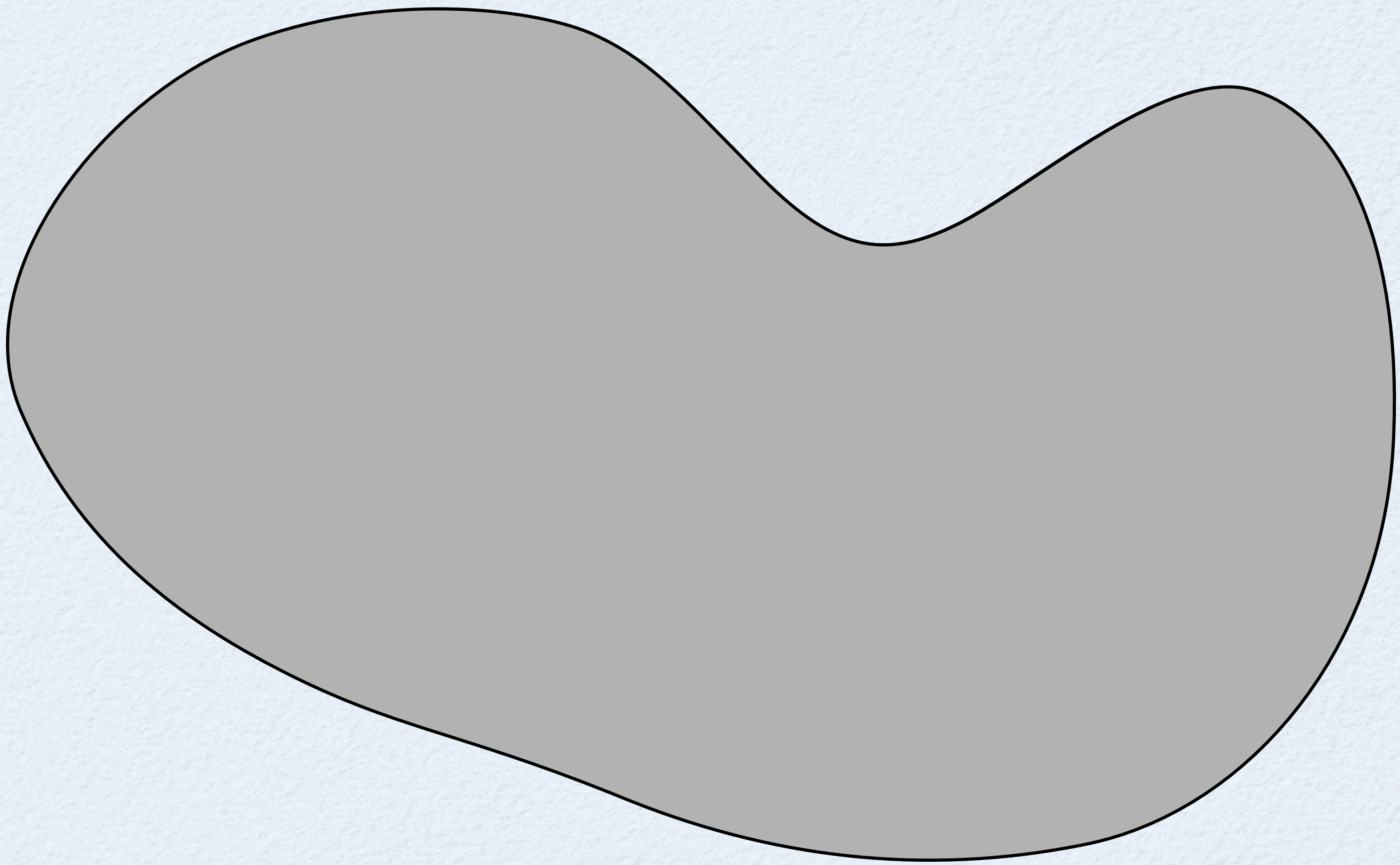

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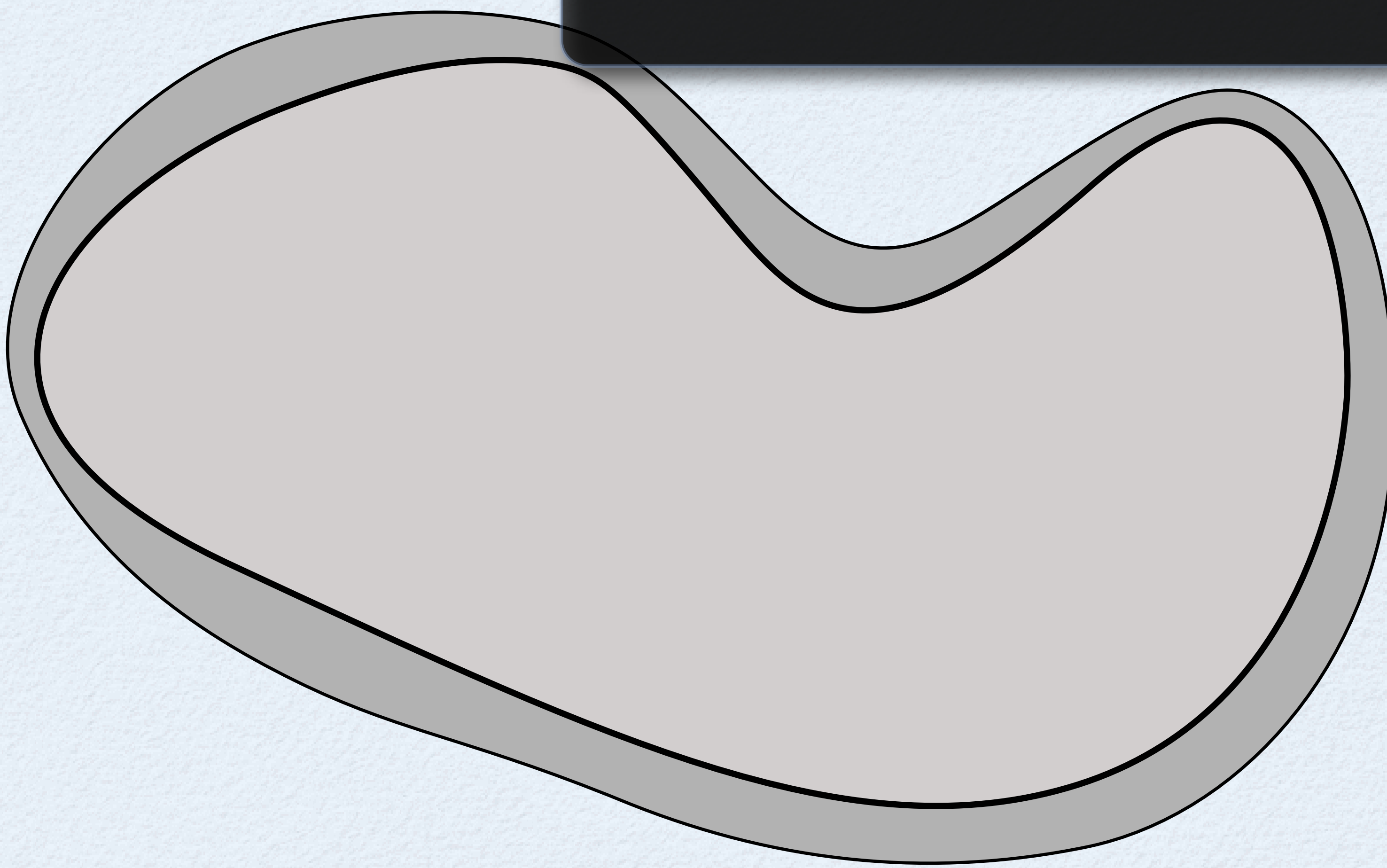

