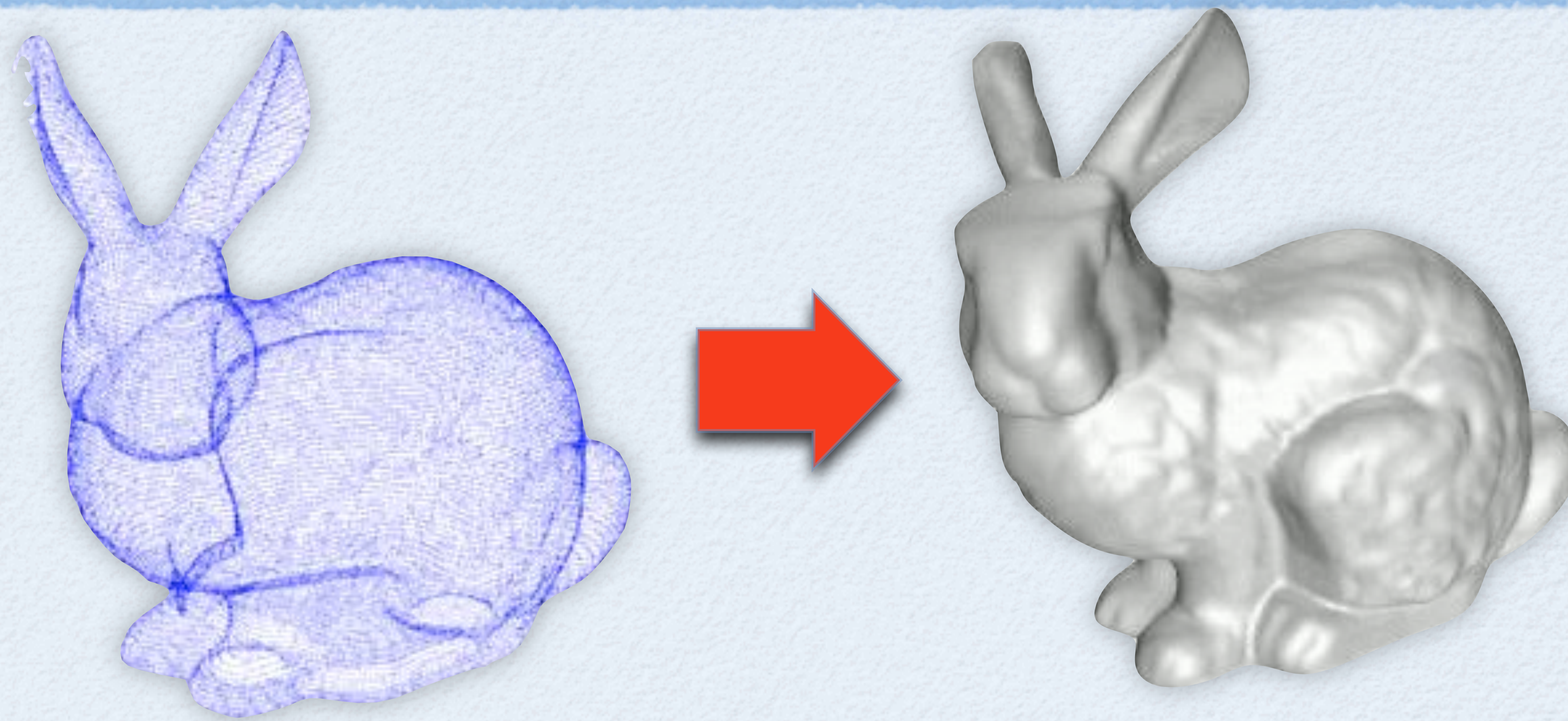
Manifold Homotopy via the Flow Complex

Bardia Sadri
Duke University

The Surface Reconstruction Problem



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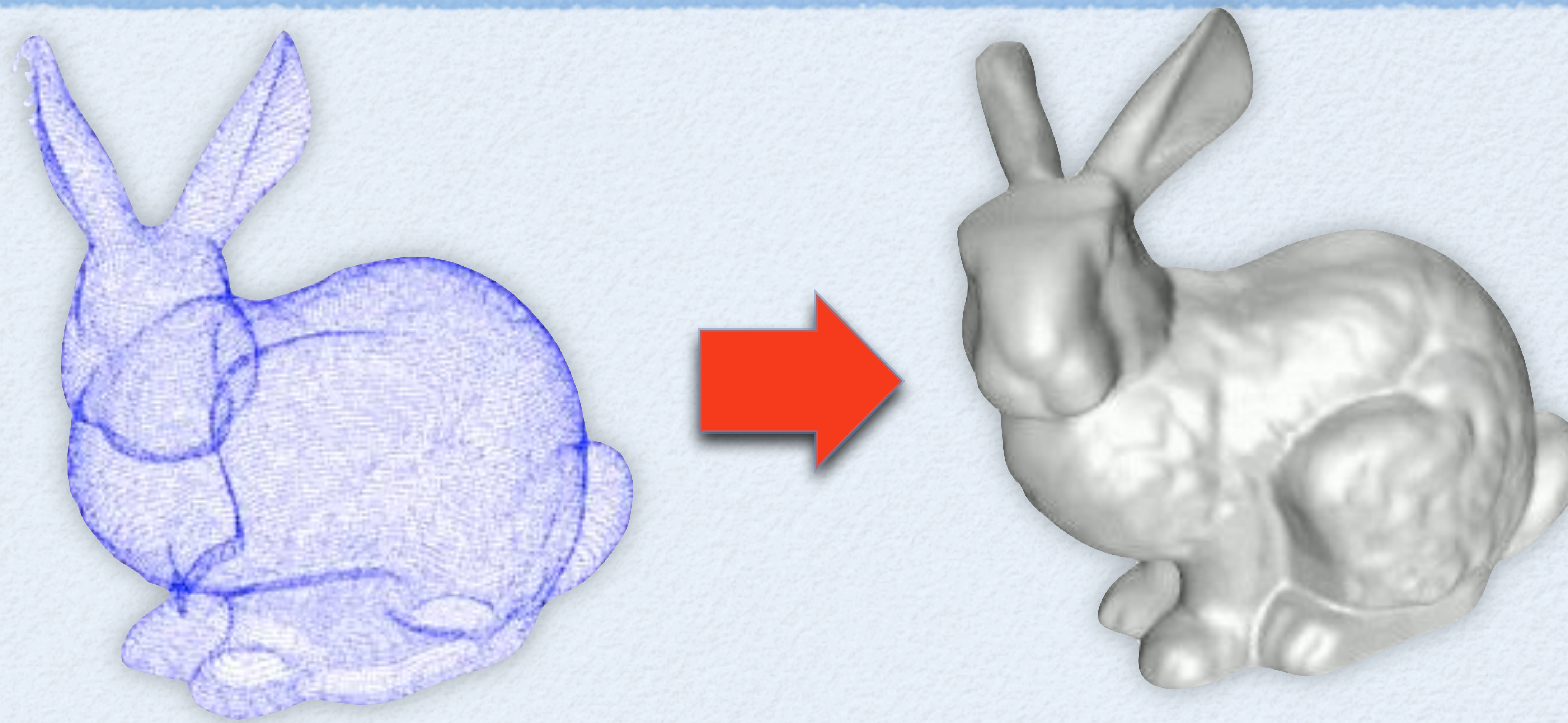

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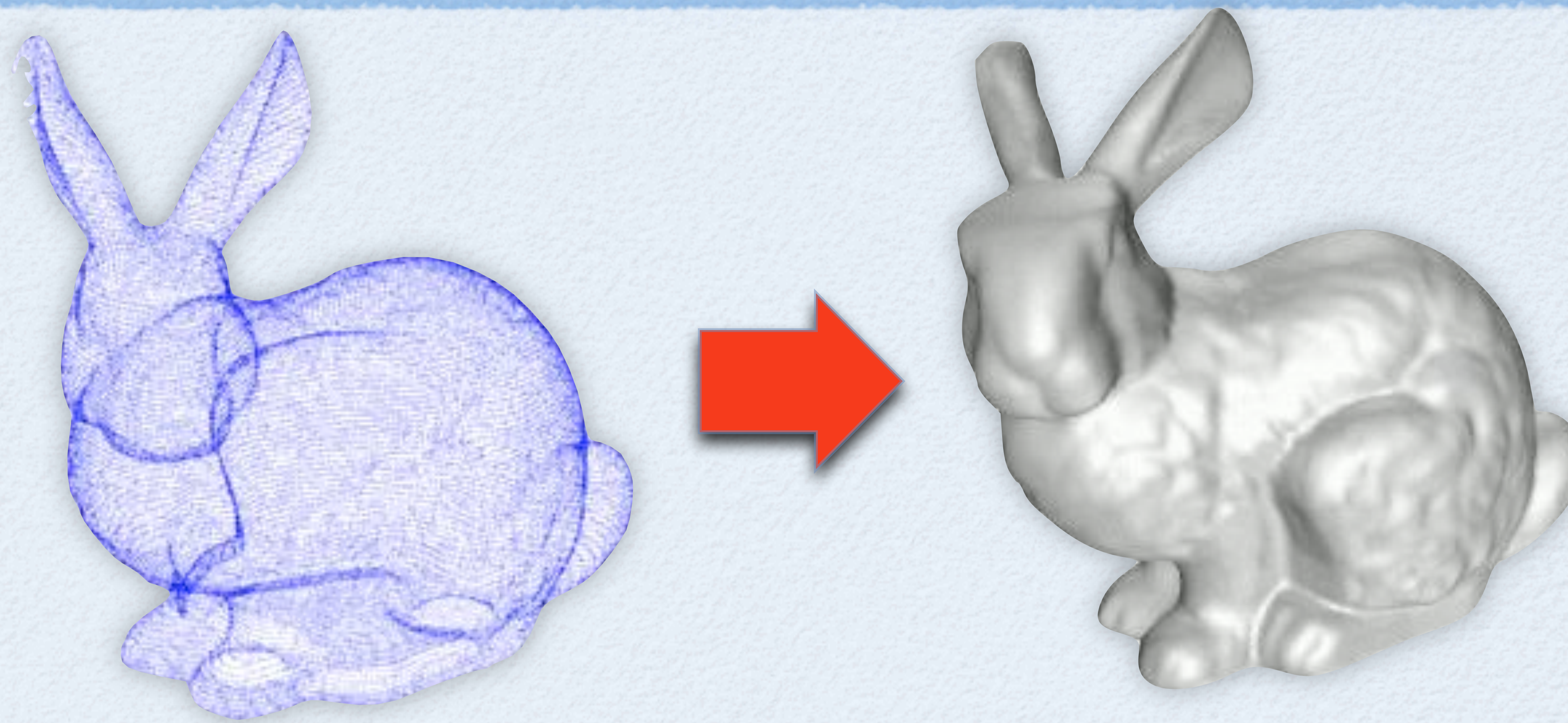

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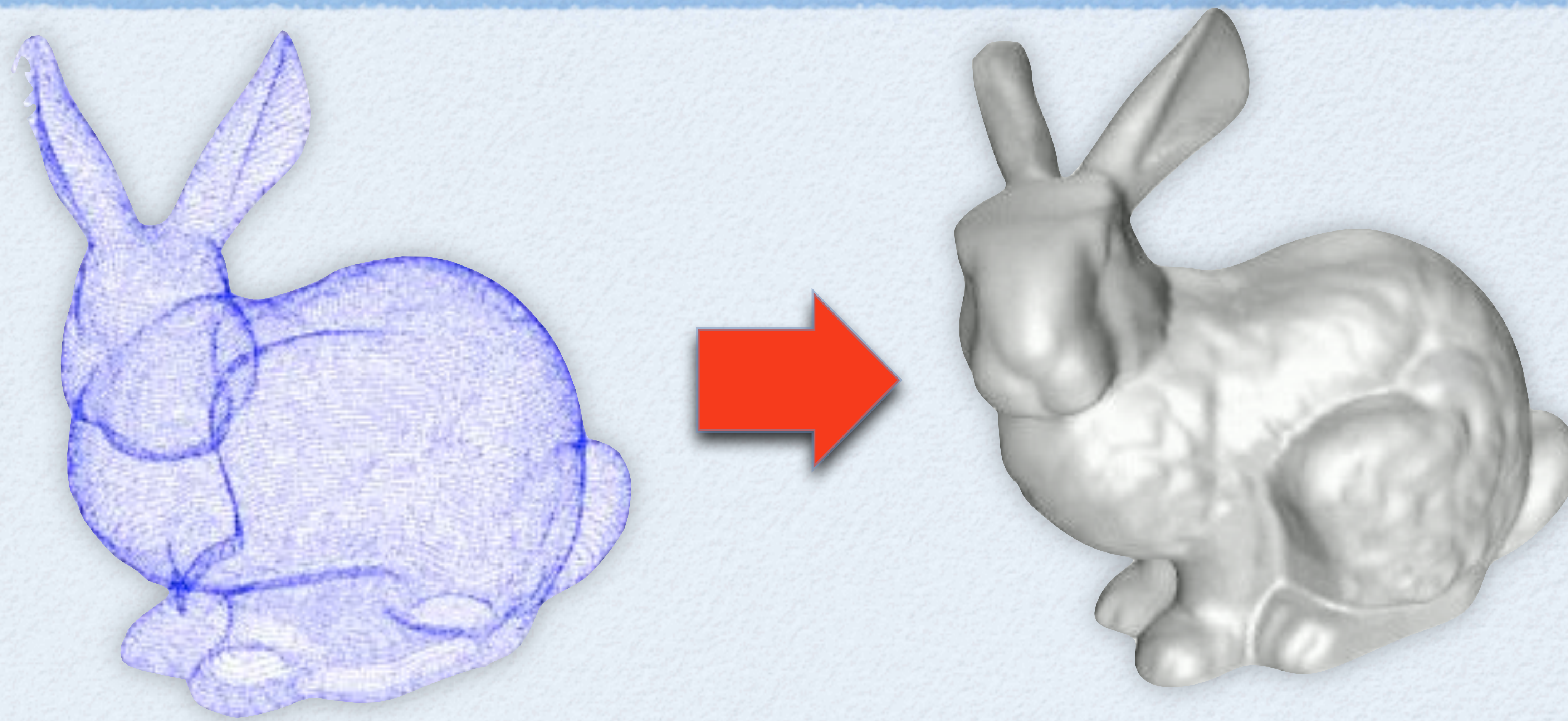

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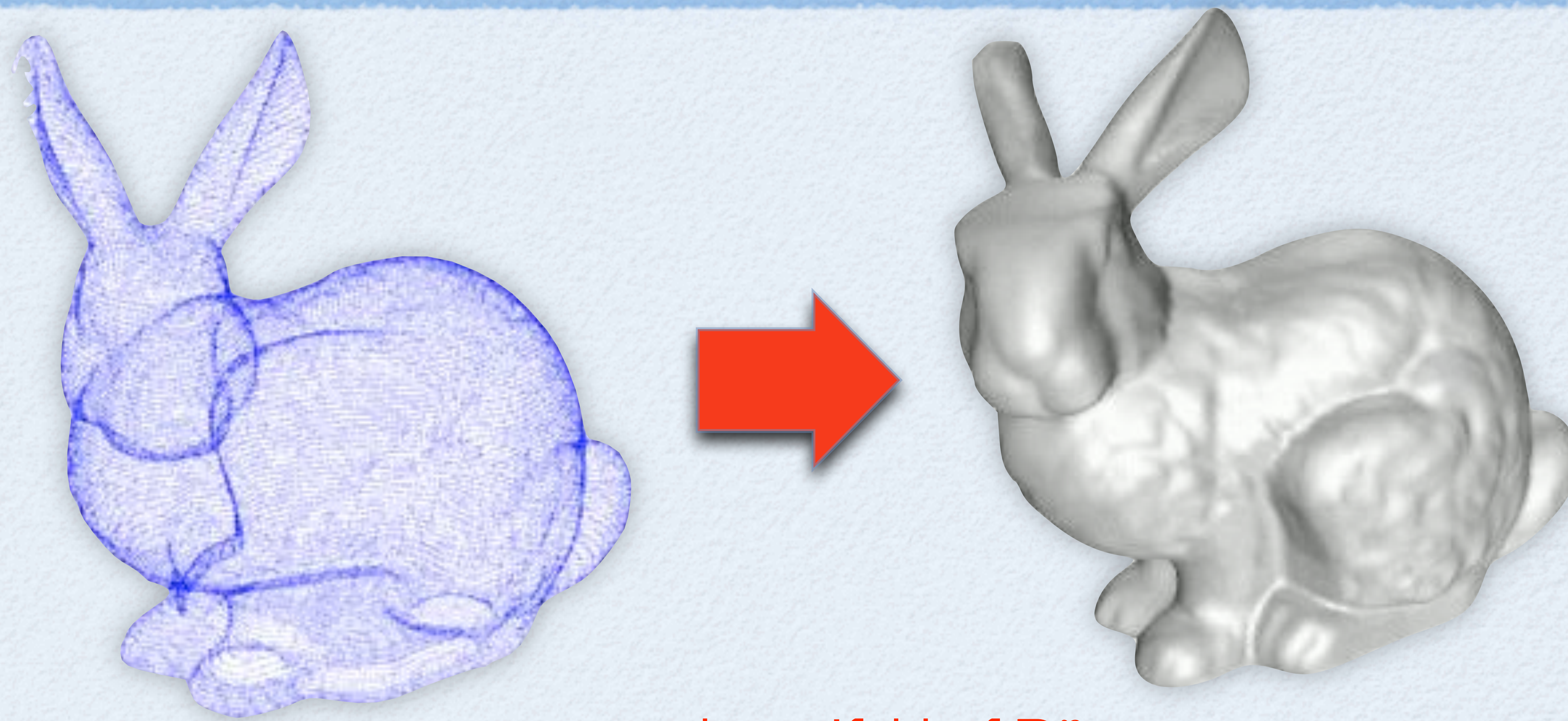

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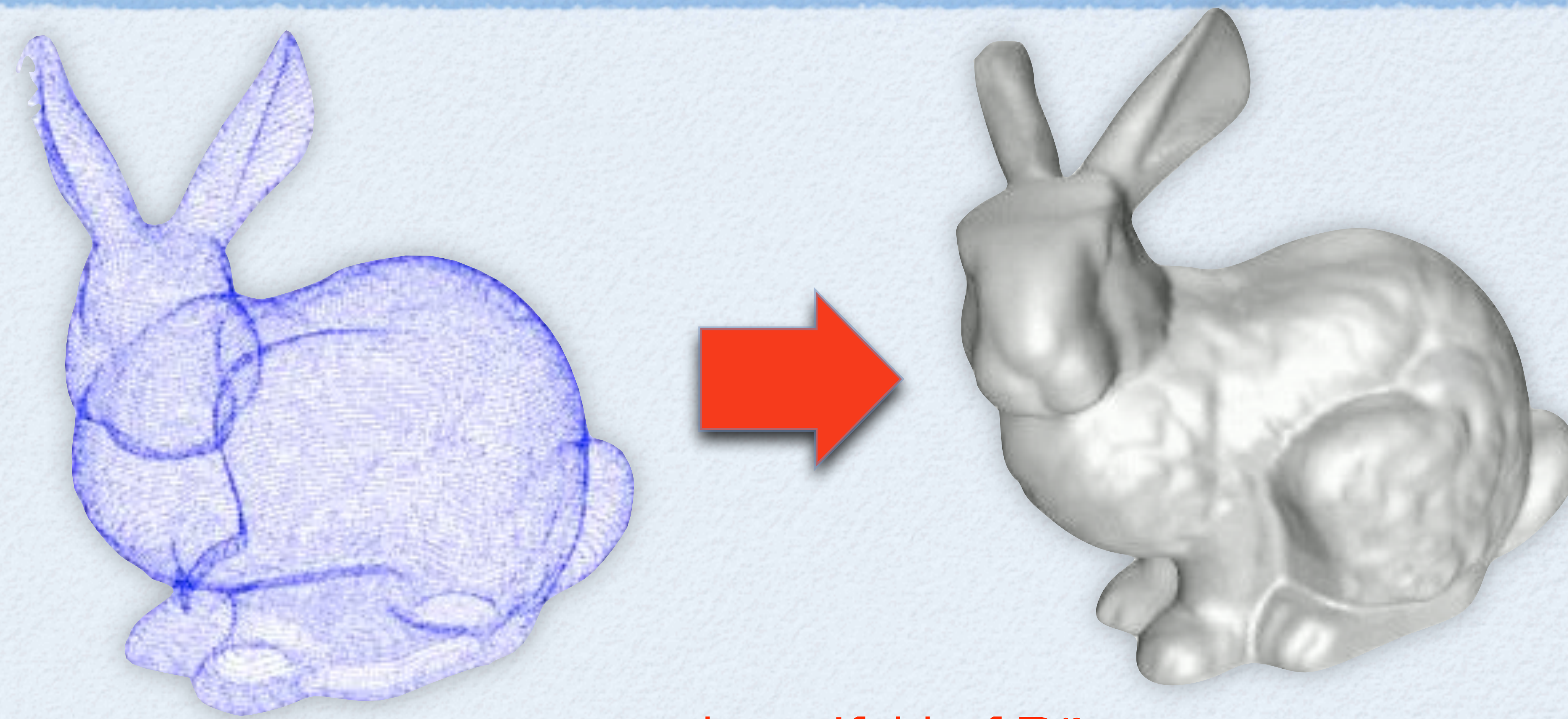

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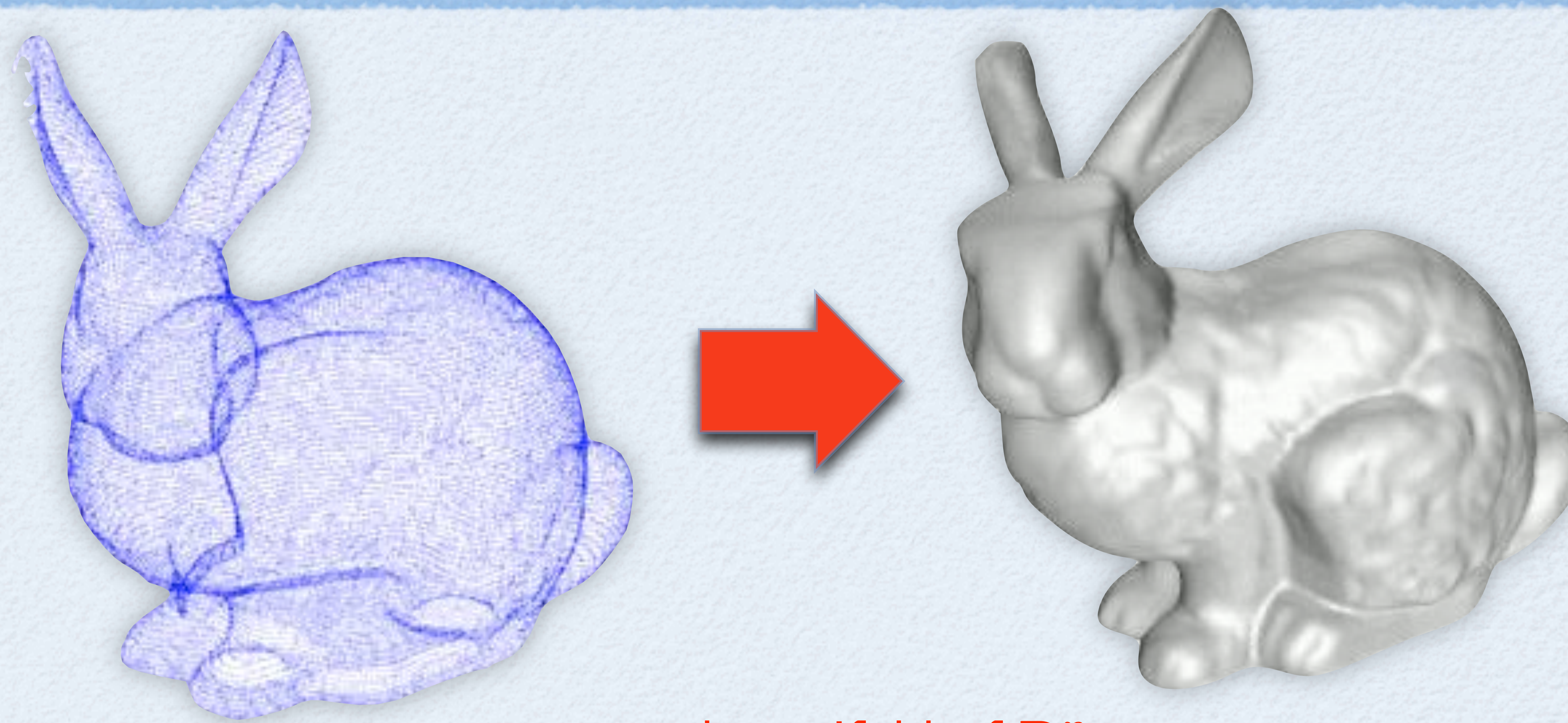

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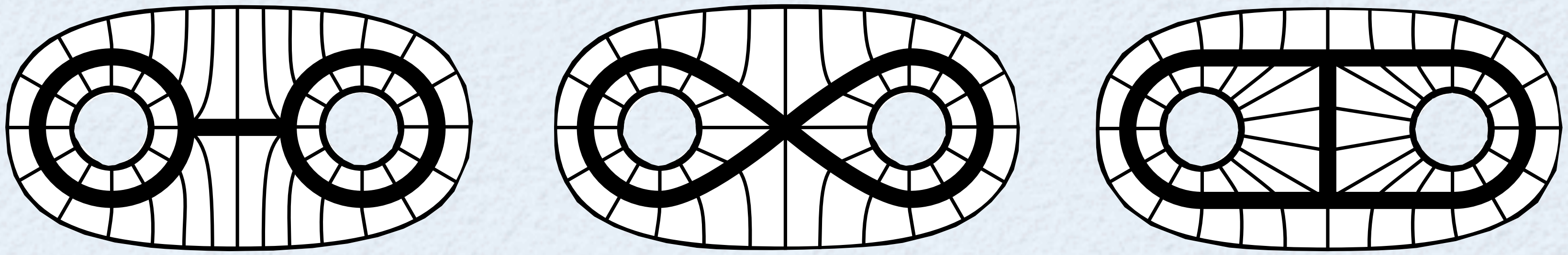
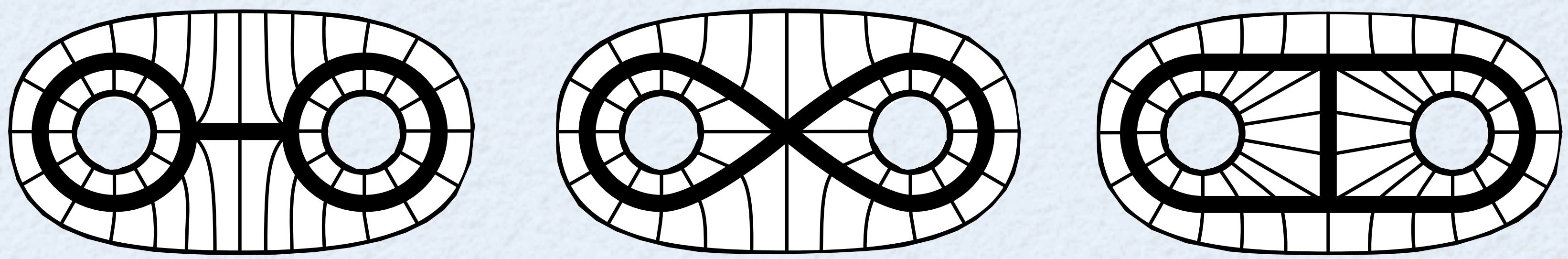
Homotopy Equivalence



Unlike homeomorphism, homotopy equivalence does not preserve dimension.

Homotopy Equivalence



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All these knots have the same homotopy type, but not their complements.

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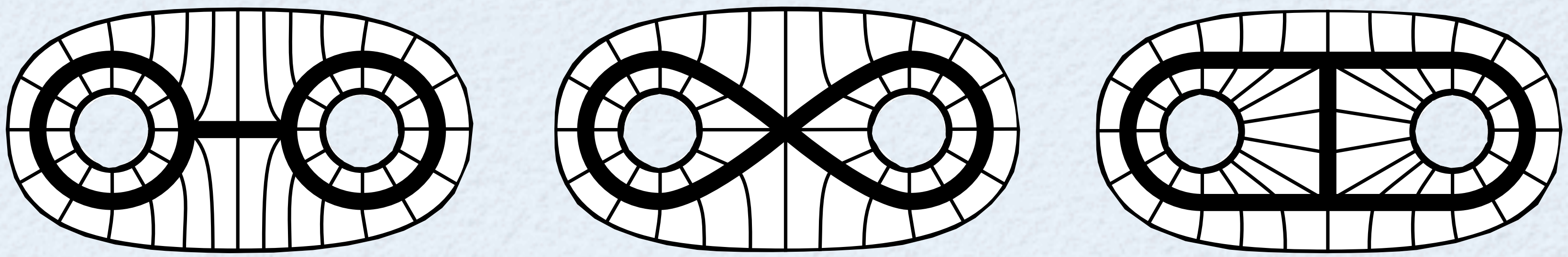

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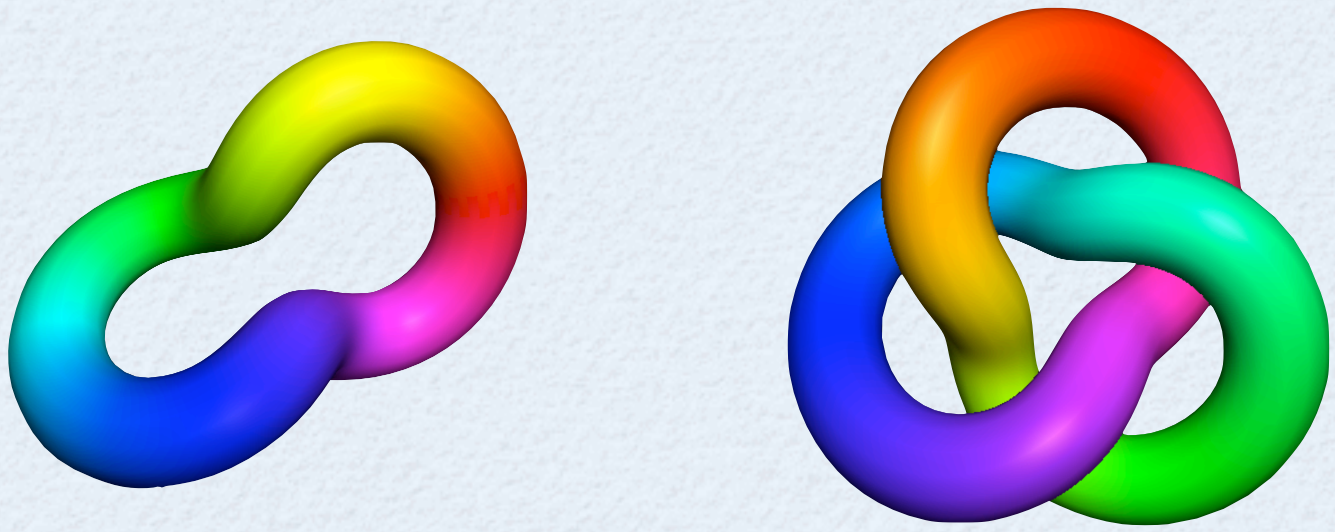
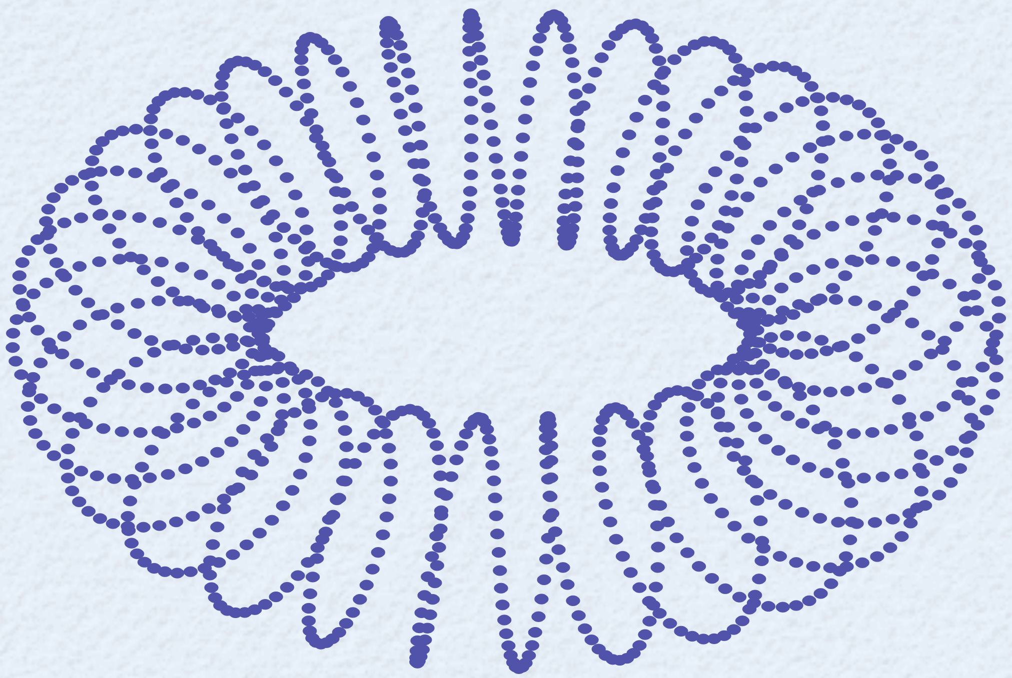


What if we don't know the submanifold's dimension?

Naturally occurring data may be generated by structured systems with much fewer degrees of freedom than the ambient dimension.



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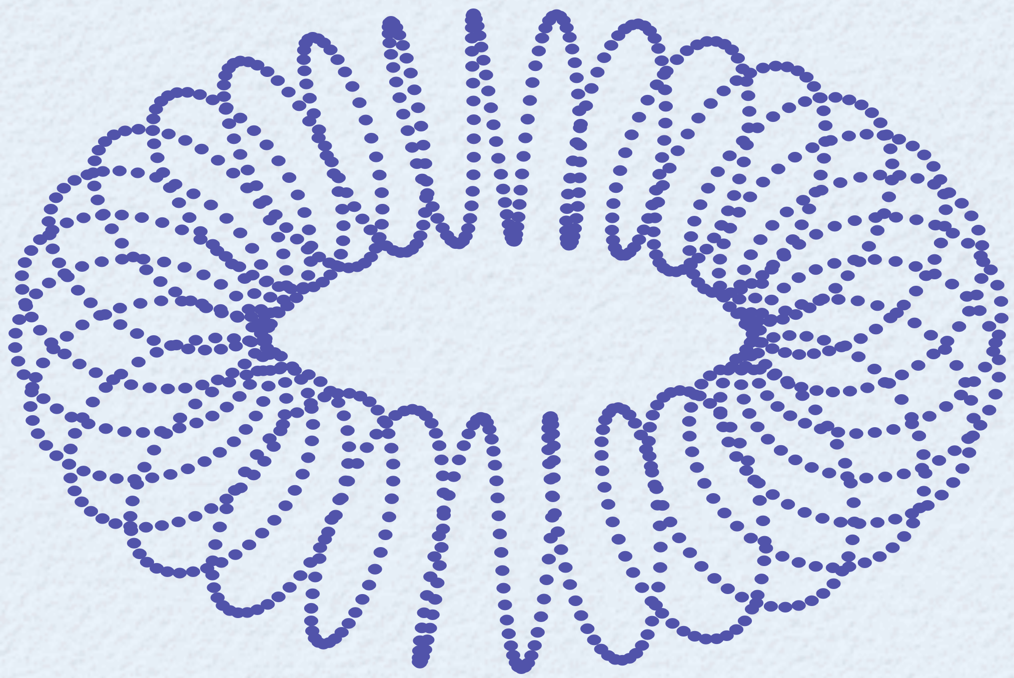
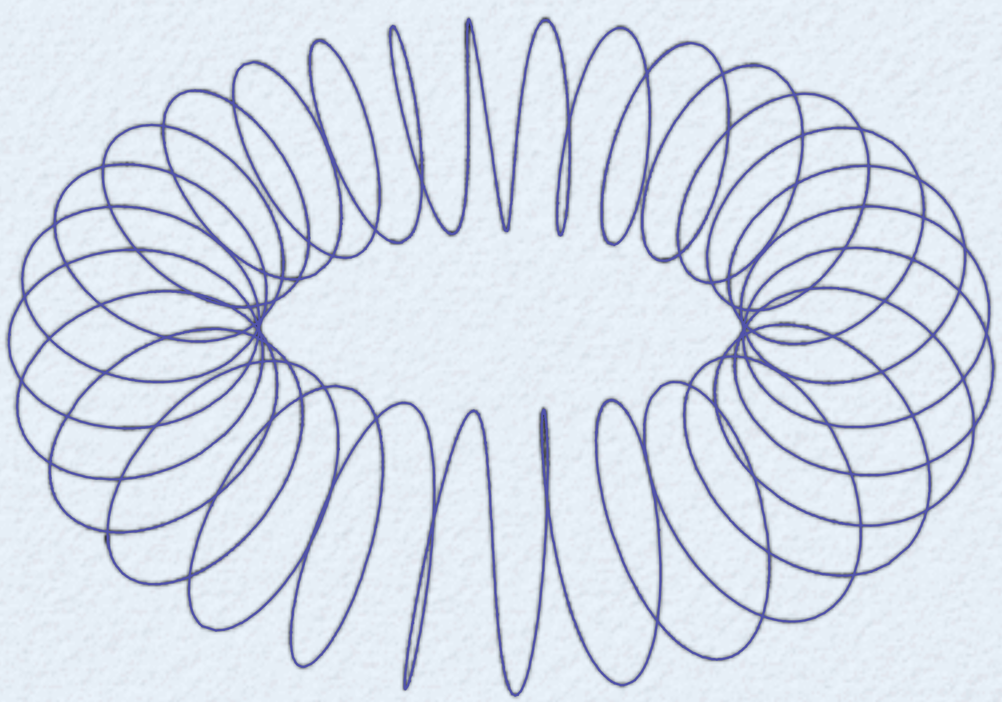
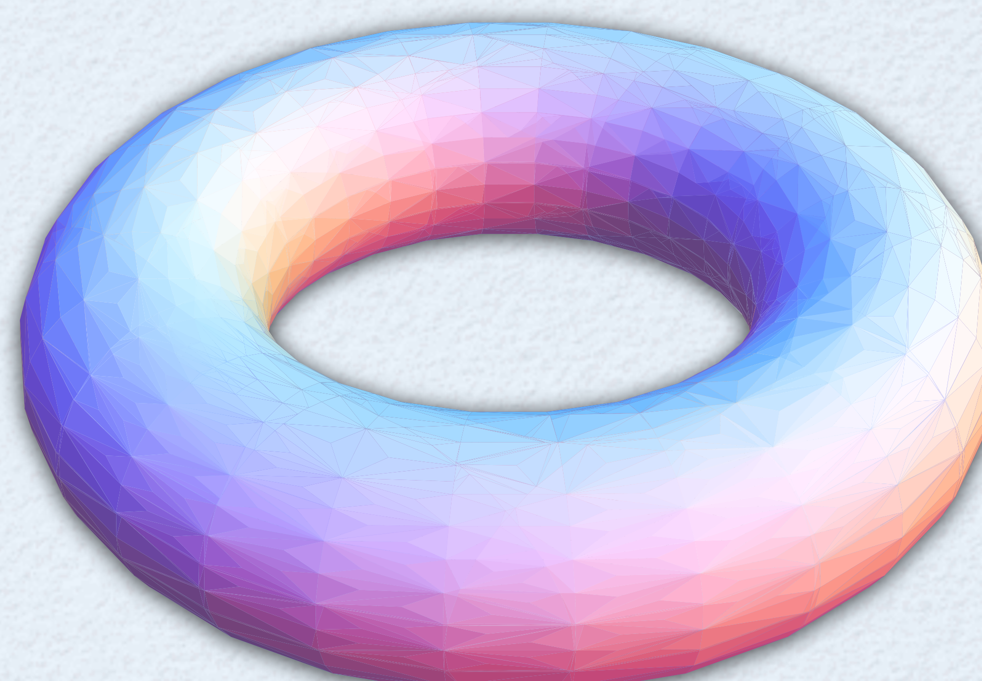
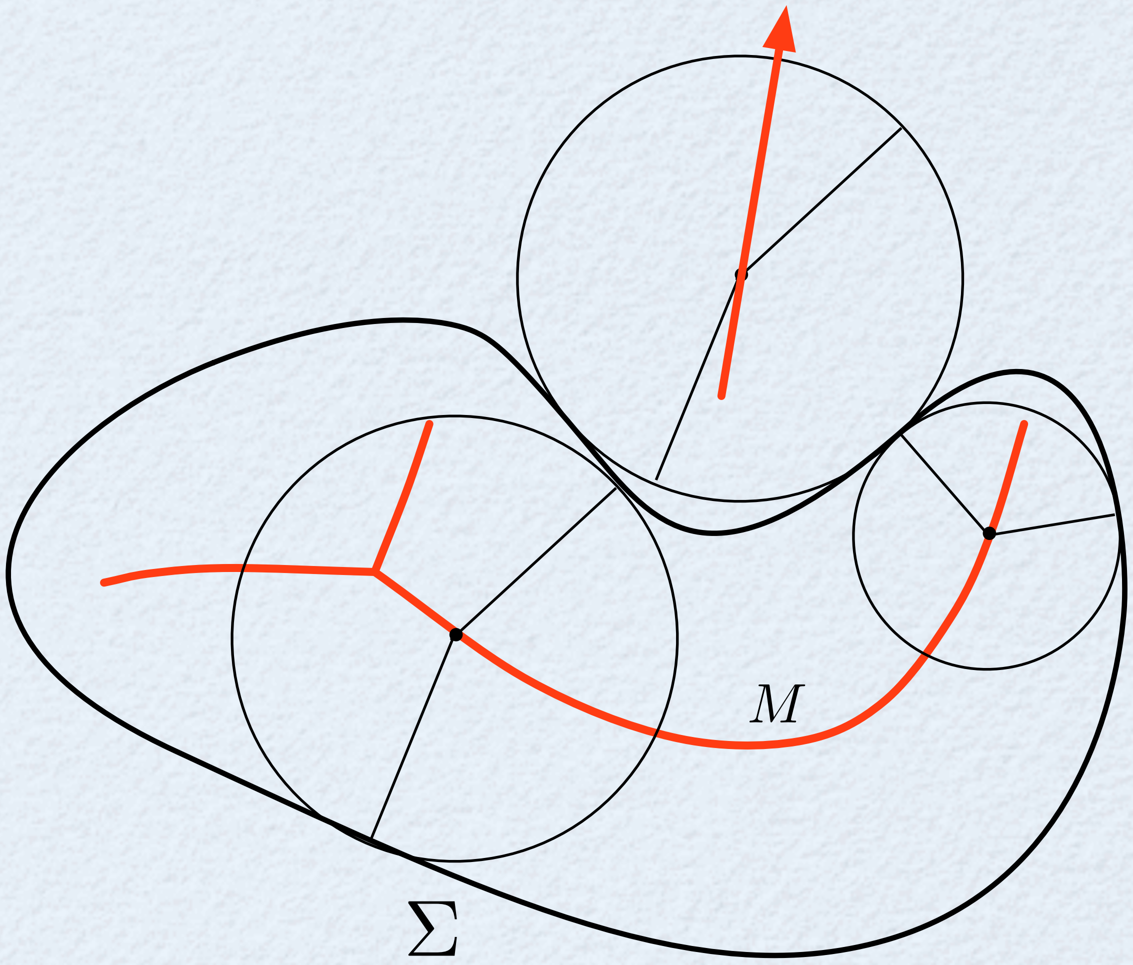






Medial Axis & Sampling Assumption

The medial axis (MA) of Σ is the set of points $M \subset \mathbb{R}^n$ that have ≥ 2 closest points in Σ.