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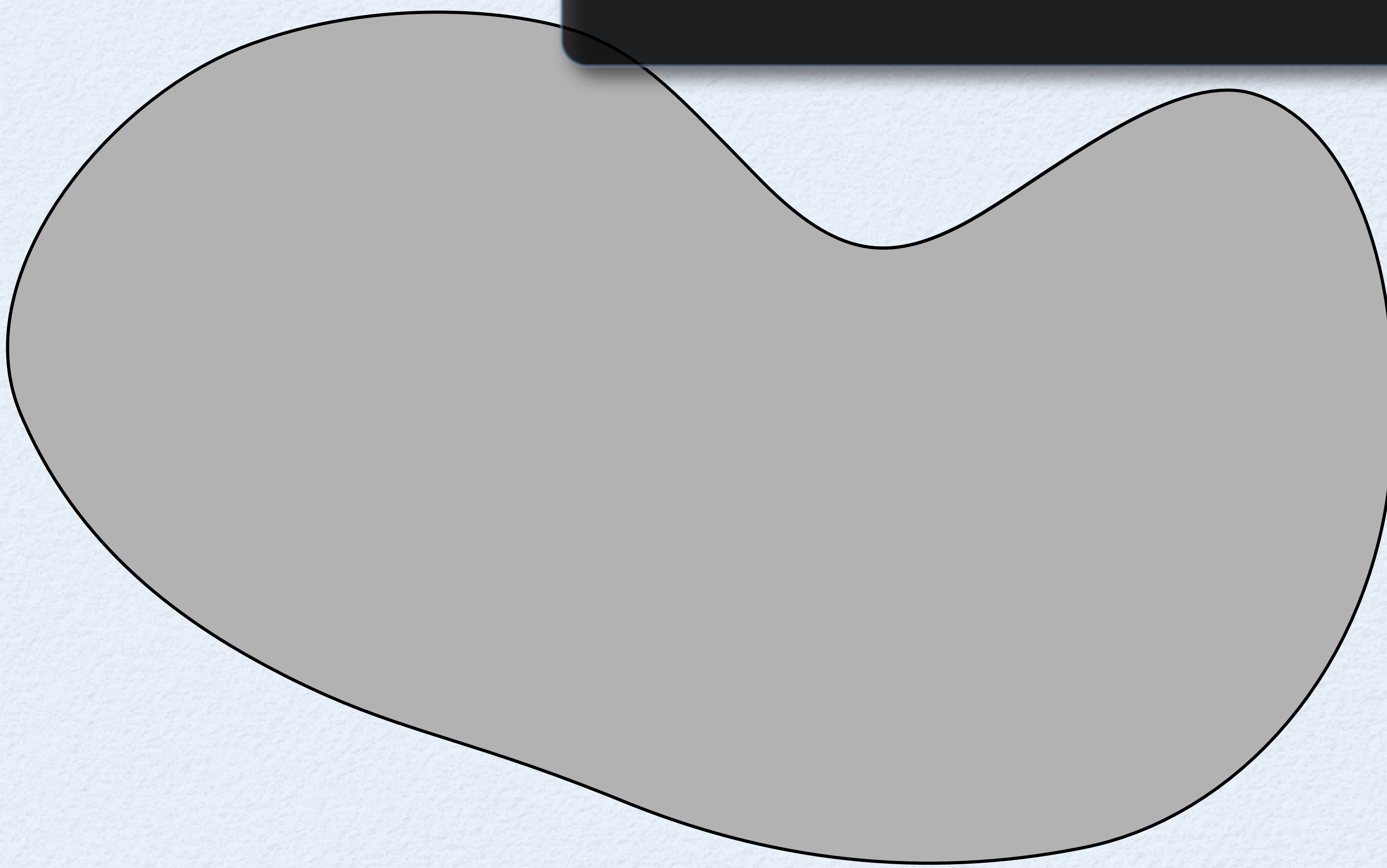