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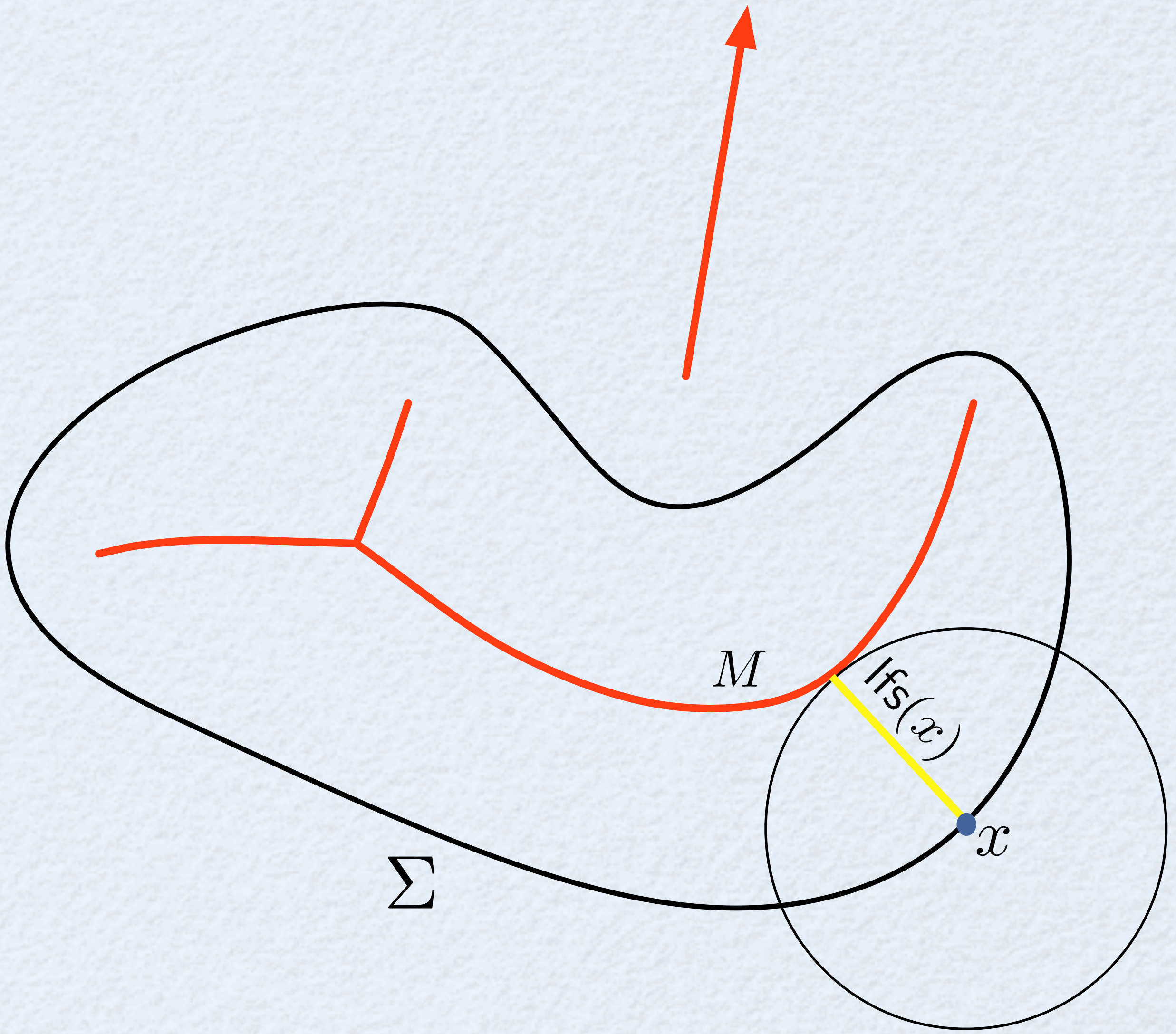

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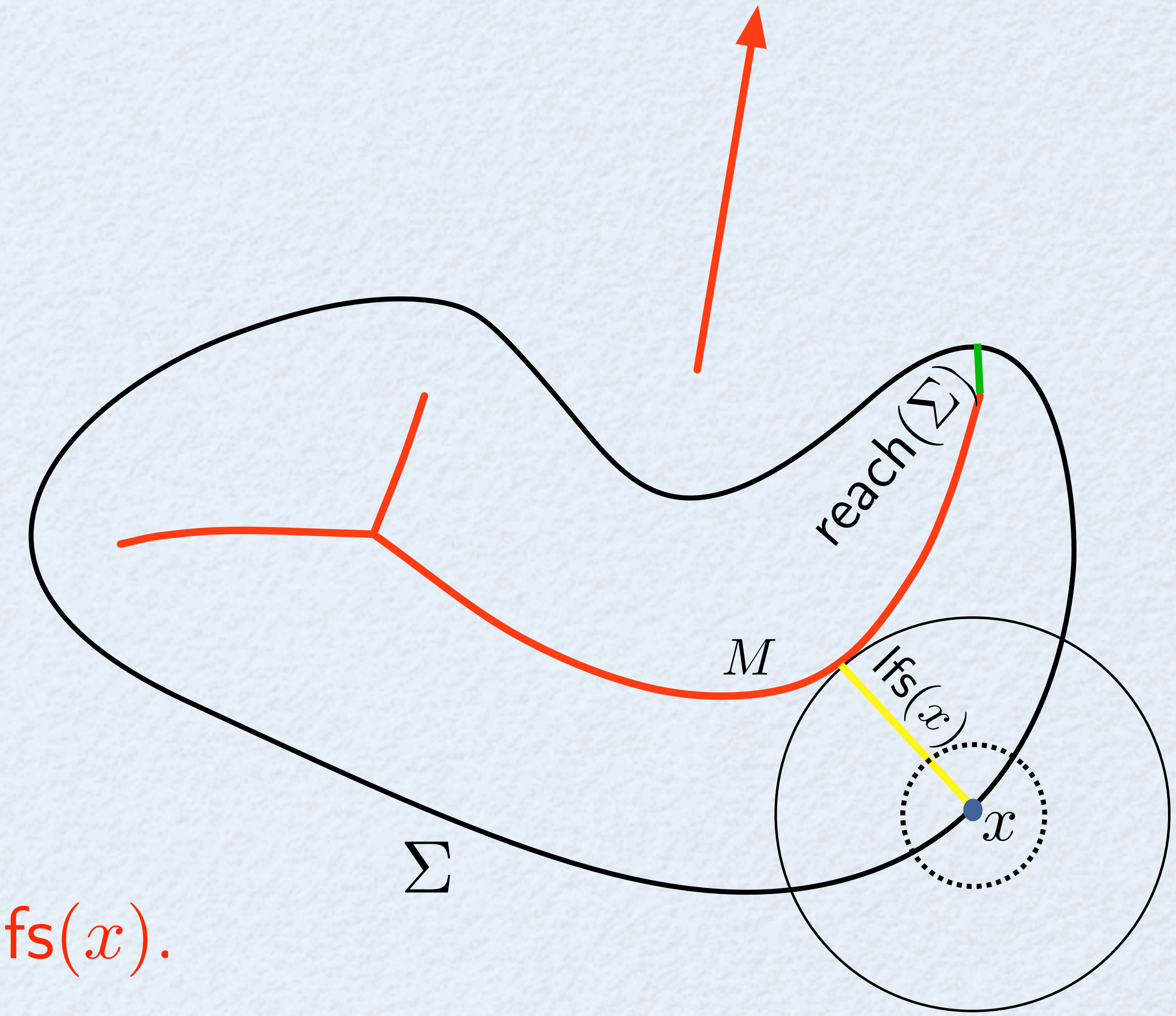

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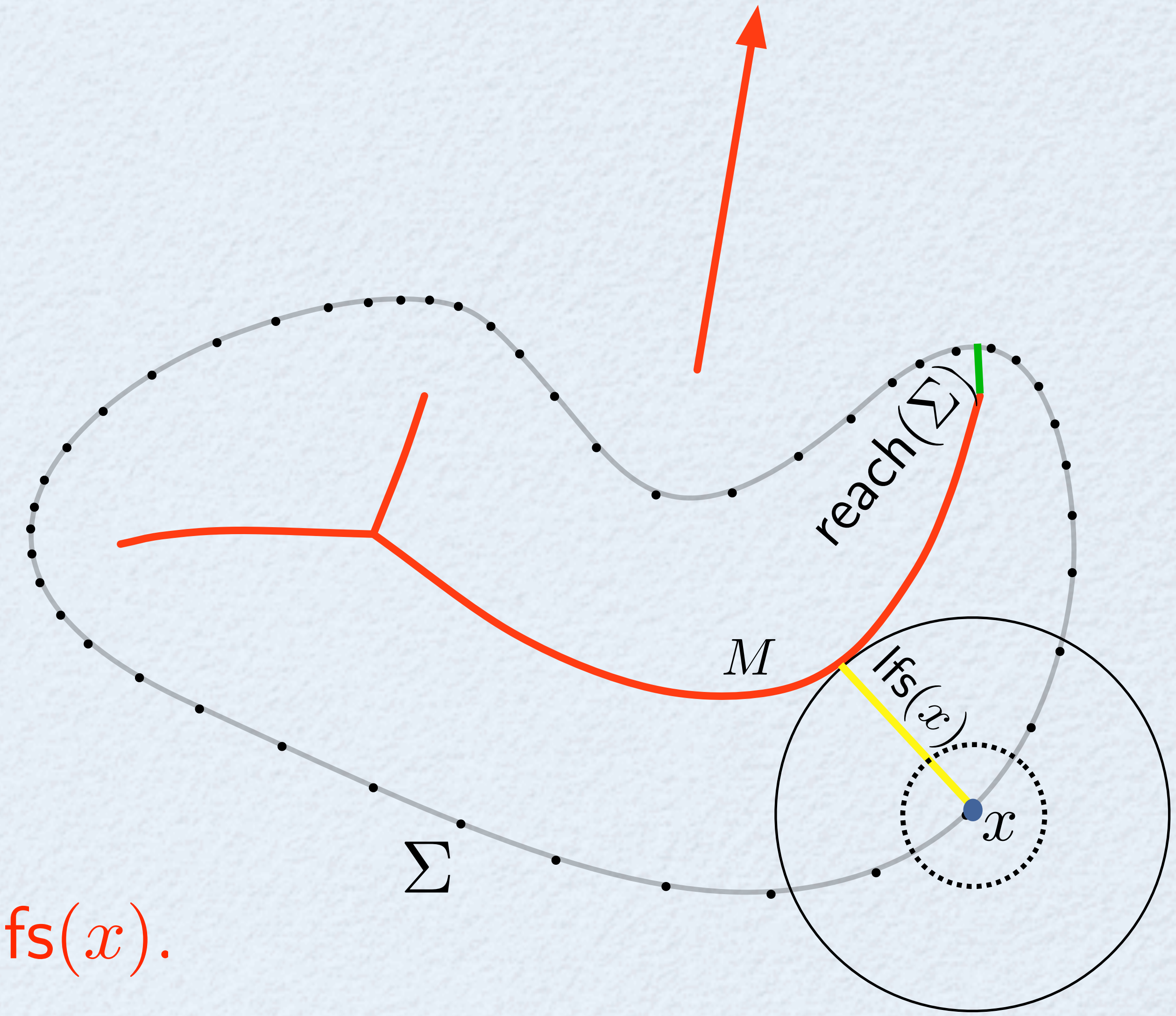

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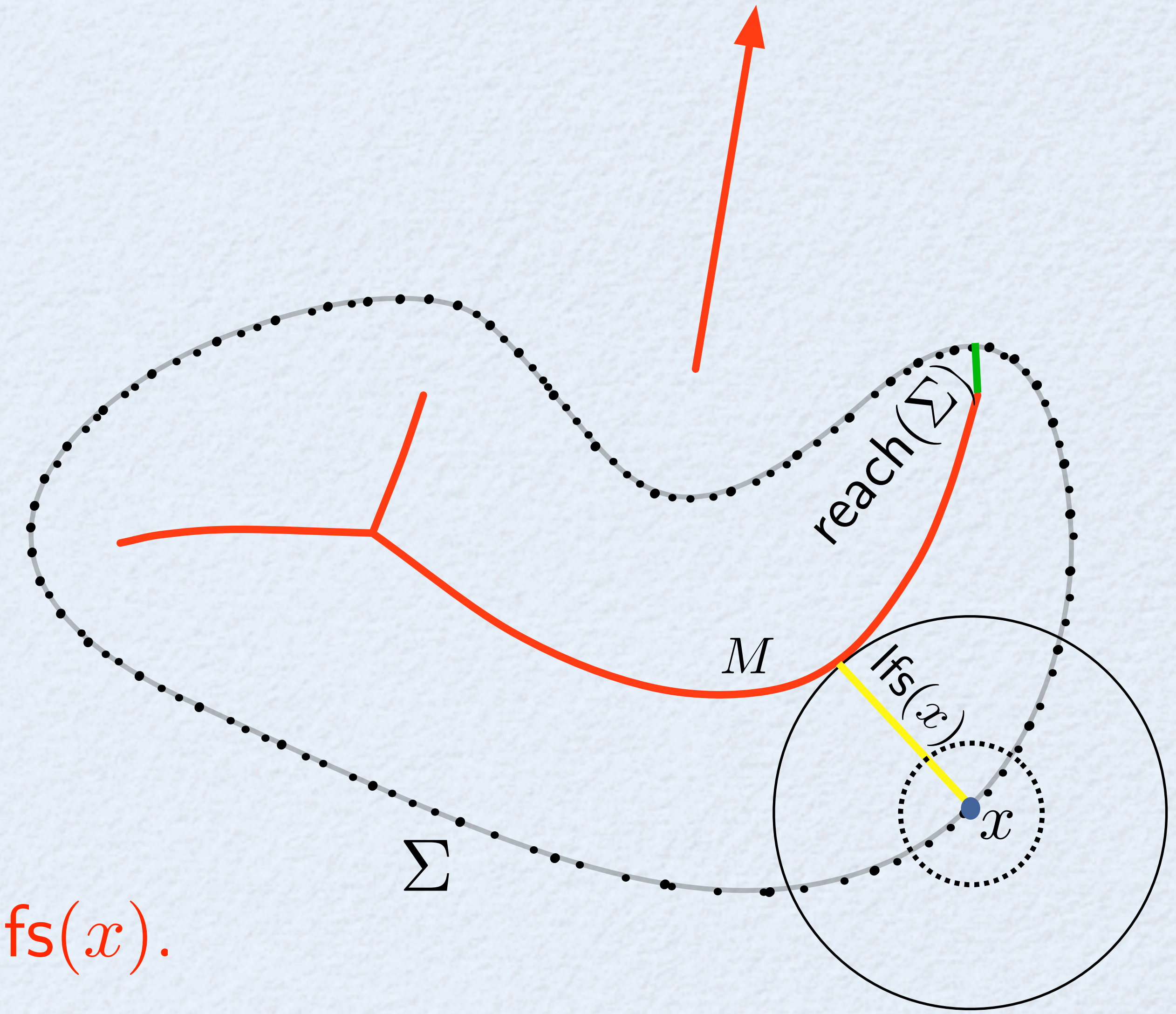

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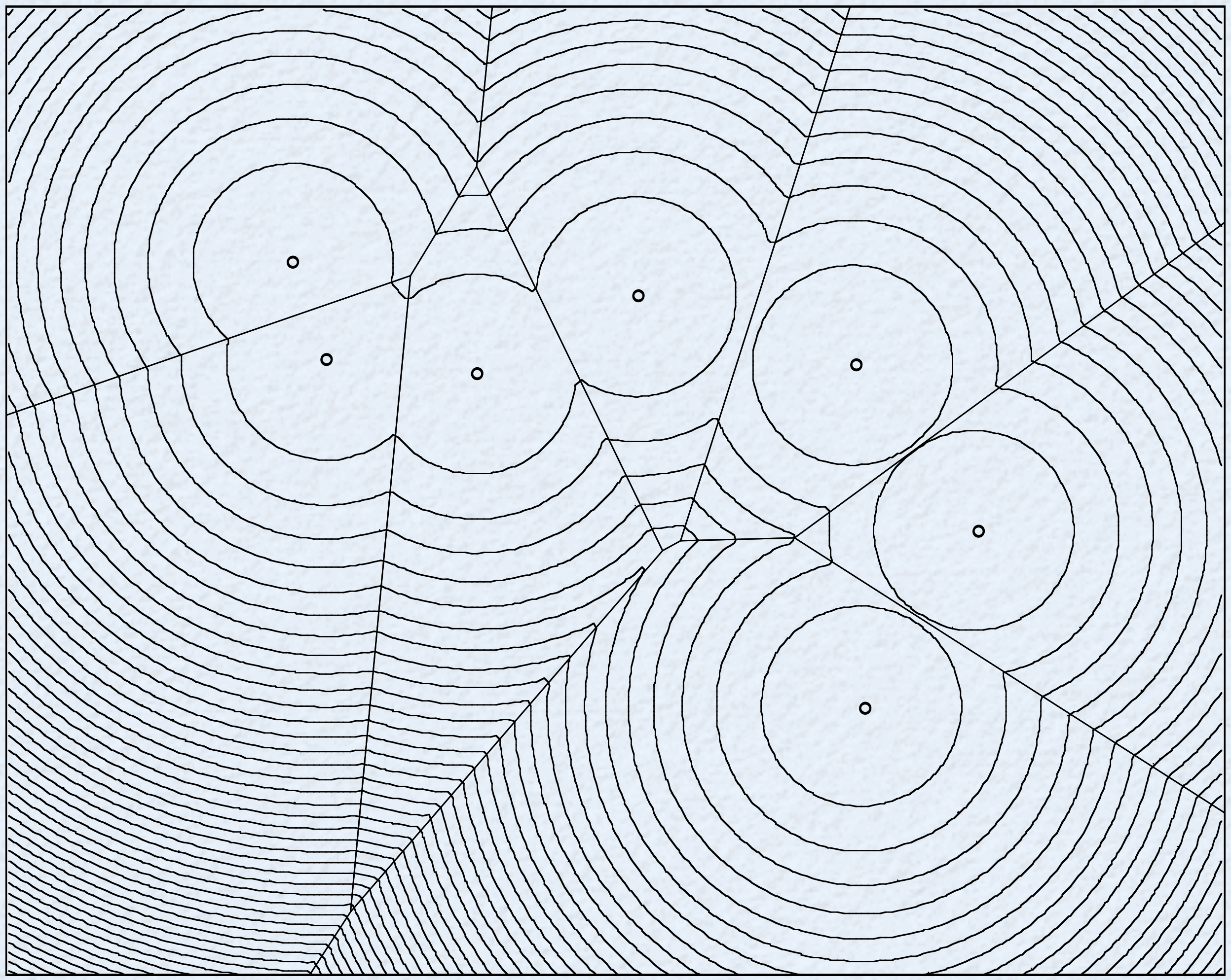
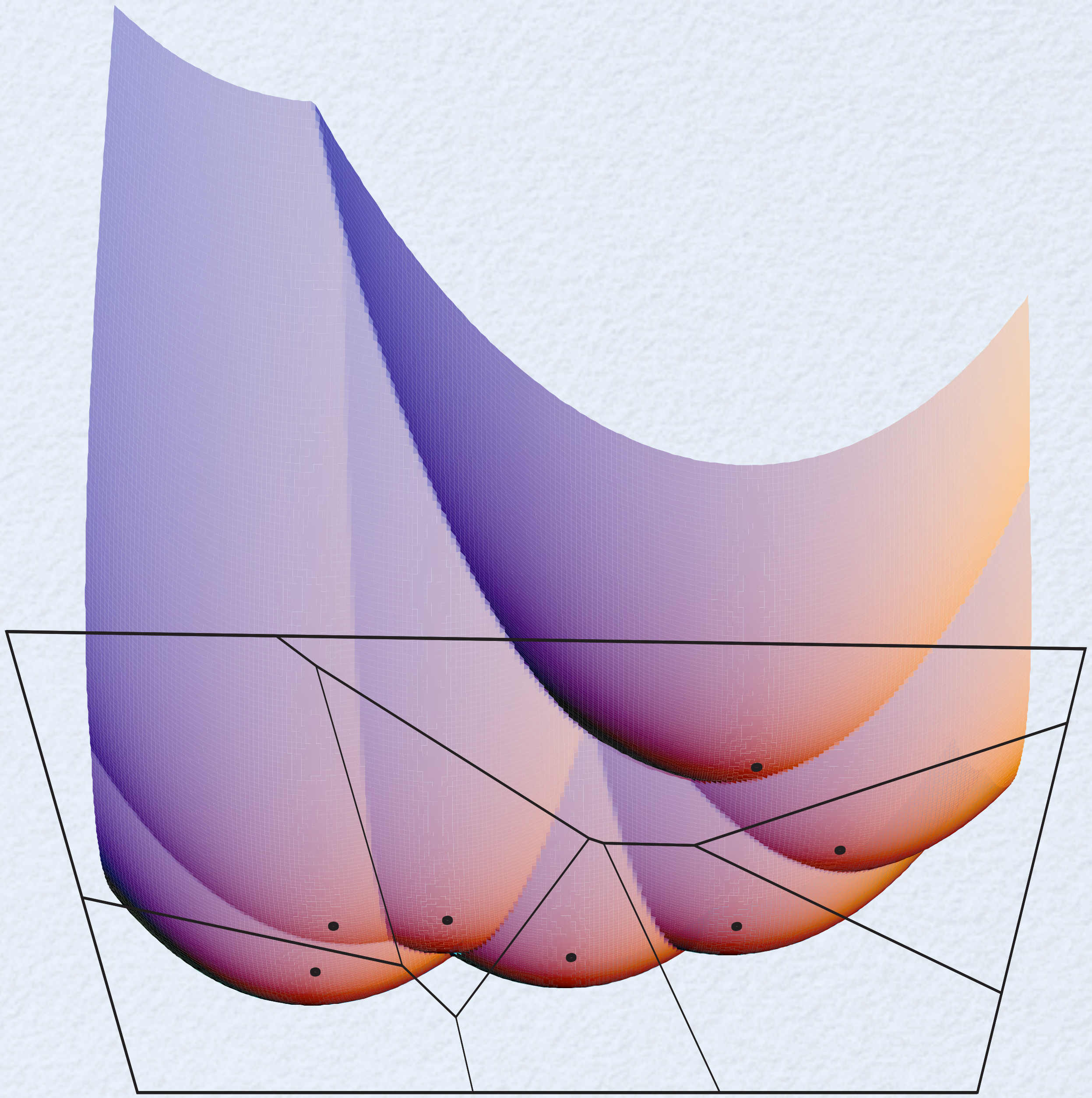
A uniform ε-sample of Σ has a point within $\varepsilon \cdot \mathsf{reach}(\Sigma)$ of every $x \in \Sigma$.