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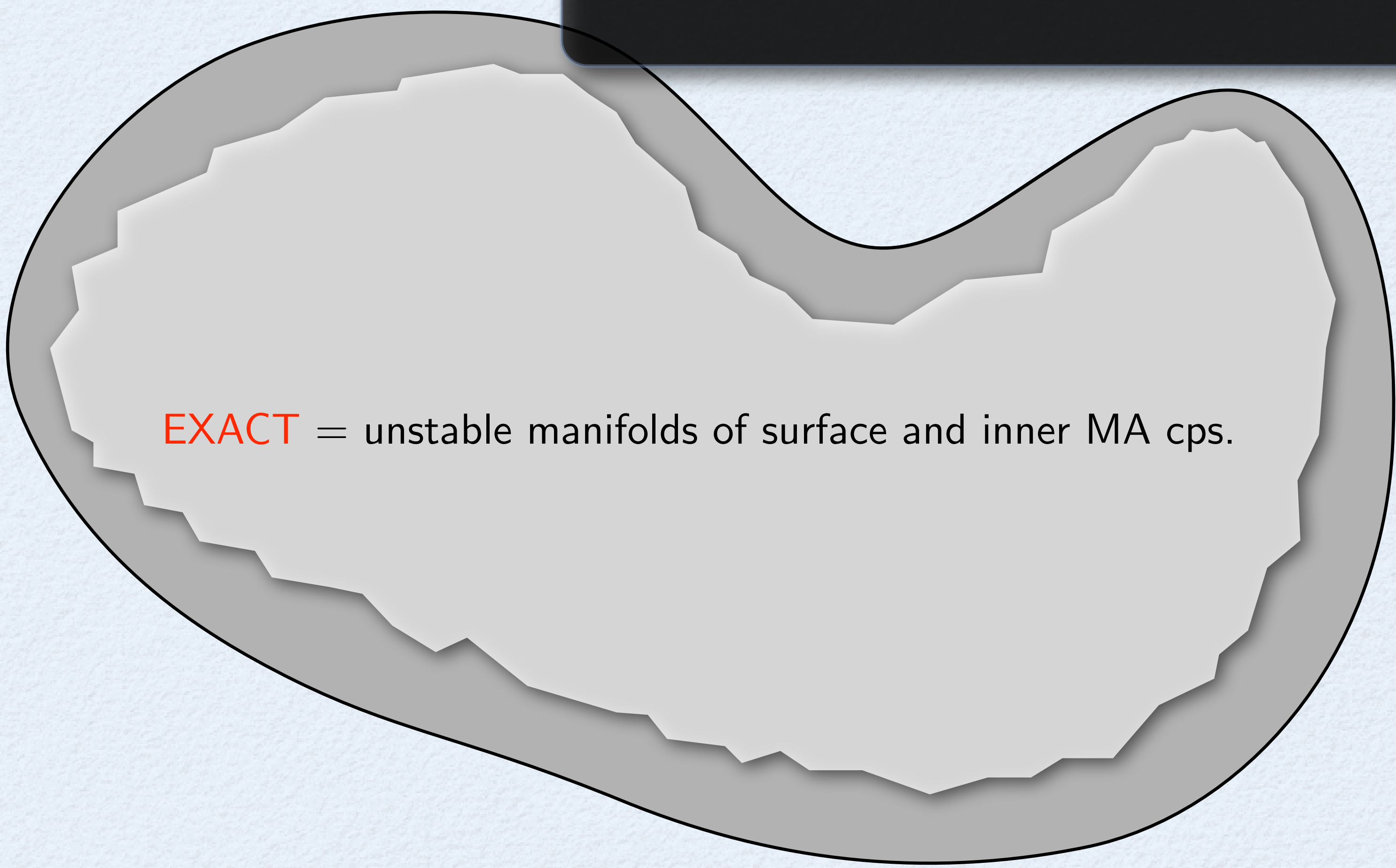