(Squared) Distance Functions



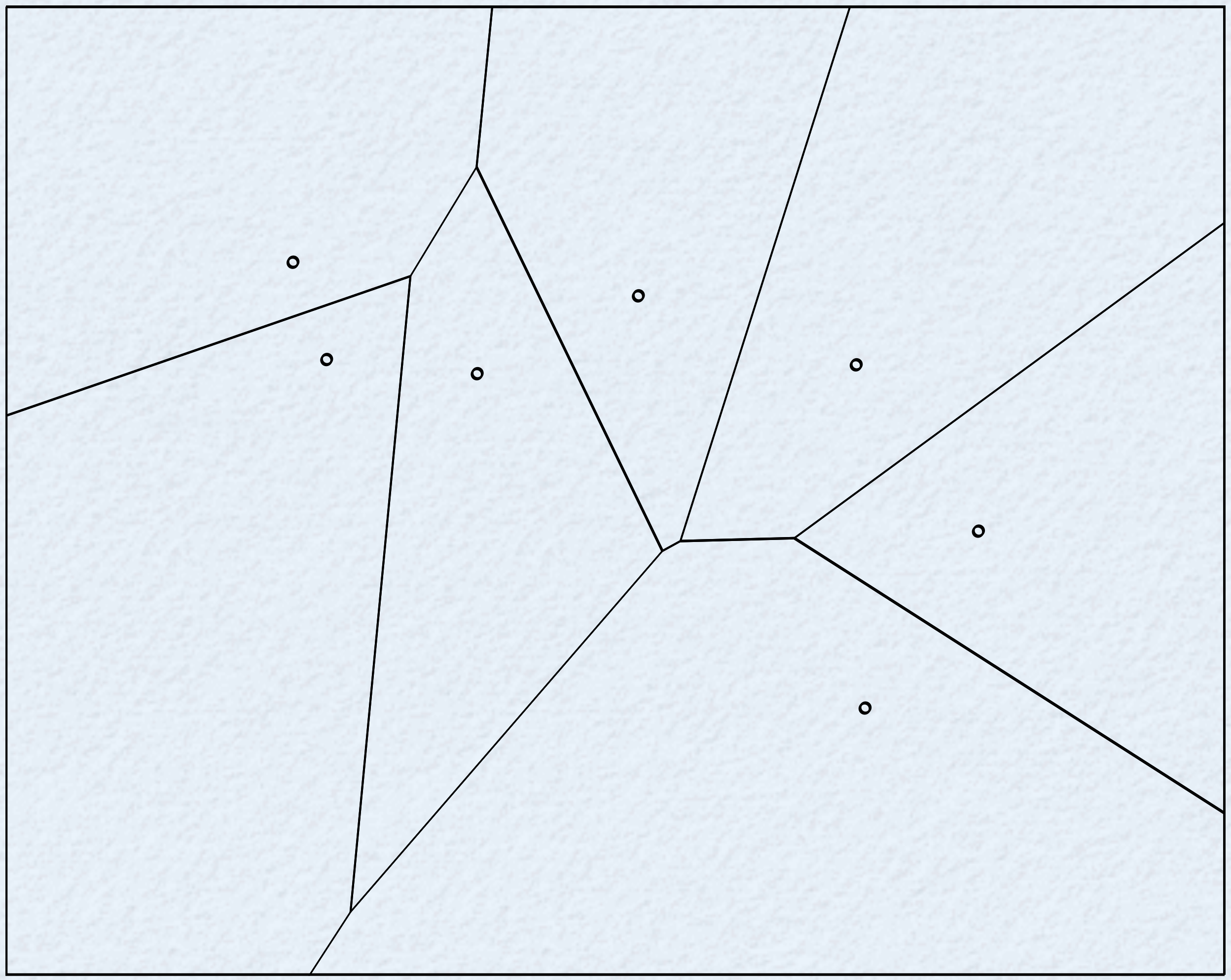


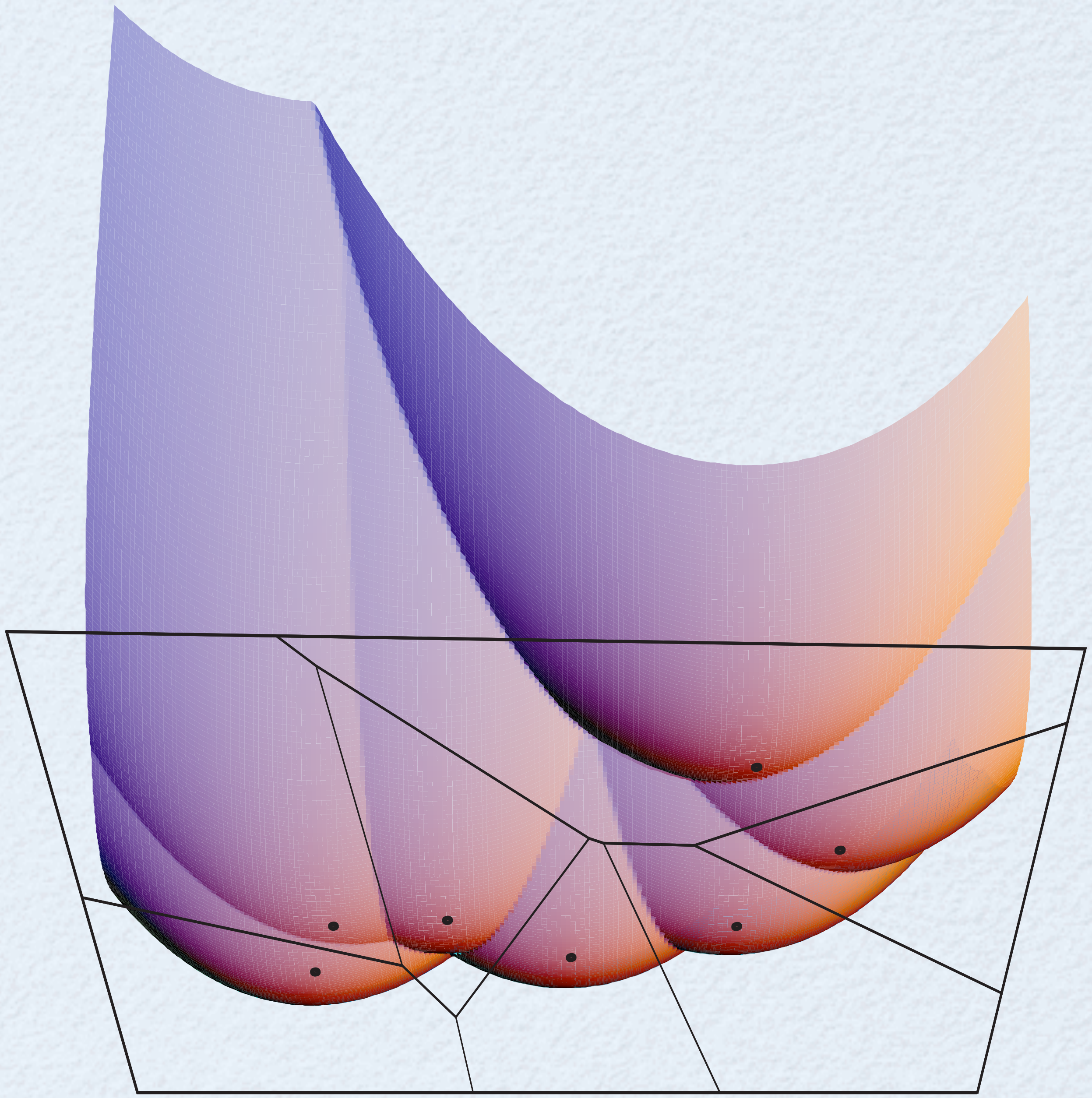
P is a discrete set of points

The squared distance function induced by P is

$$h(x) := \min_{p \in P} \|x - p\|^2$$

Generalized Gradient





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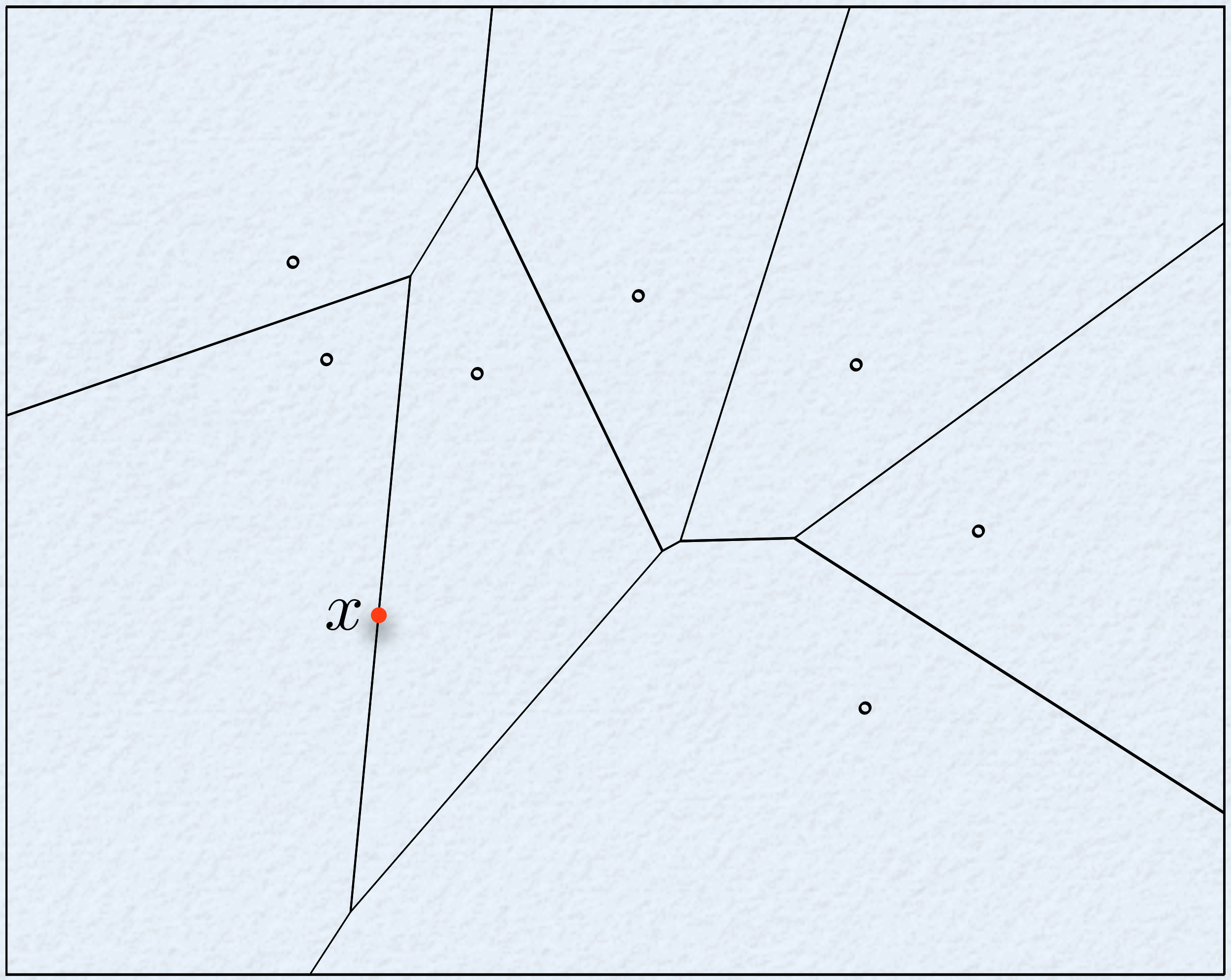

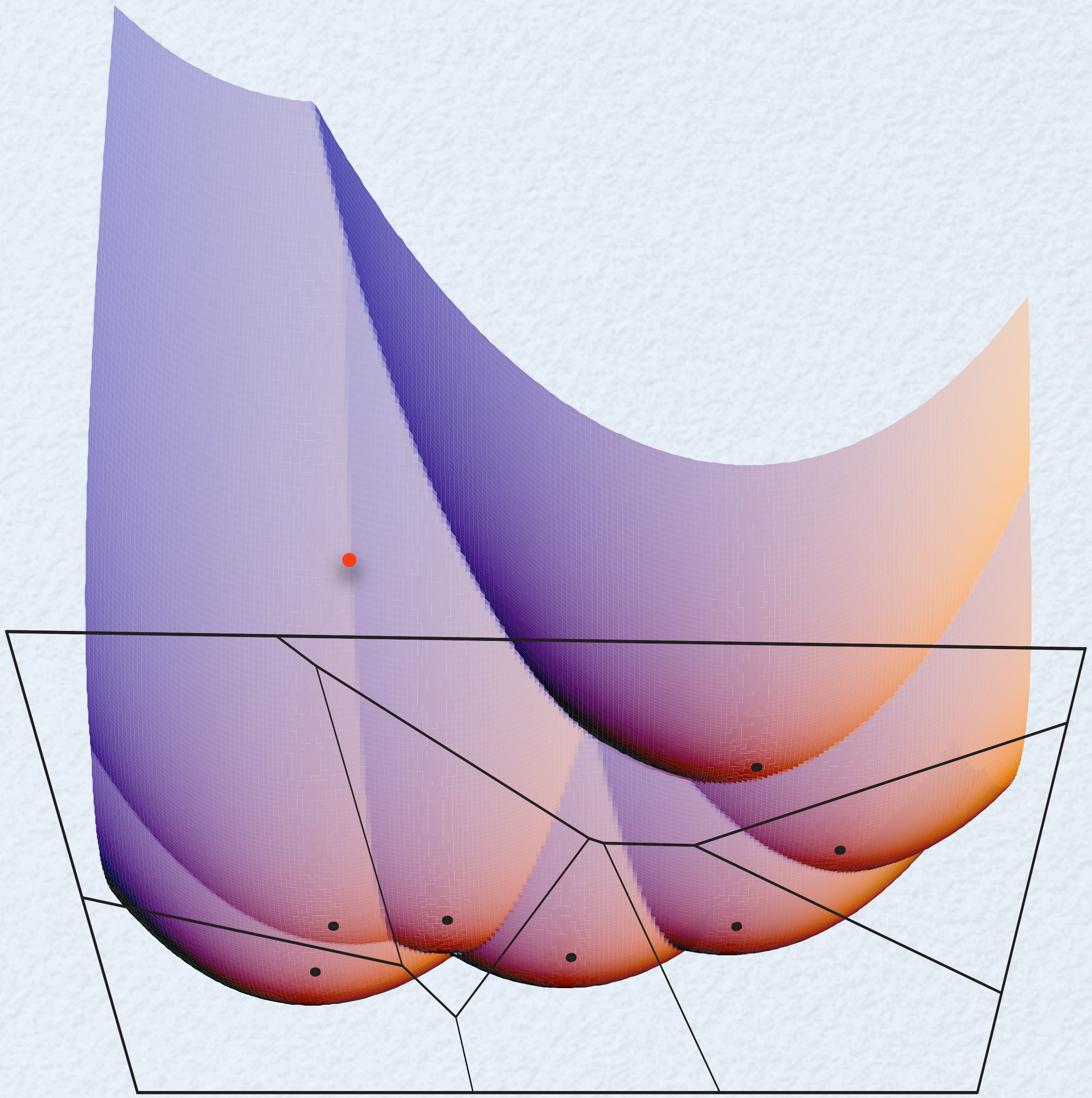

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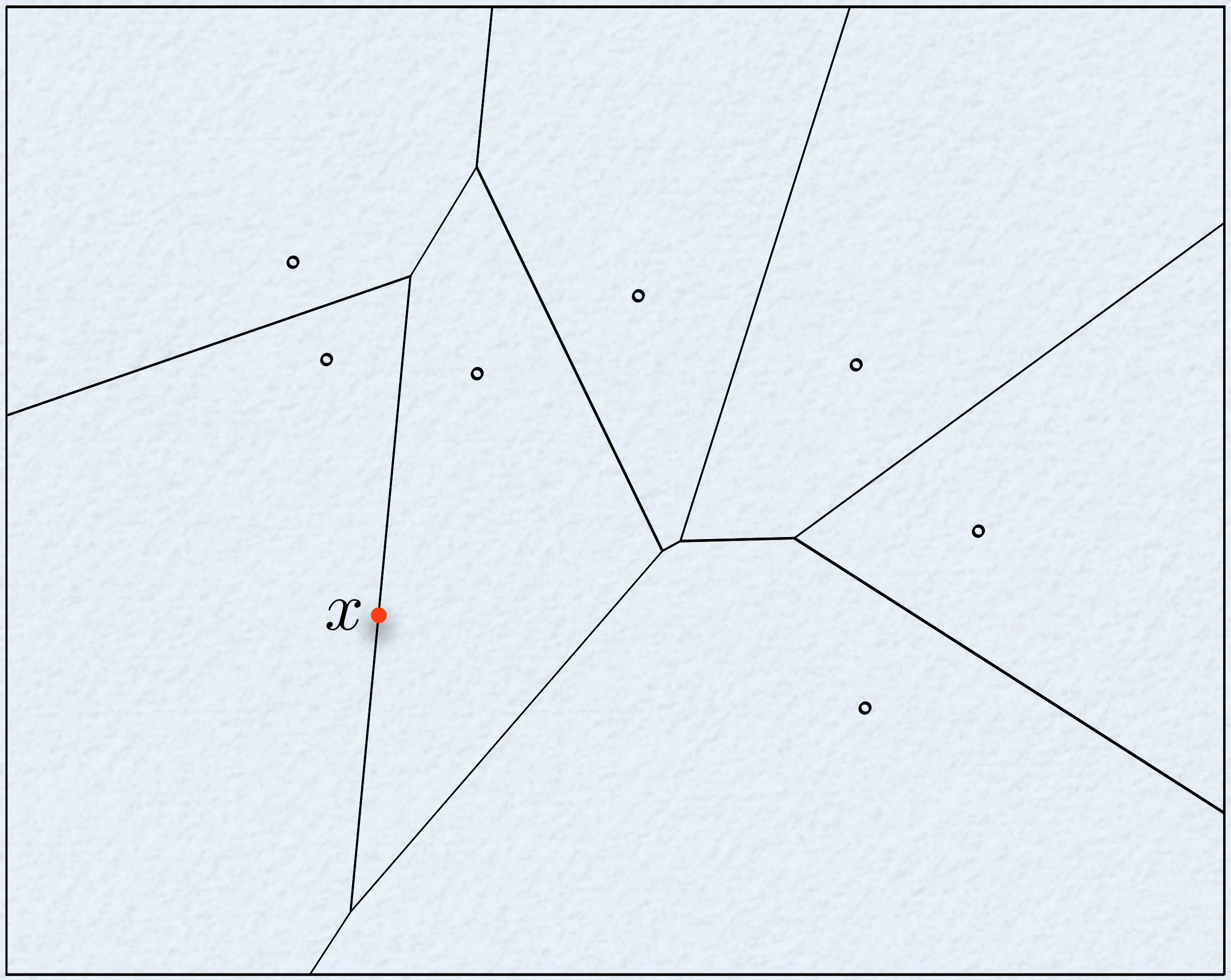

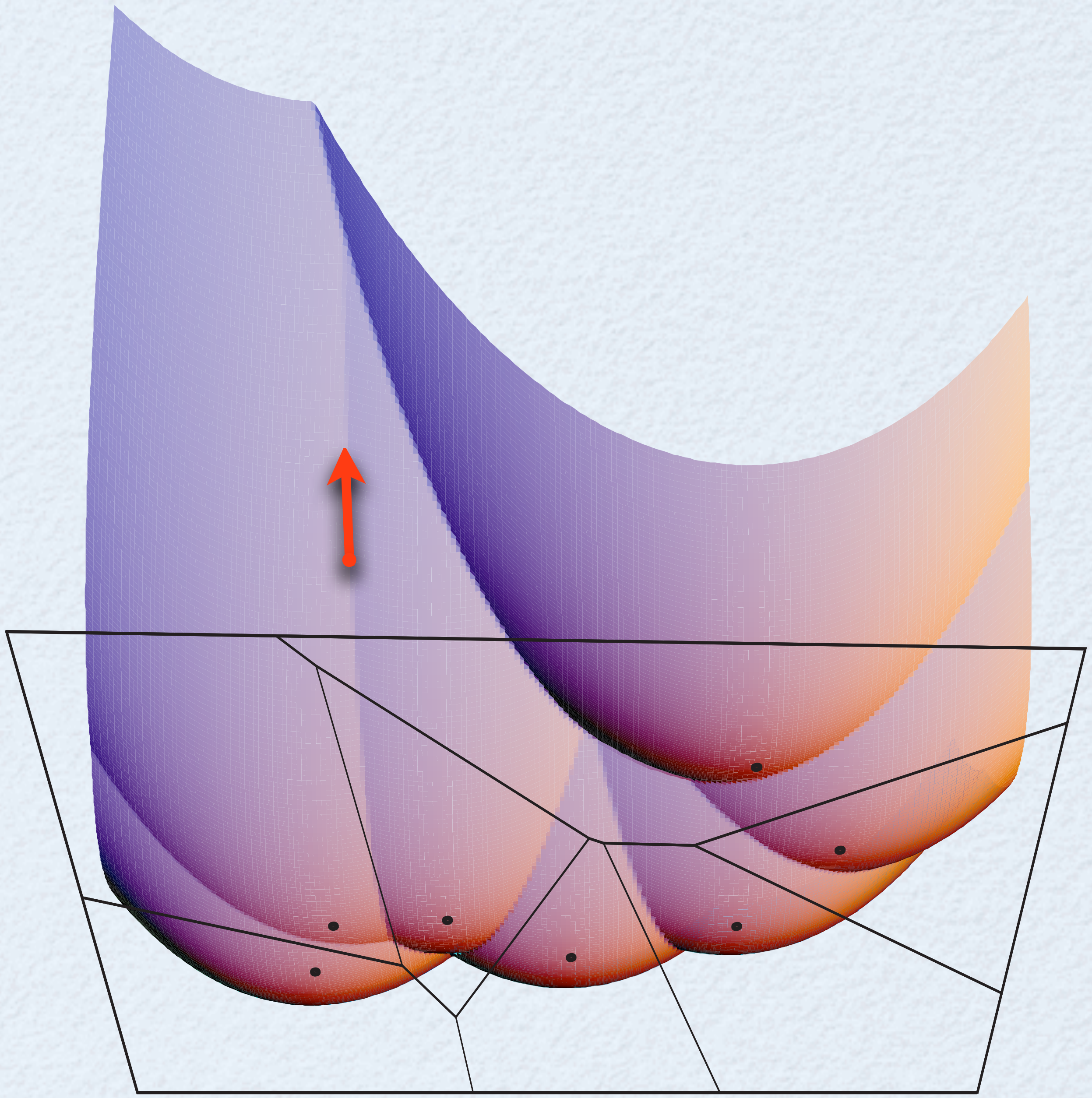

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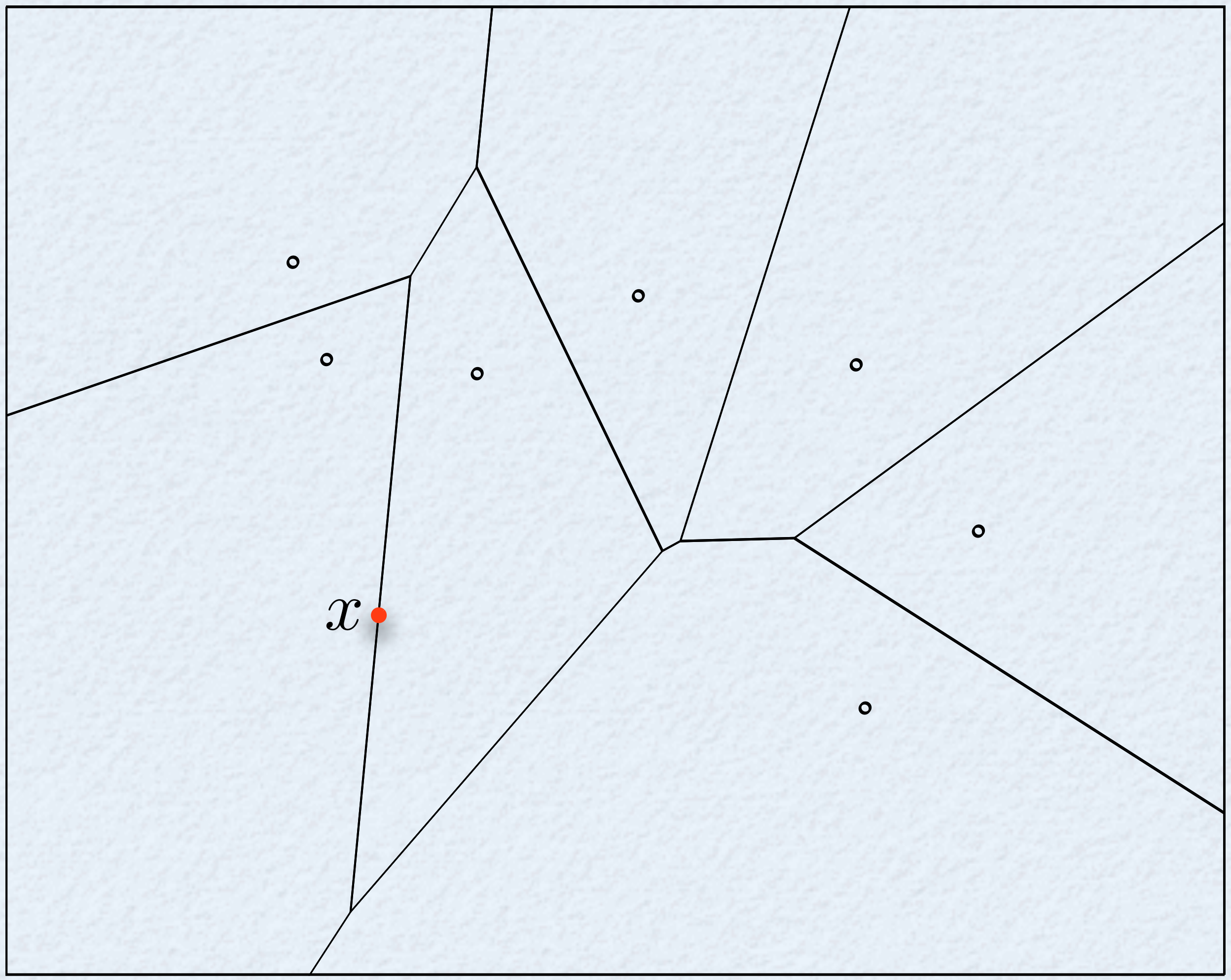