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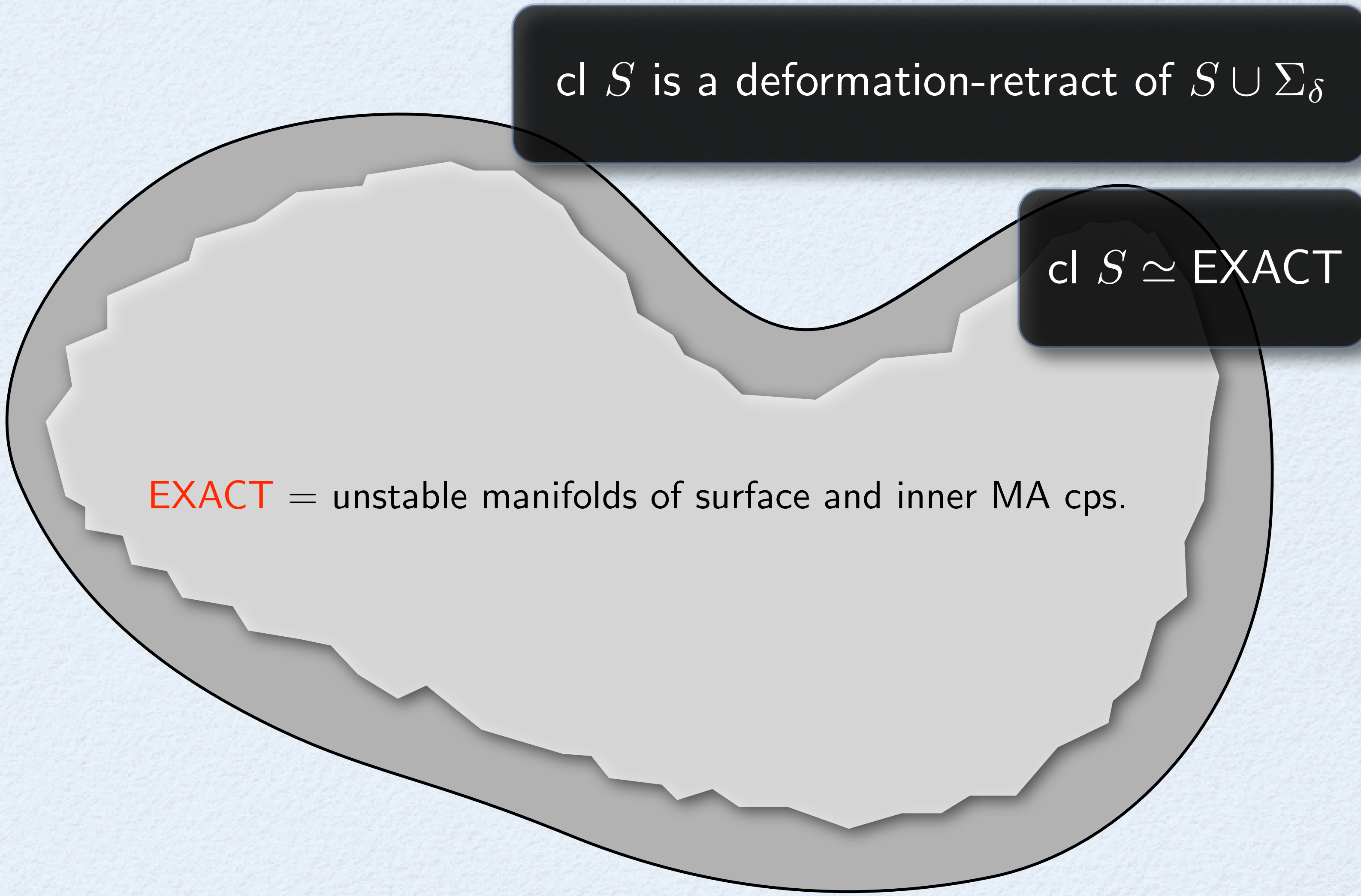