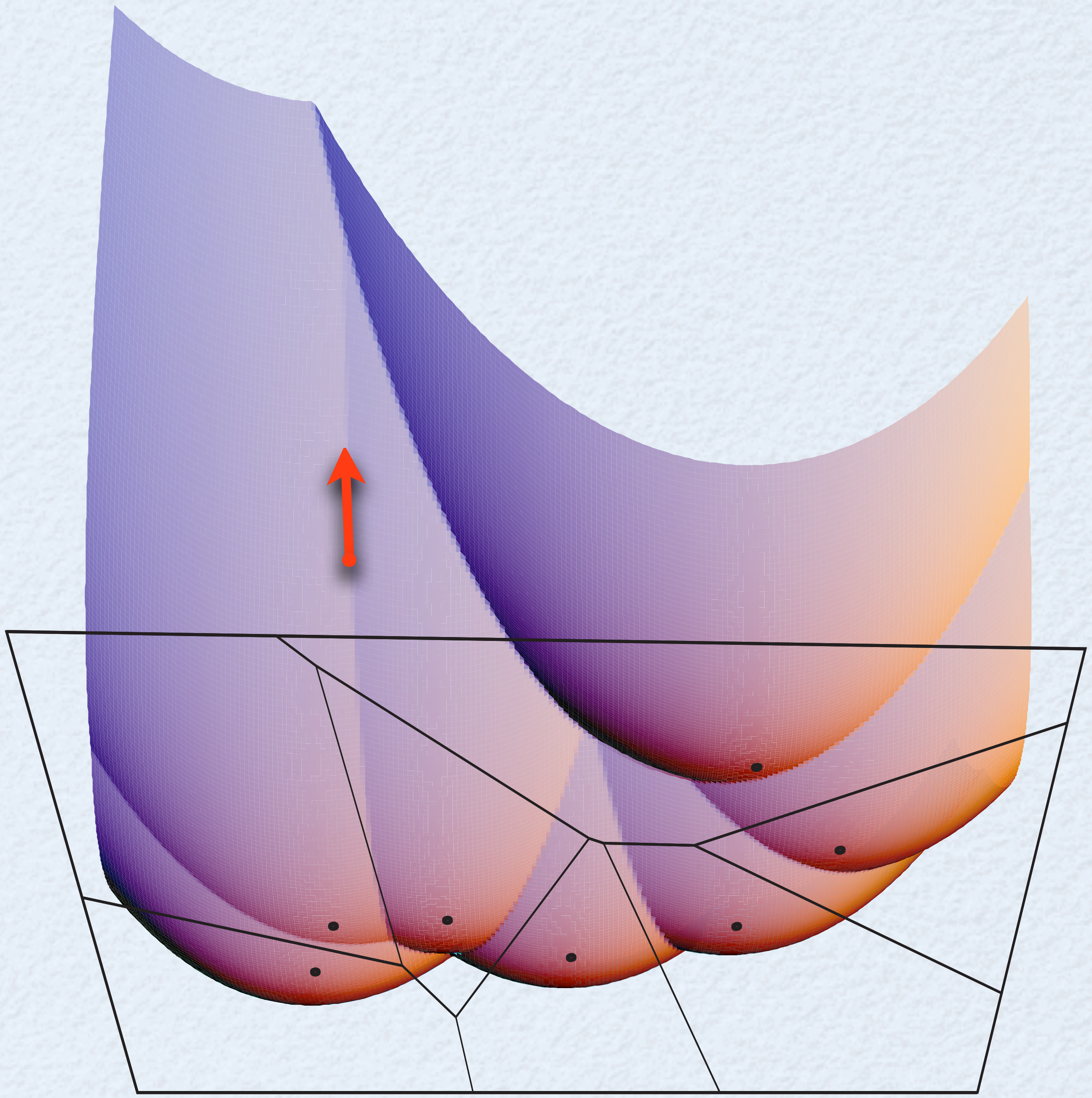

$V(x)$: lowest-dimensional Voronoi face containing x.

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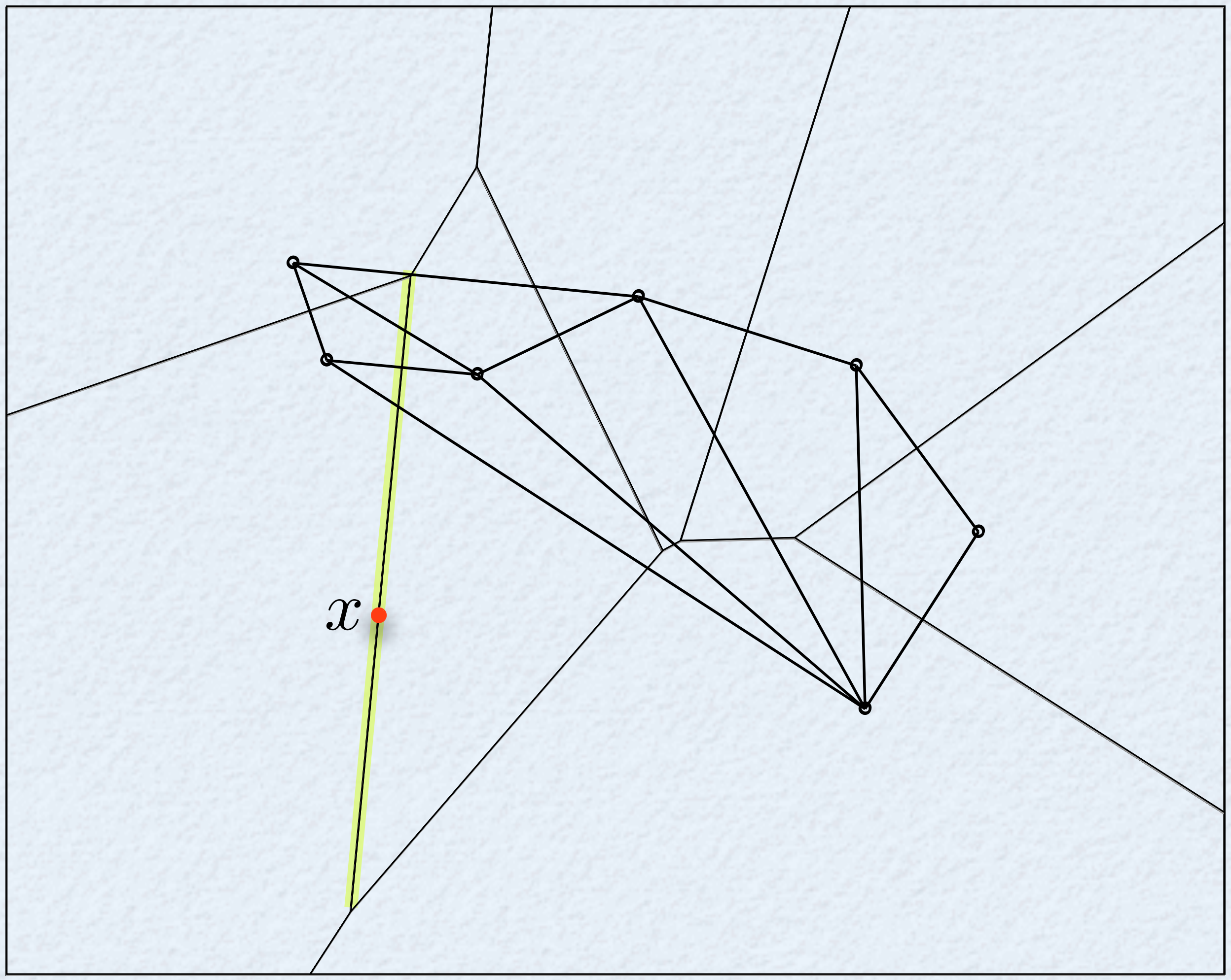

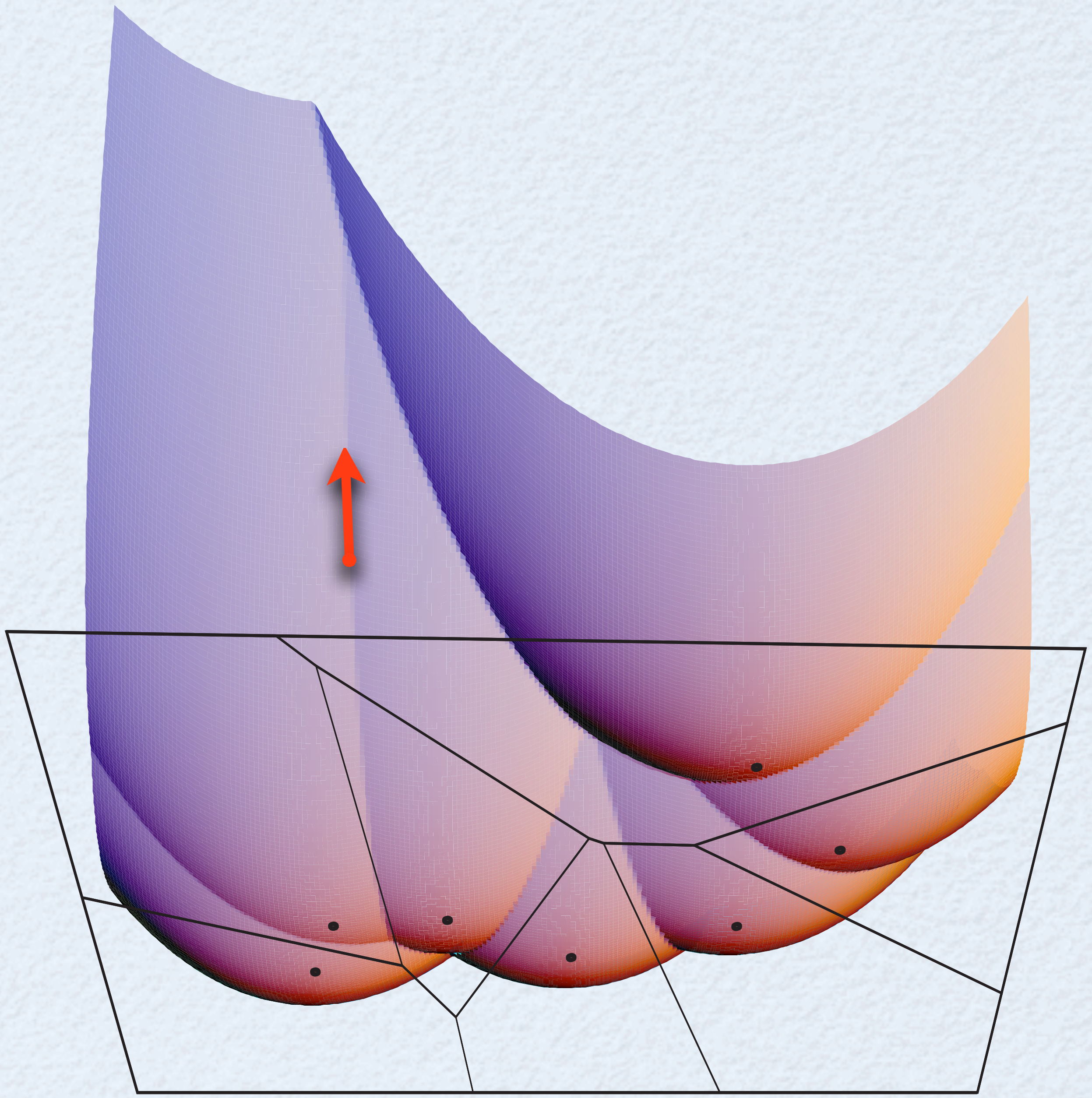

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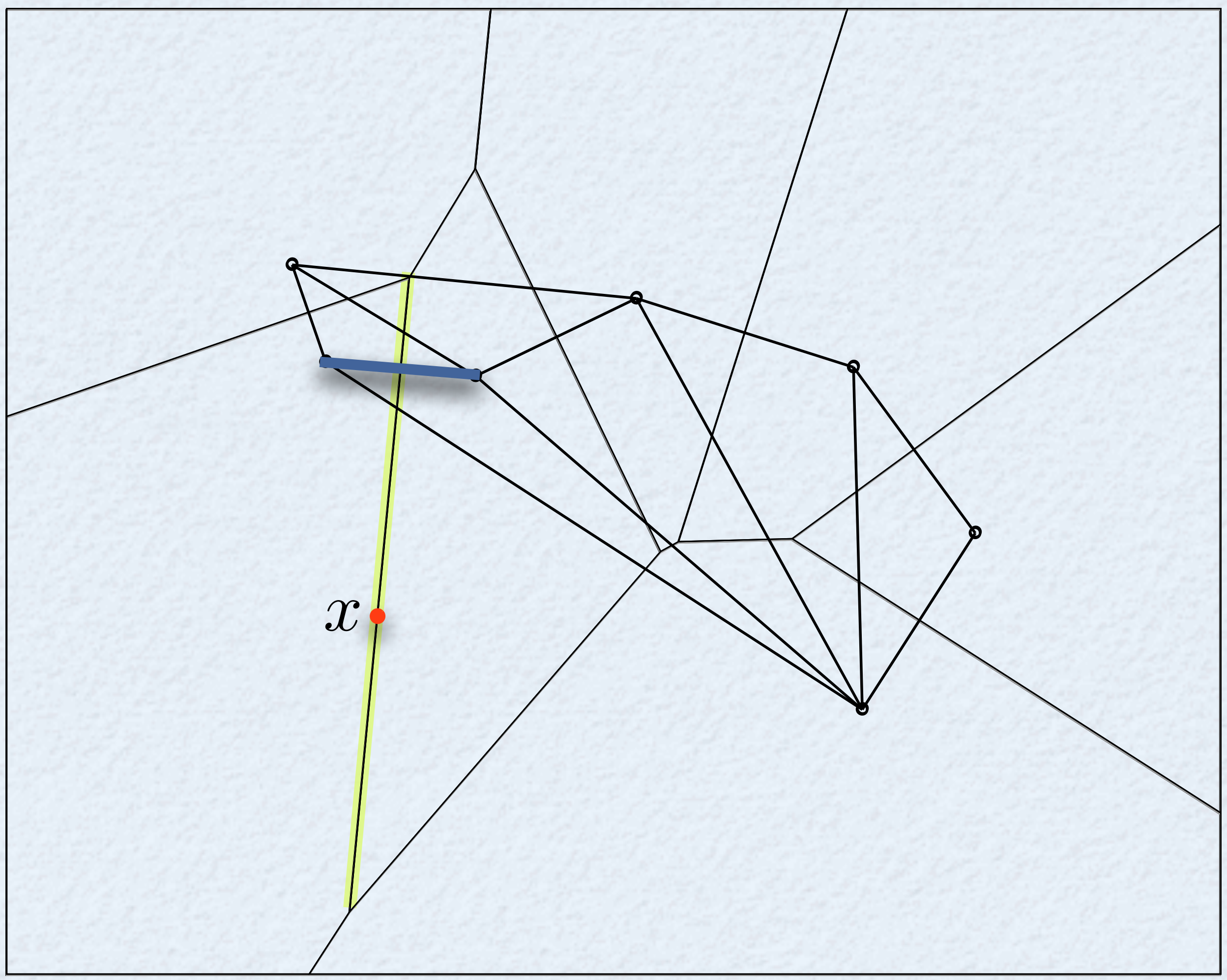

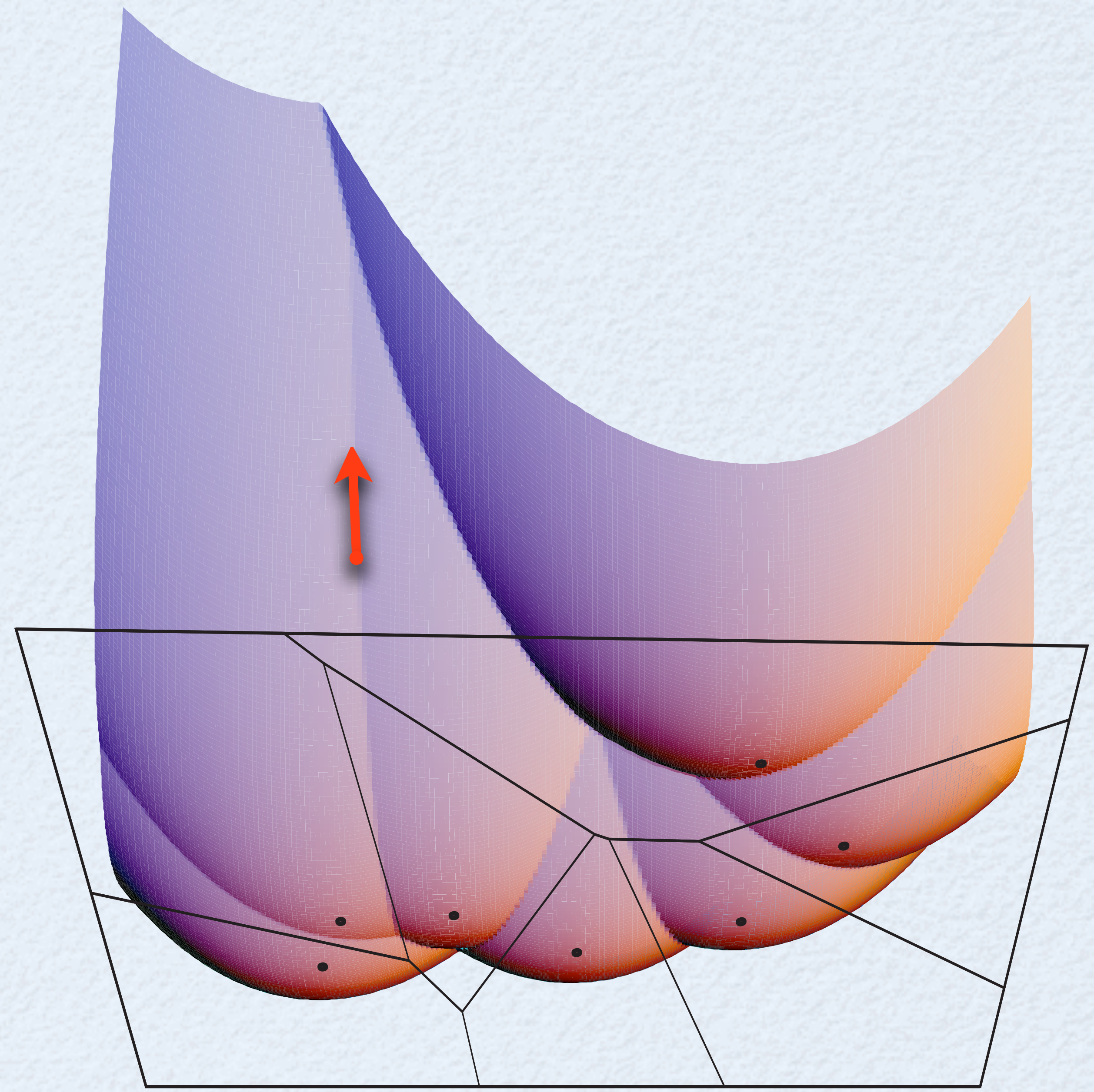

$V(x)$: lowest-dimensional Voronoi face containing x.

$D(x)$: Delaunay dual to $V(x)$.