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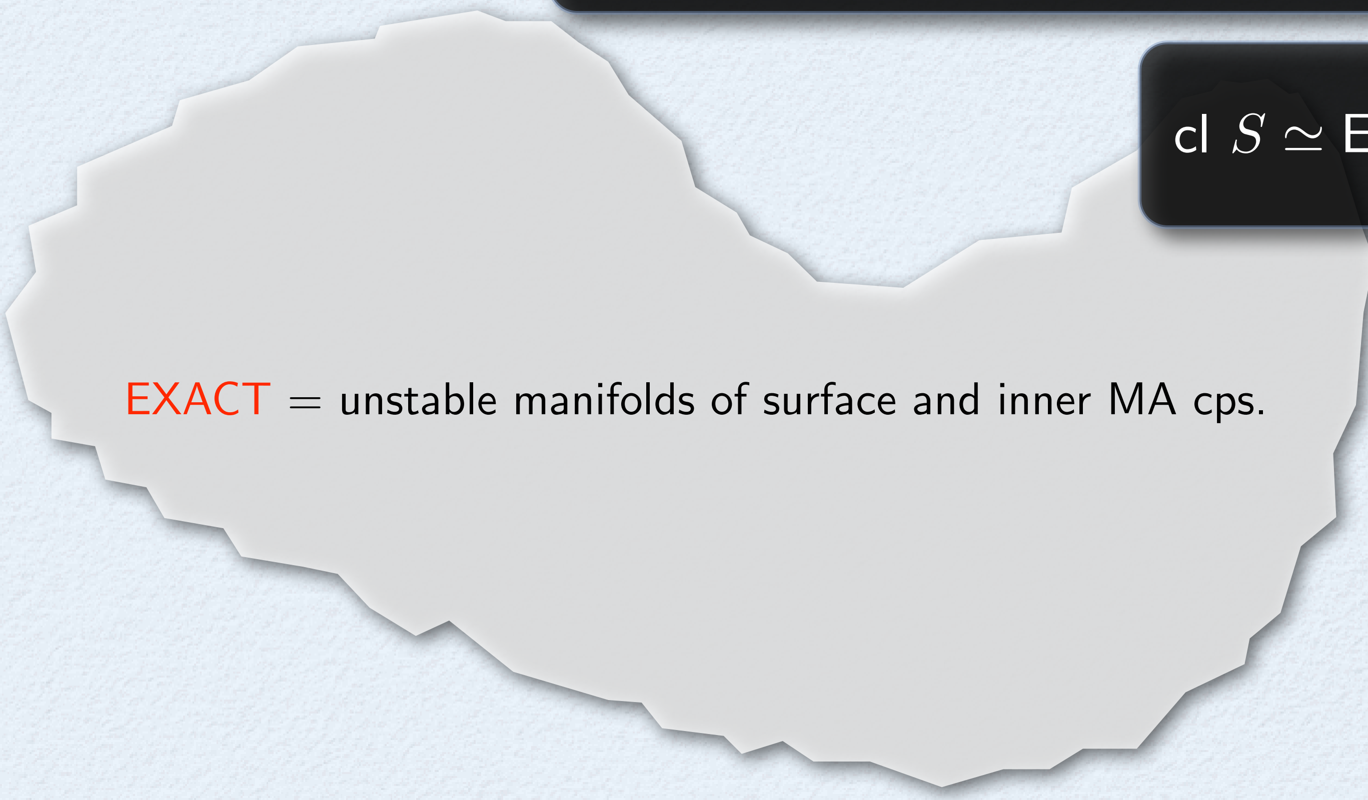

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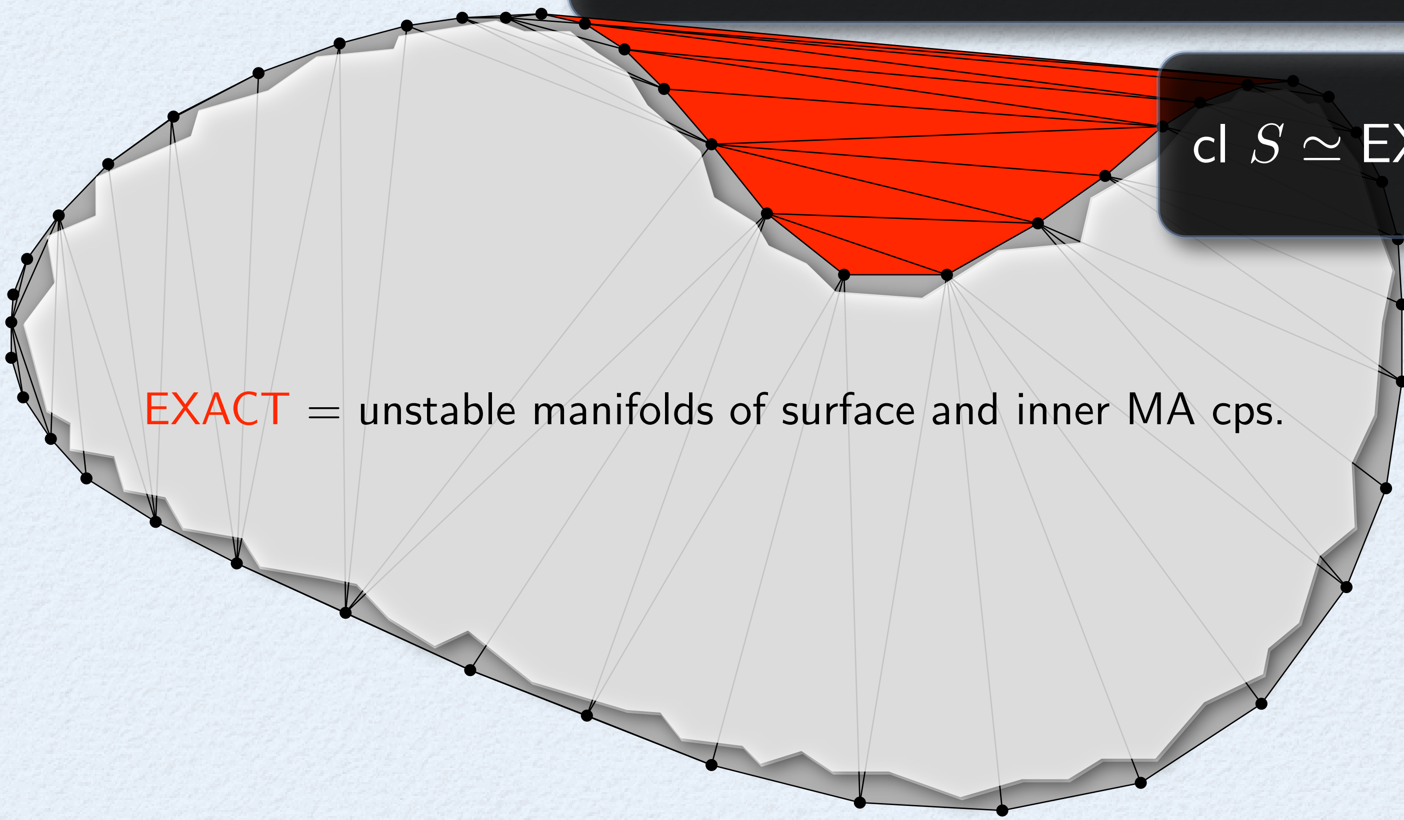