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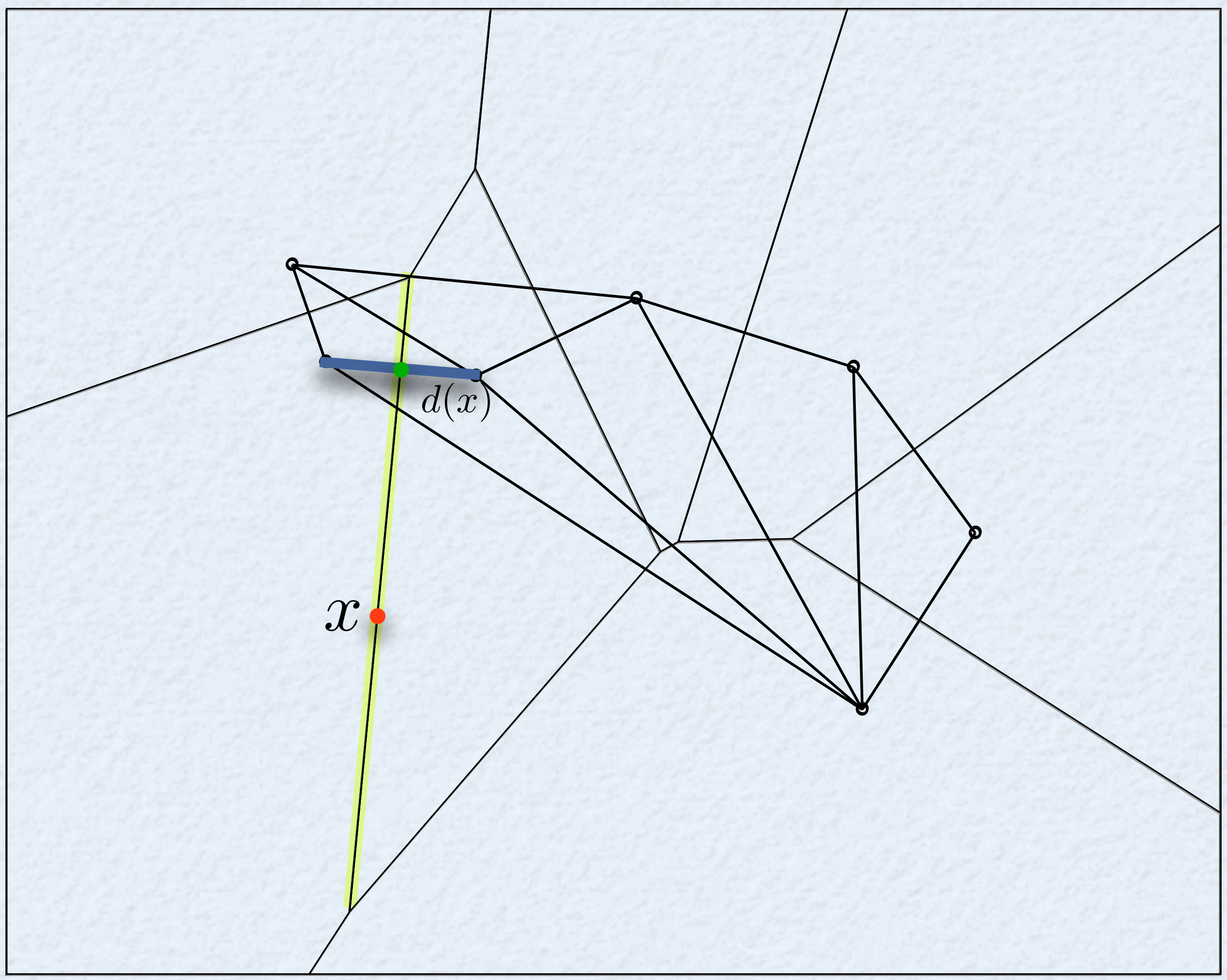

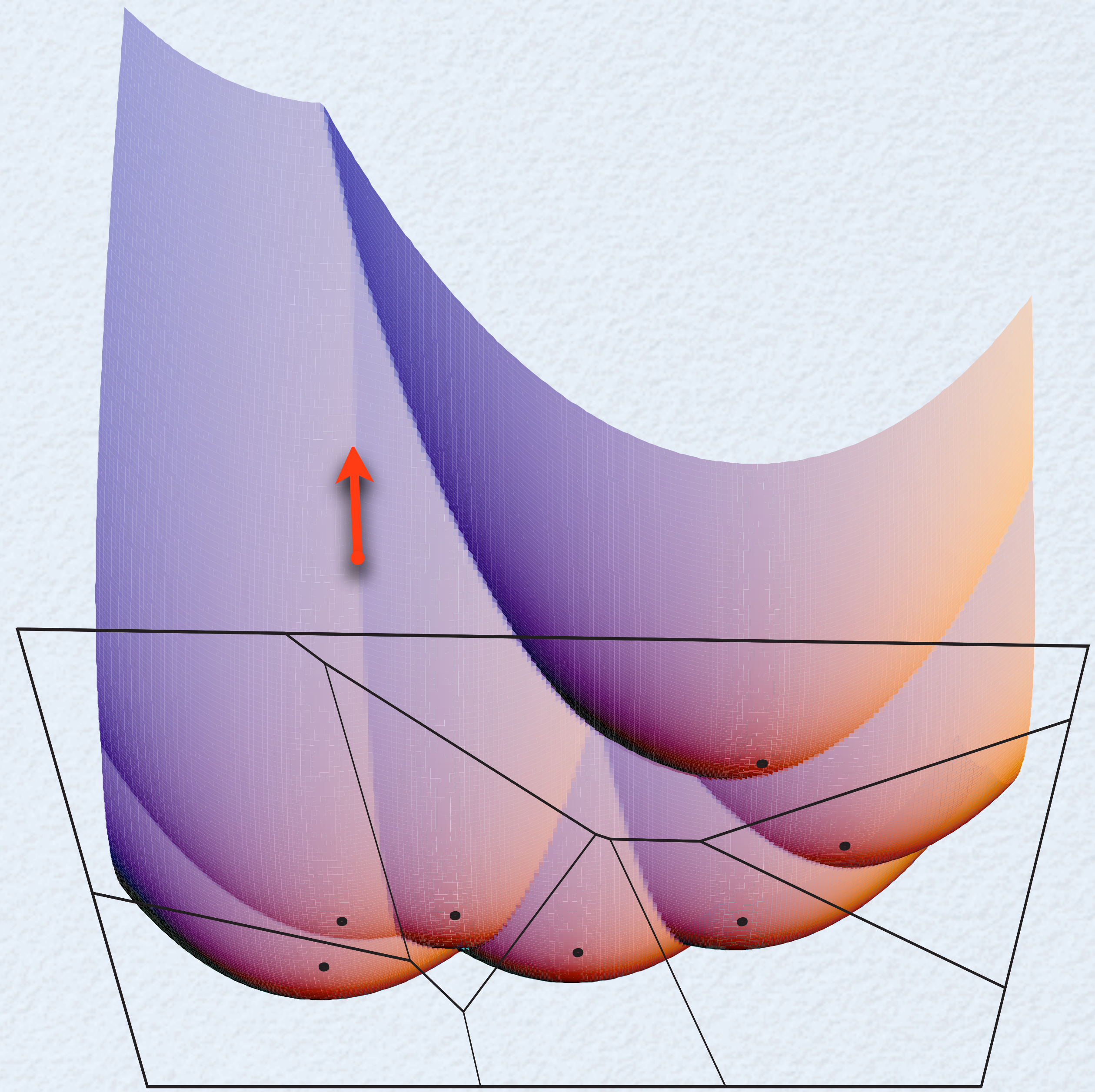

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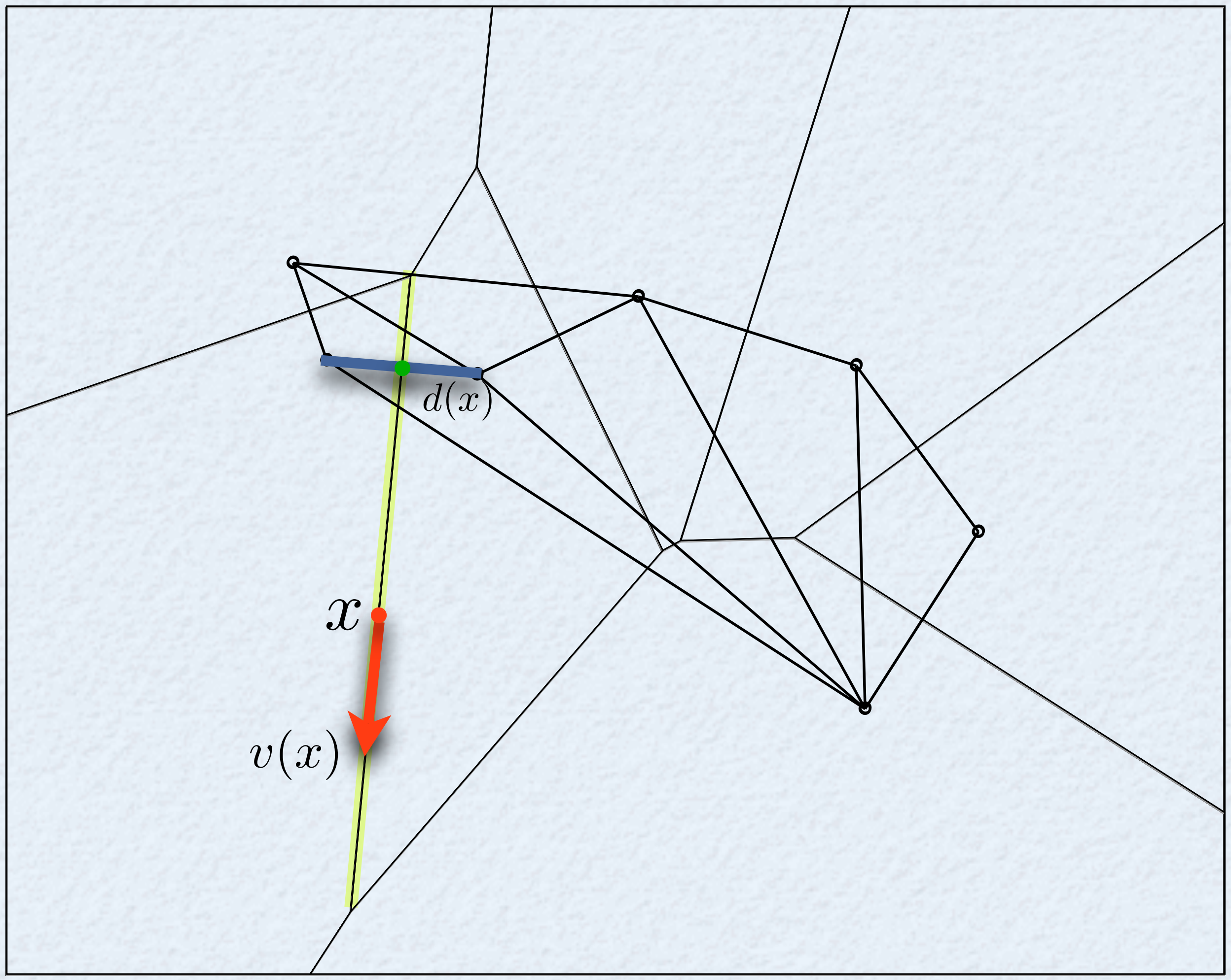

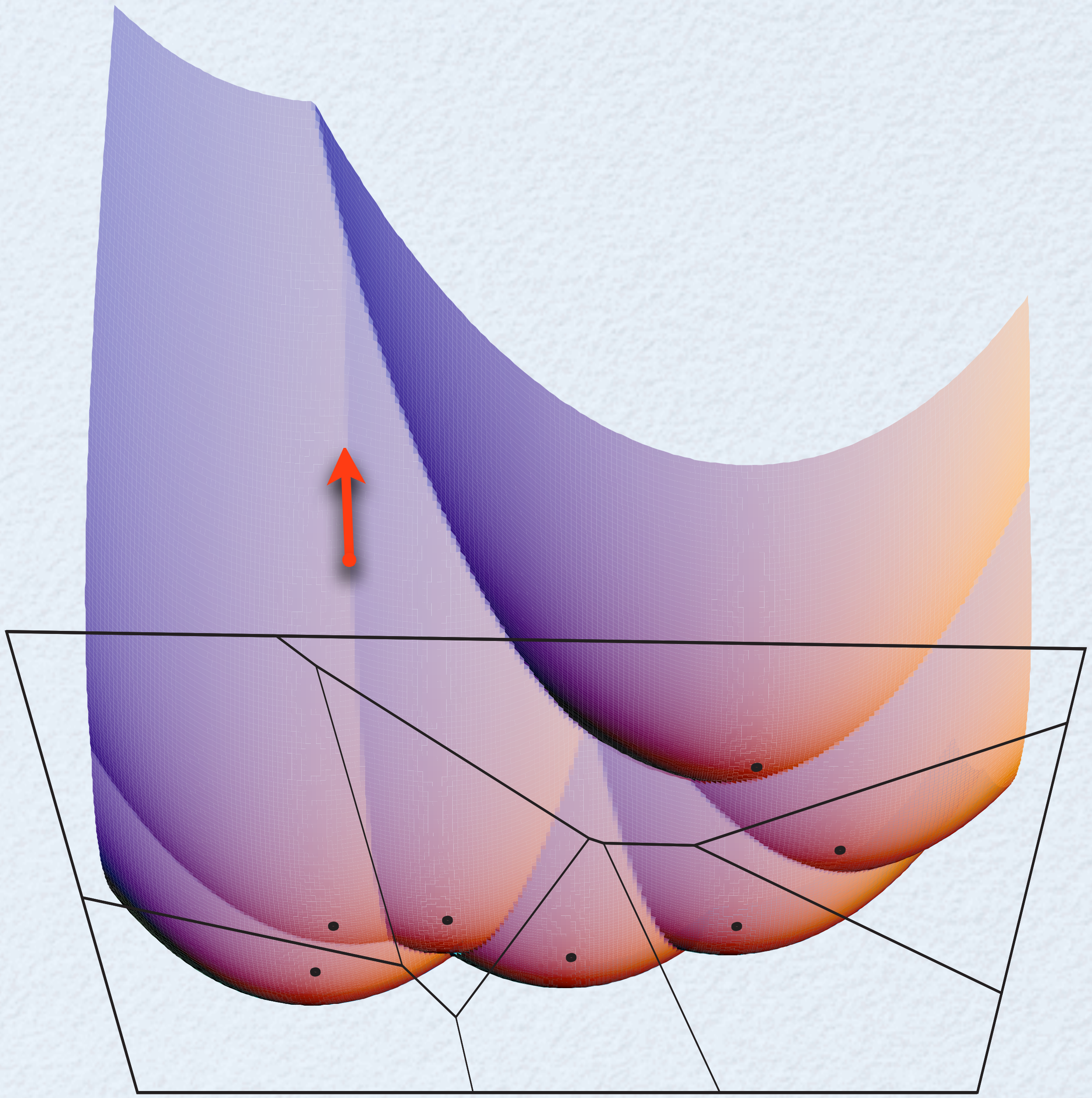

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$$v(x) = 2(x - d(x))$$

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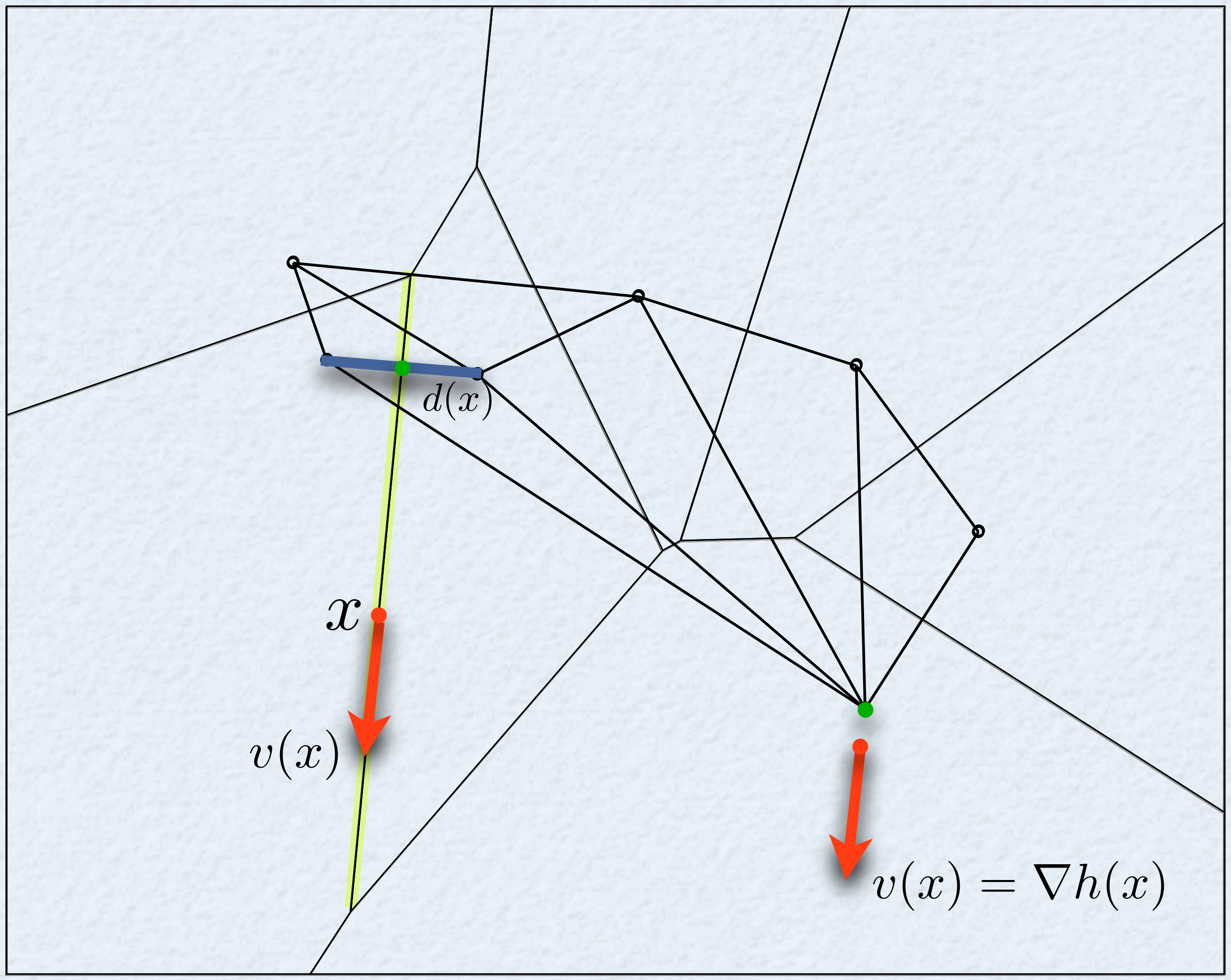

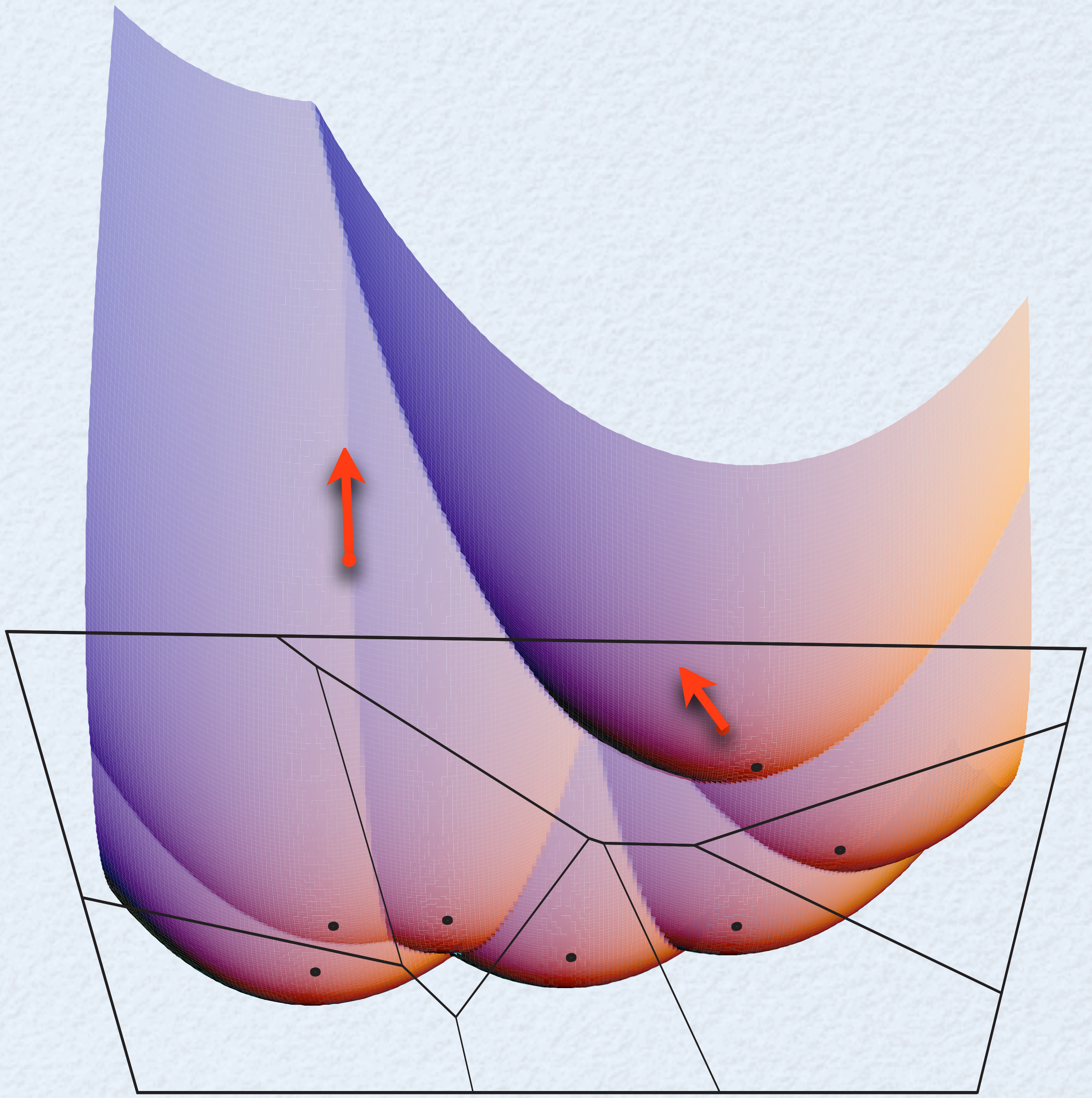

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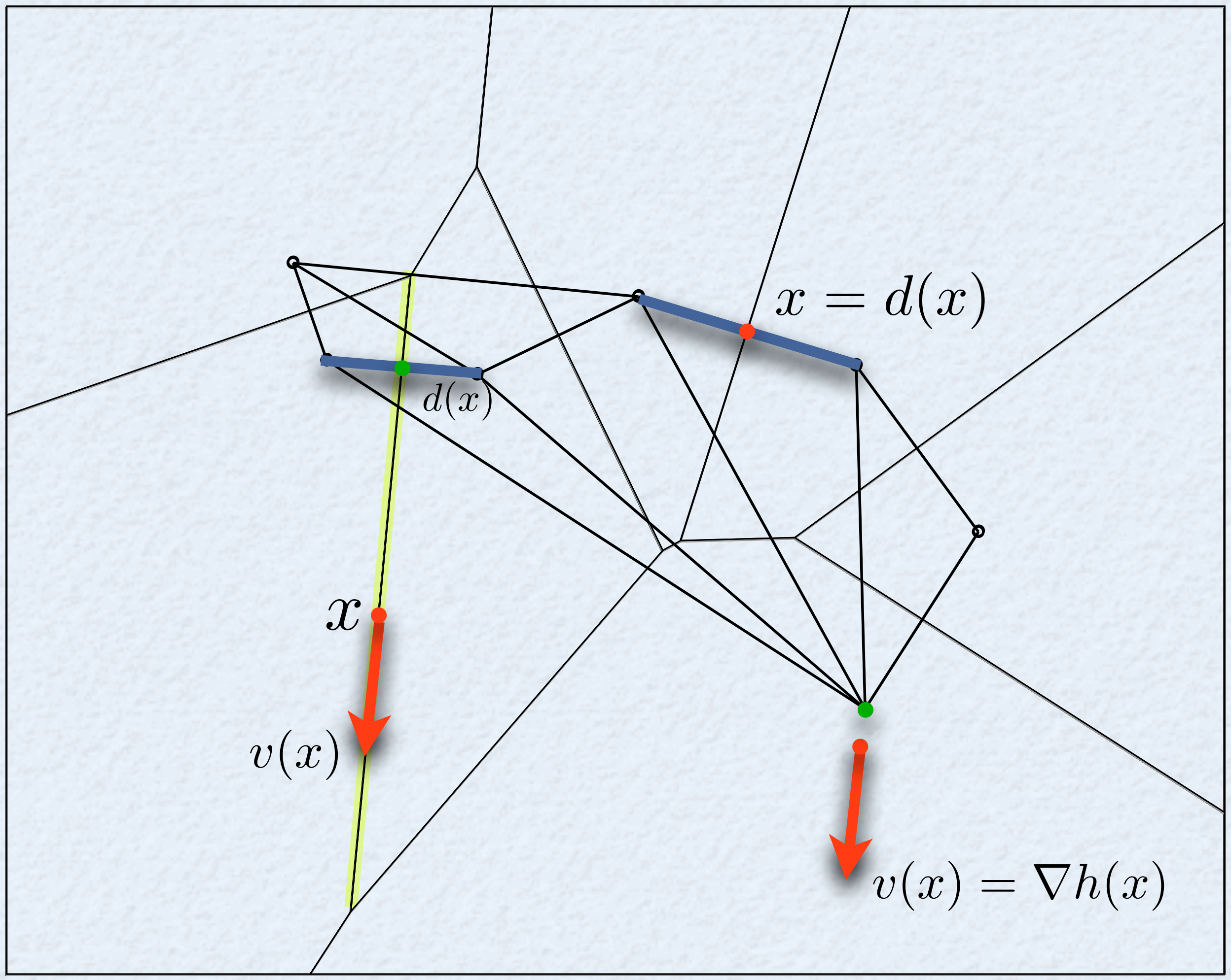