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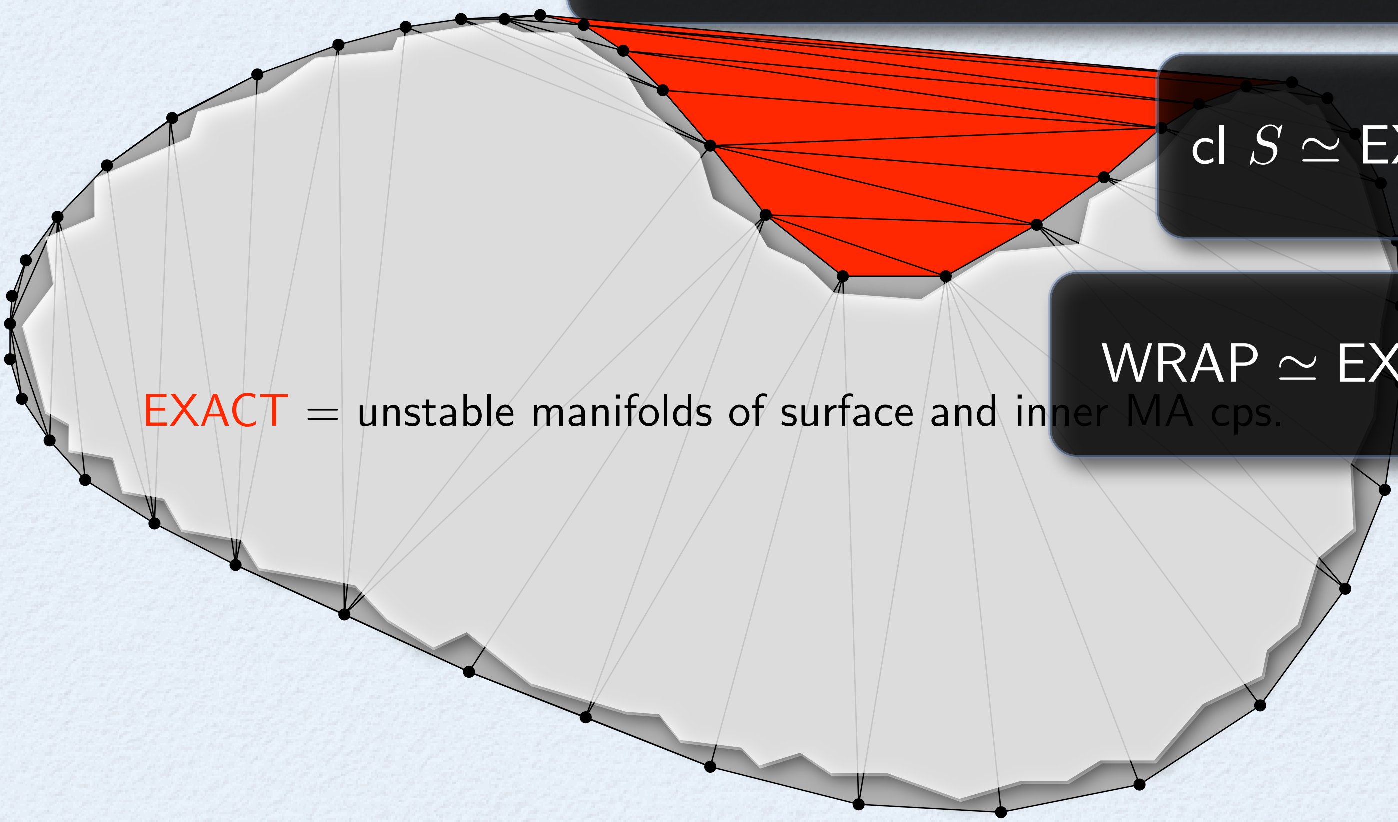

Analysis of WRAP

Theorem. WRAP and closure of shape are homotopy equivalent. (in 3D)

Lemma. $\phi^*(\Sigma_\delta) = \Sigma_\delta$.

cl S is a deformation-retract of $S \cup \Sigma_\delta$

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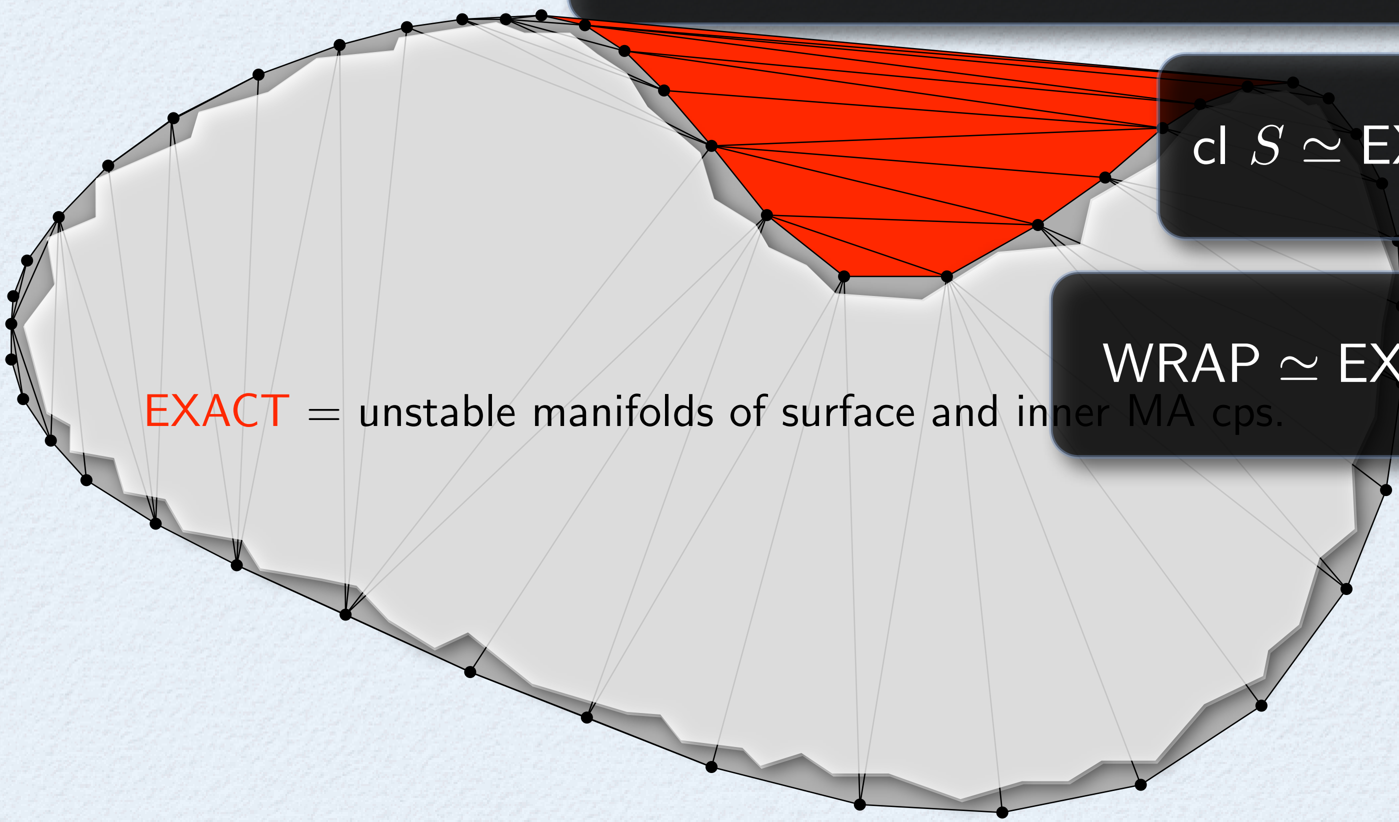

EXACT = unstable manifolds of surface and inner MA cps.

Summary of Results

Using the distance flow maps induced by samples,

- We showed that the critical points of the distance function induced by an ε-sample of a surface are concentrated close to the surface or to the medial axis and these two types can be distinguished algorithmically.

- We gave an algorithm that reconstructs the shape homotopically (and its boundary homeomorphically in 3D) and approximates these closely in geometric terms.

- We introduced the notion of the CORE for medial axis approximation and established its homotopy equivalence with the medial axis. We also showed how the core can be extended by any other MA-approximation algorithm. We also bounded the rate of degradation of geometric approximation of MA in taking flow closures.

- We modified Edelsbrunner's WRAP reconstruction algorithm and proved that this modified version captures the topology of the sampled shape (for ε-samples in 3D and for uniform samples in any dimension).

Some Open Questions

- In RECONSTRUCTION, is the union of stable manifolds of surface critical points also homotopy equivalent to surface?
- The "primal" analog of WRAP corresponds to an approximation of CORE by a subcomplex of Vor P. A geometric analysis showing this approximation is close to MA is enough to prove the same topological guarantee for the approximation.
- Can these ideas (especially WRAP) be generalized for reconstruction of shapes with non-smooth surfaces? How should the sampling condition be defined? (some work done in [Lieutier-Chazal'06])
- In general when (and in what sense) can stable and unstable manifolds of critical points be approximated by sub-complexes of Vor P or Del P.
- Can the proof of existence of a continuous flow map be generalized to non-discrete sets of weighted points (generalization of Lieutier's result)?

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Thank You!