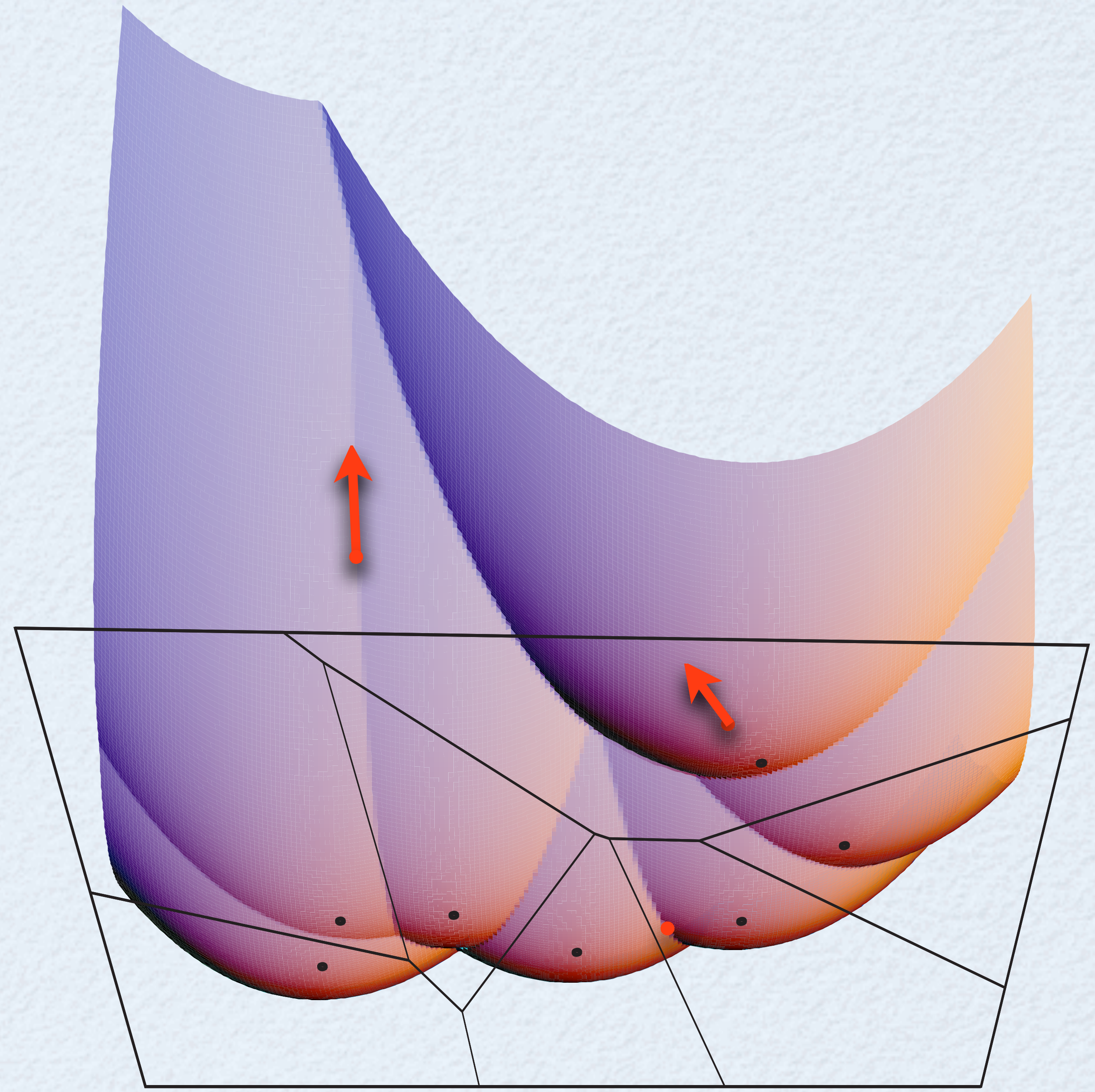

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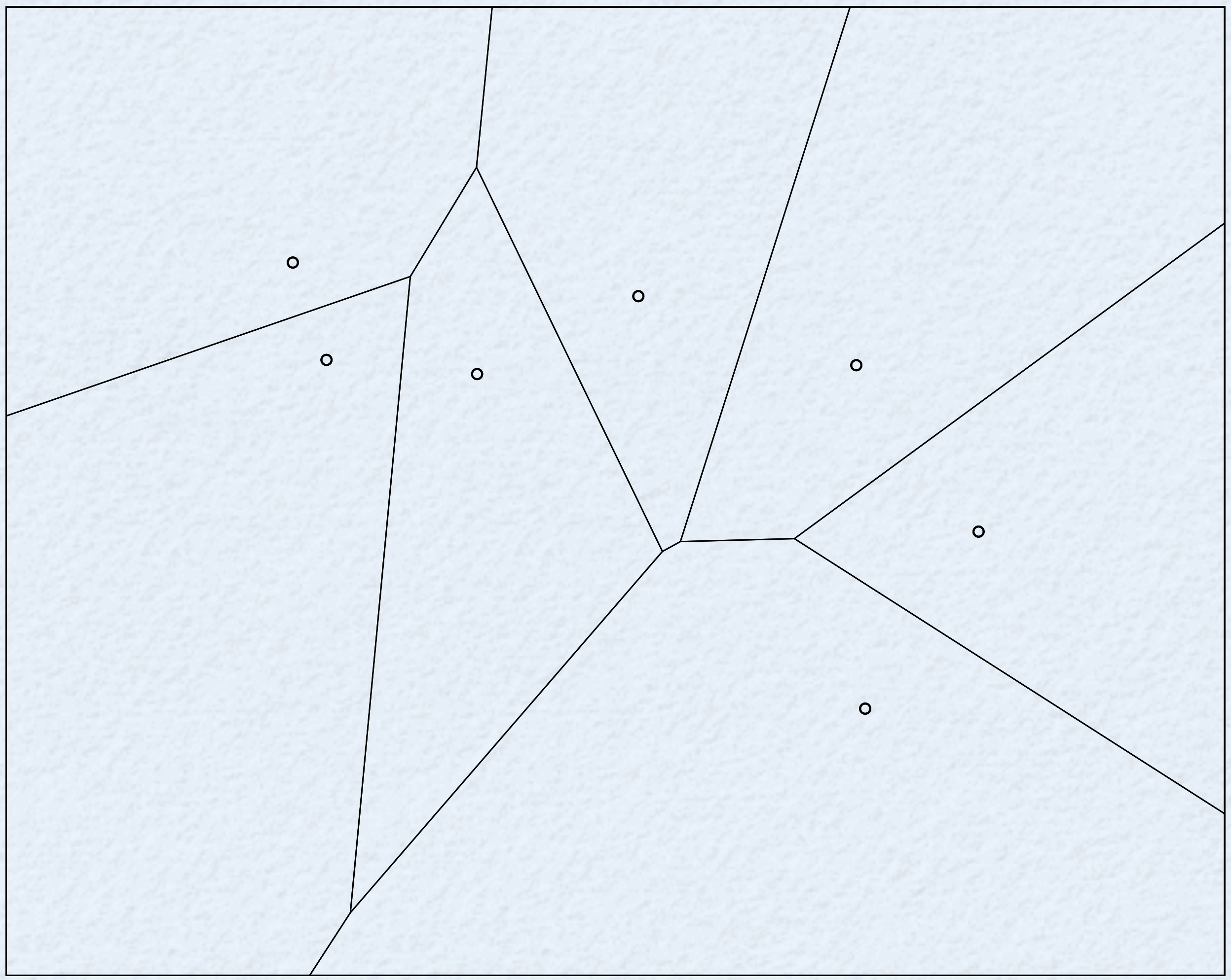

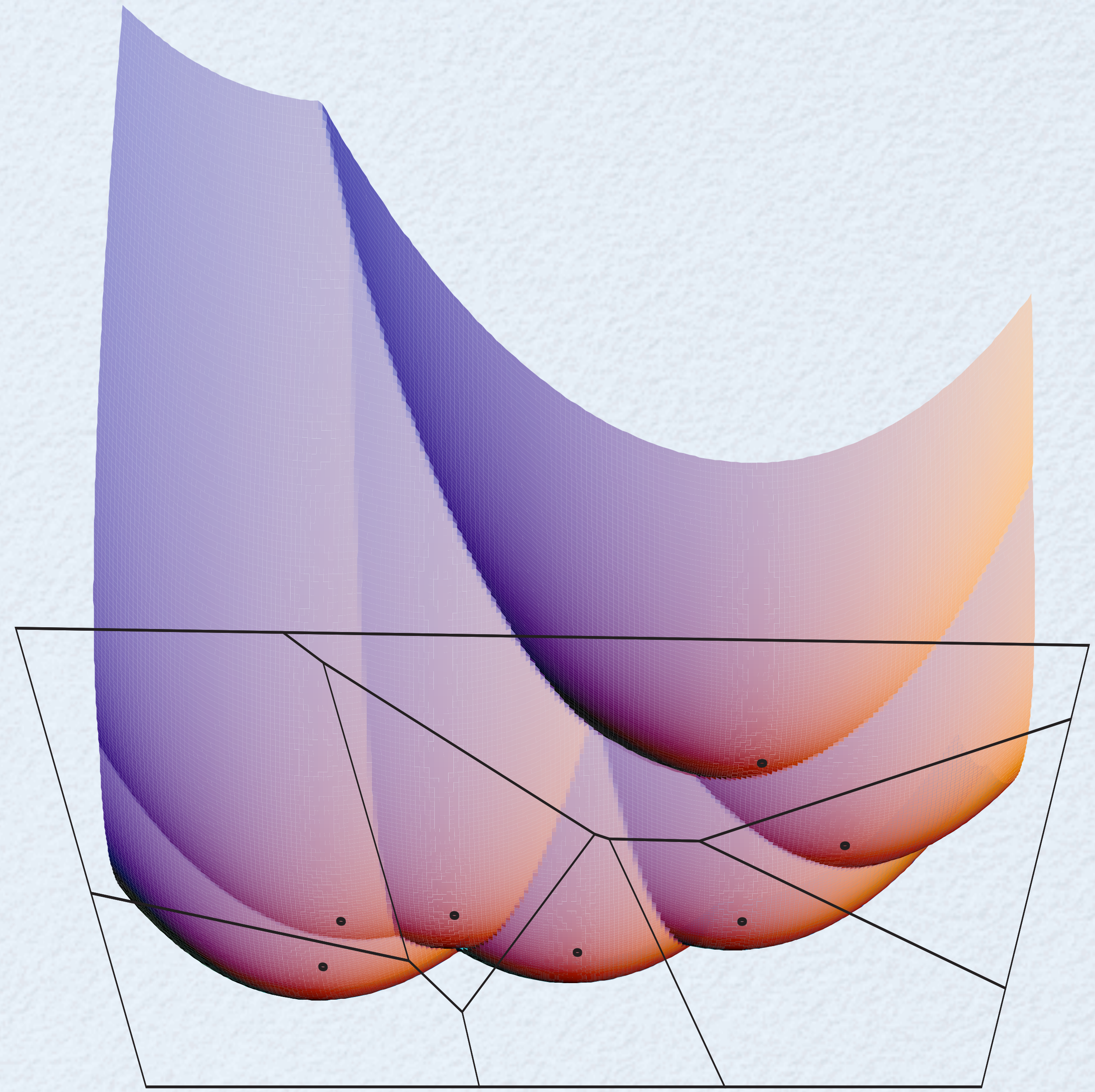

Moving at point x with speed $v(x)$ results a flow map $\phi : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$.

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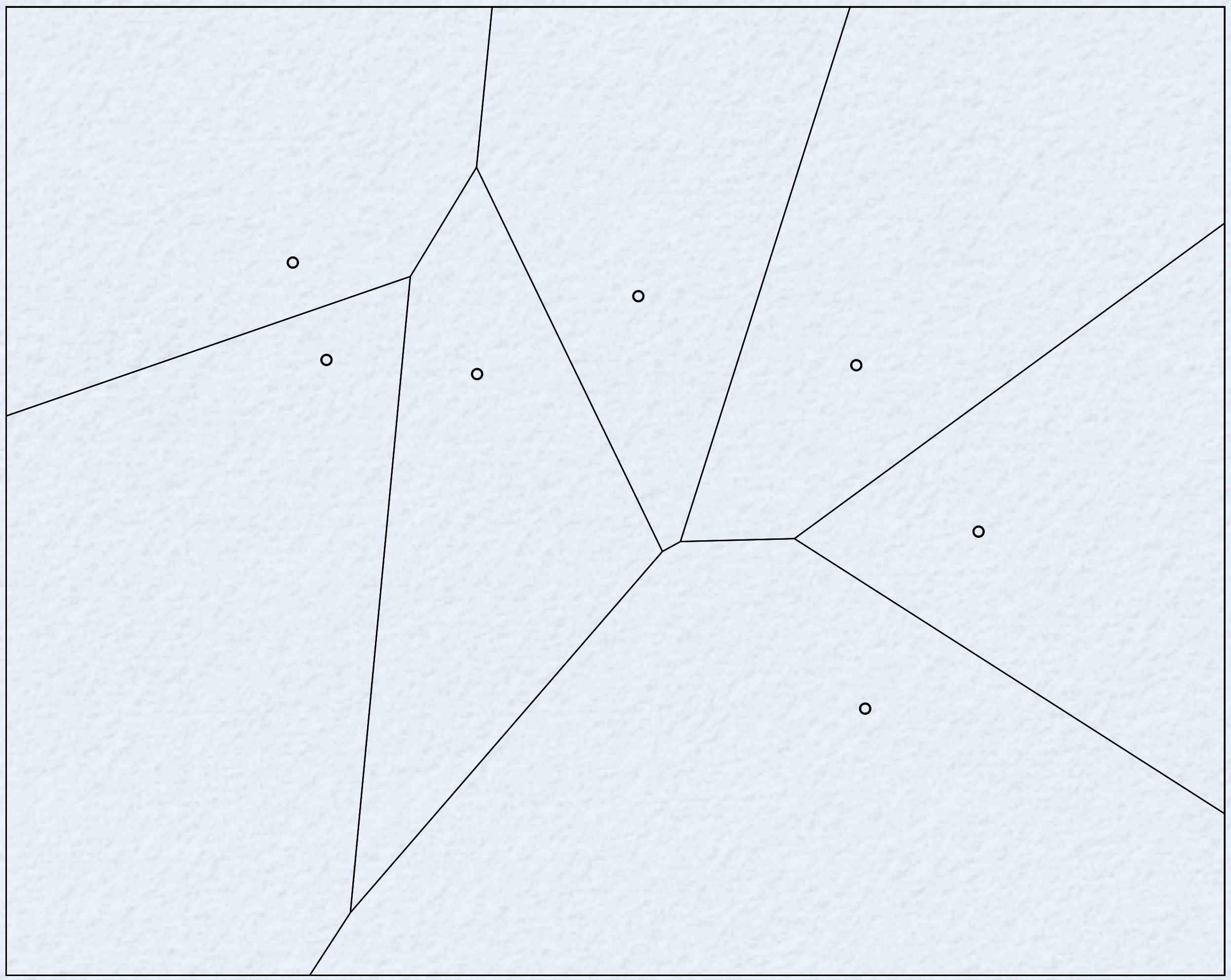

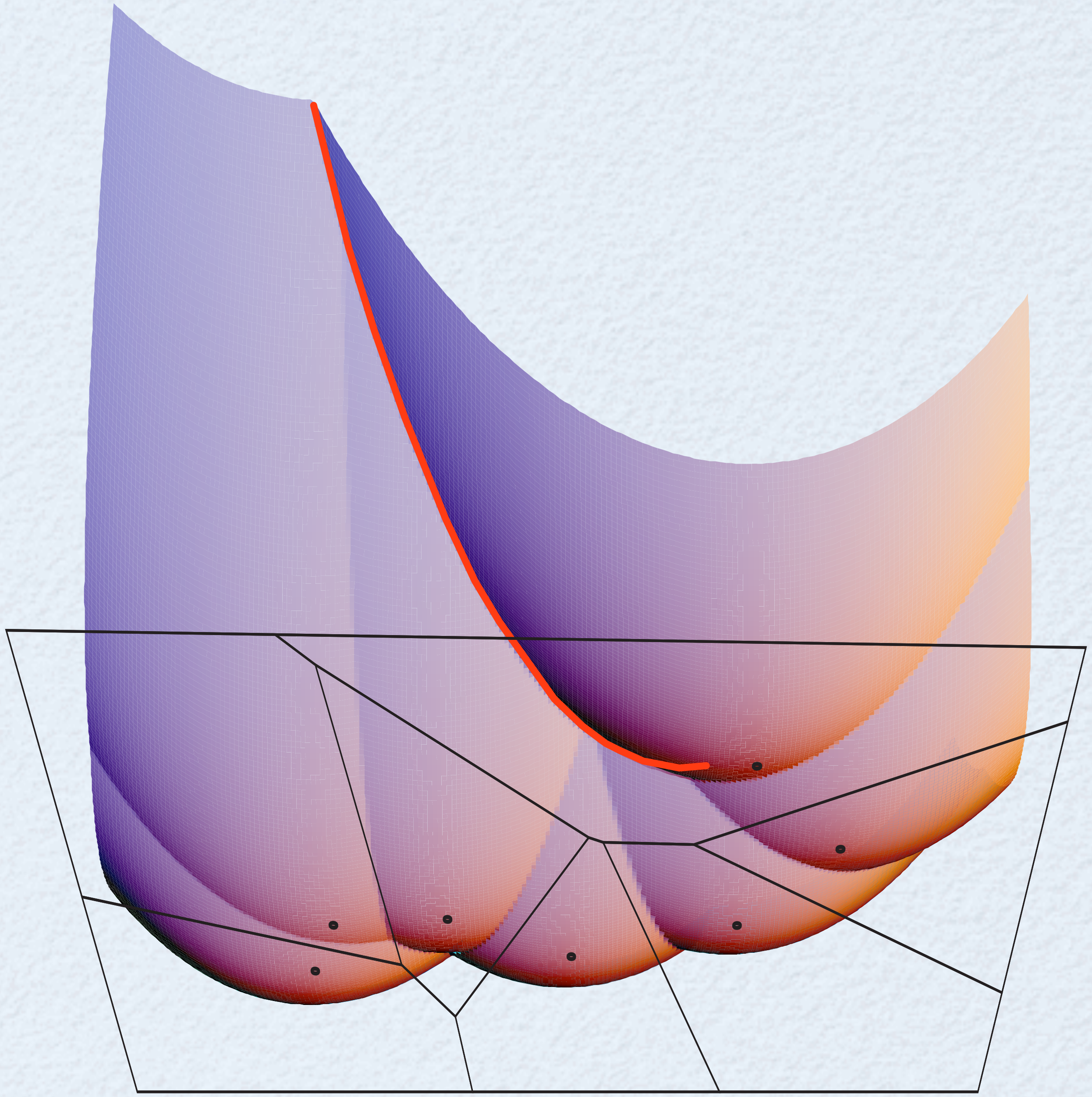

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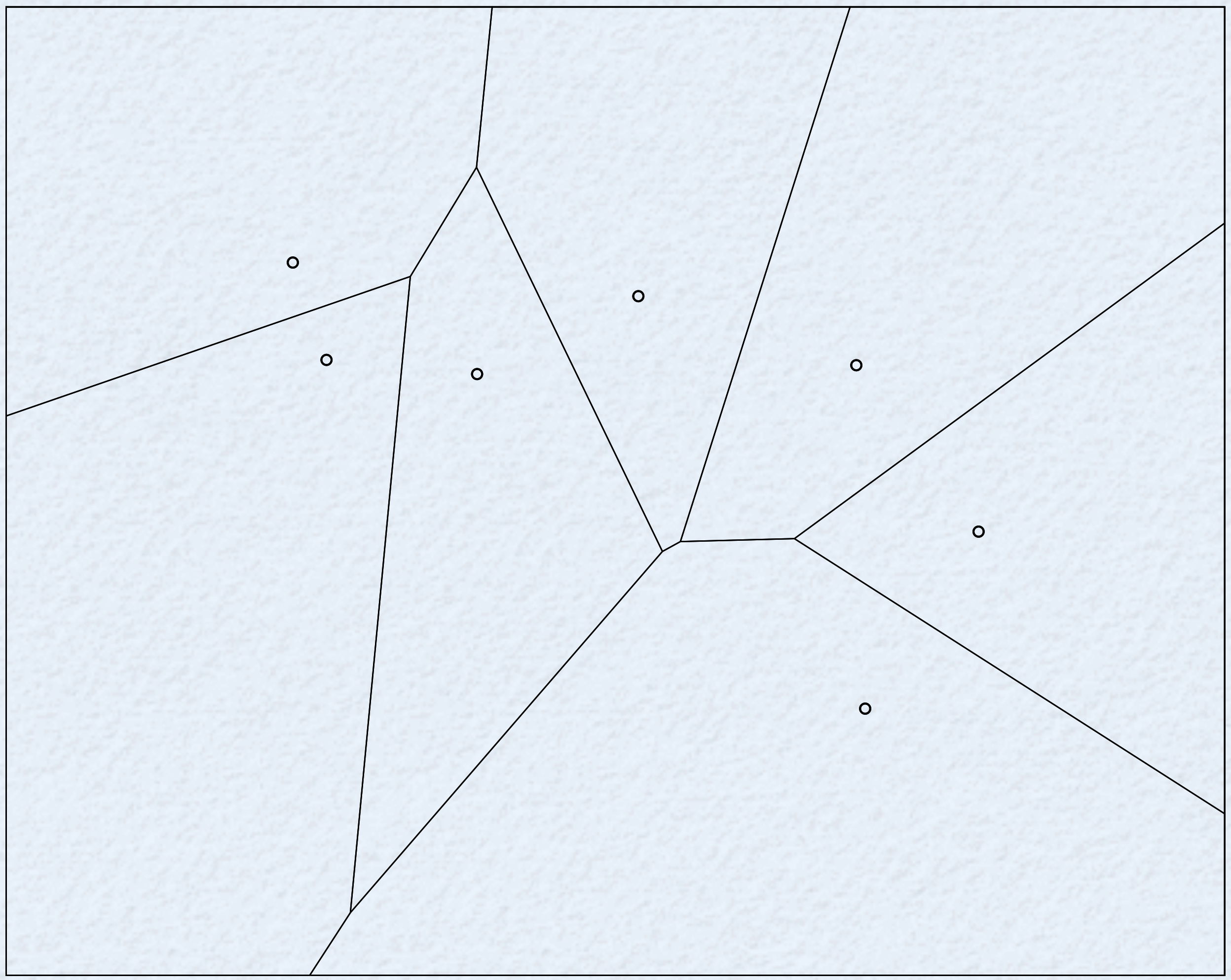

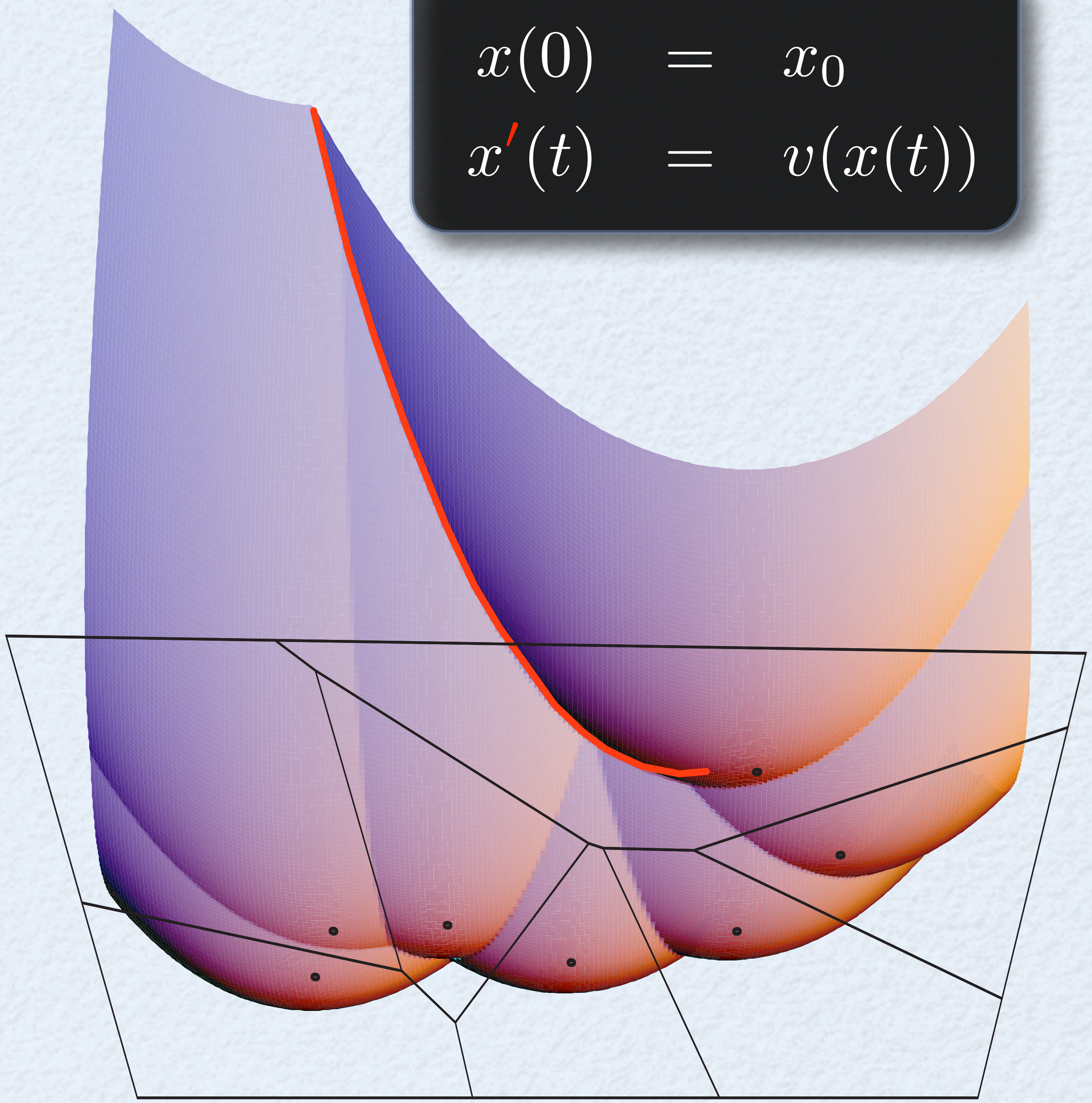

$$x(0) = x_0$$
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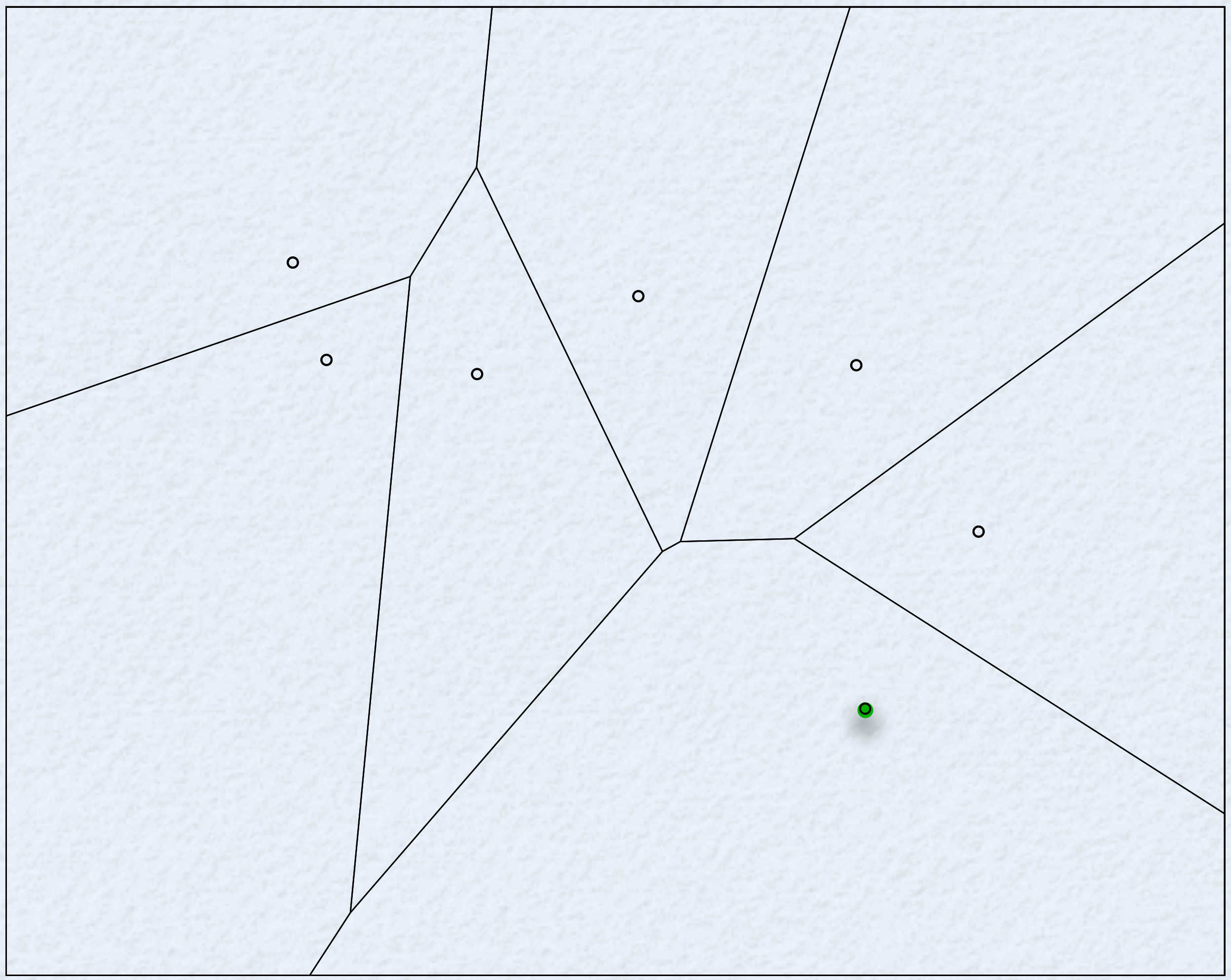

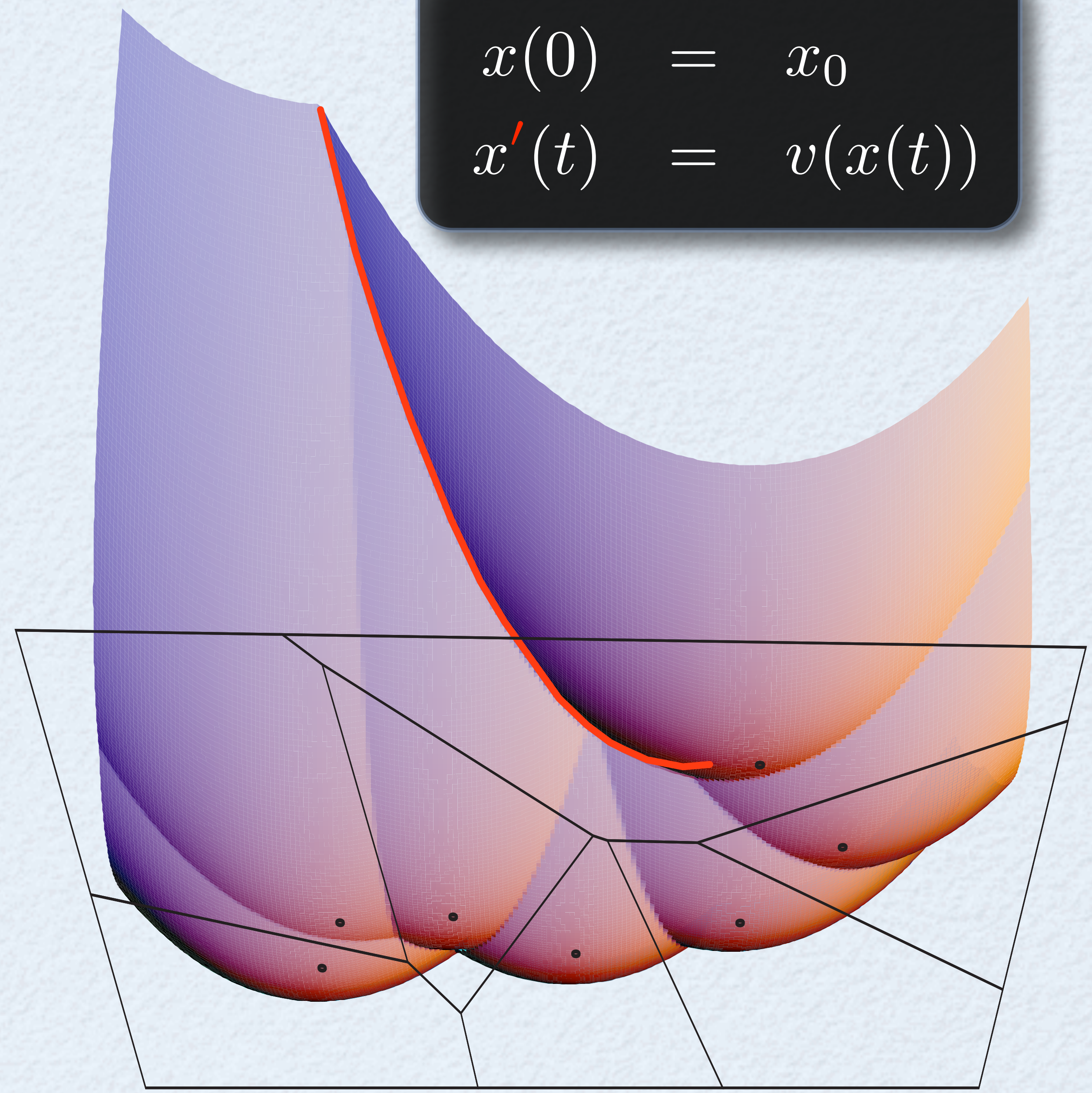

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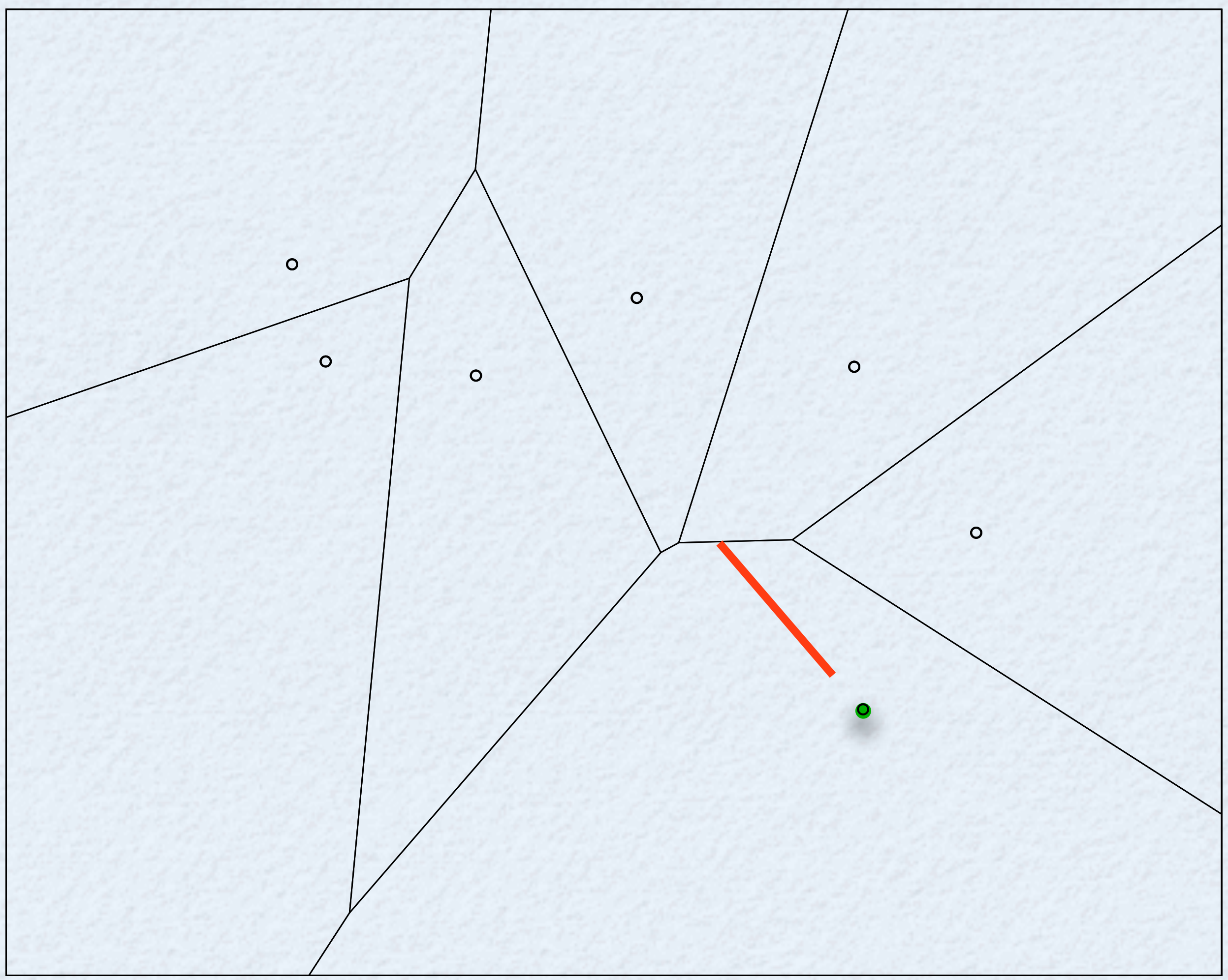

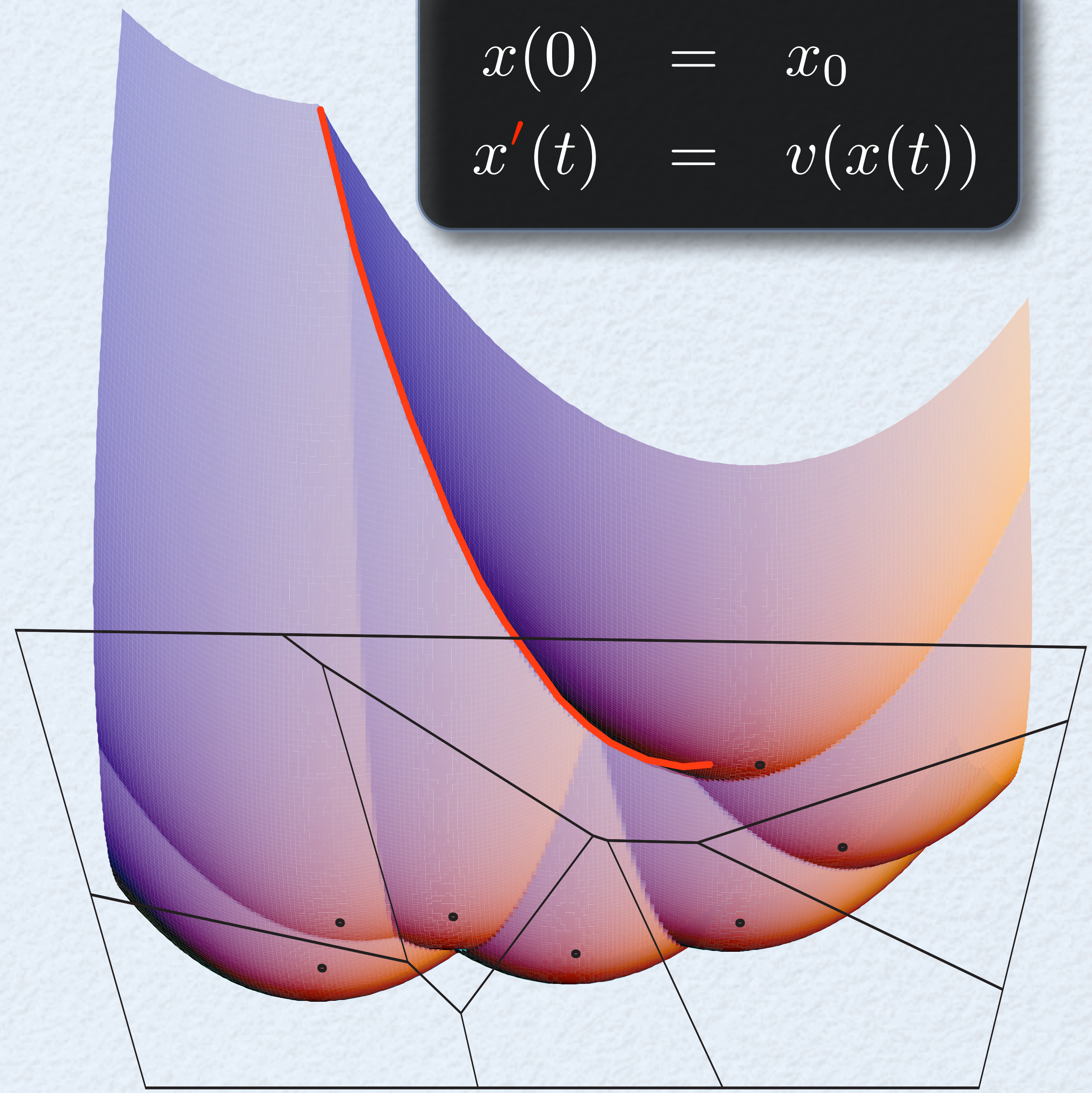

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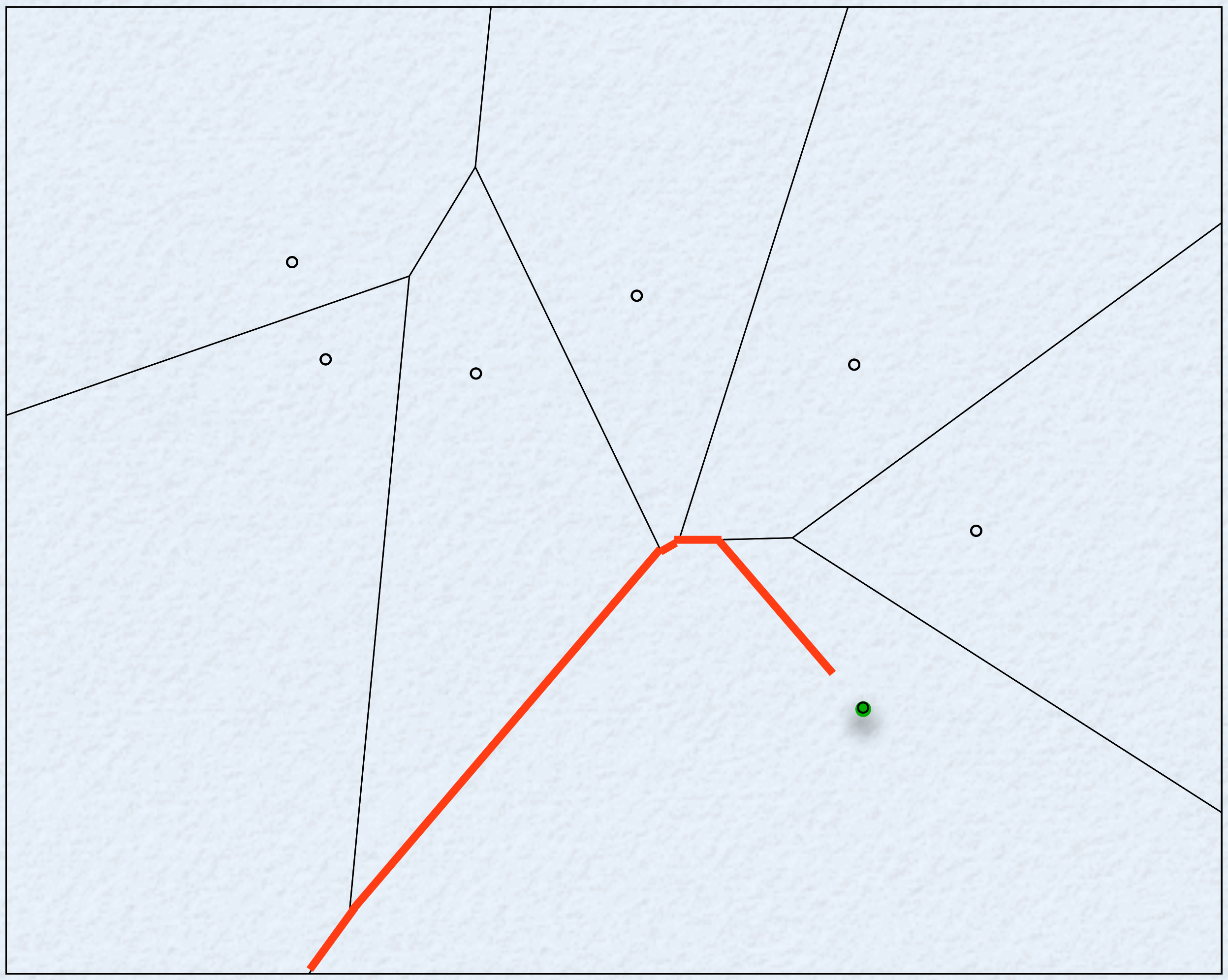

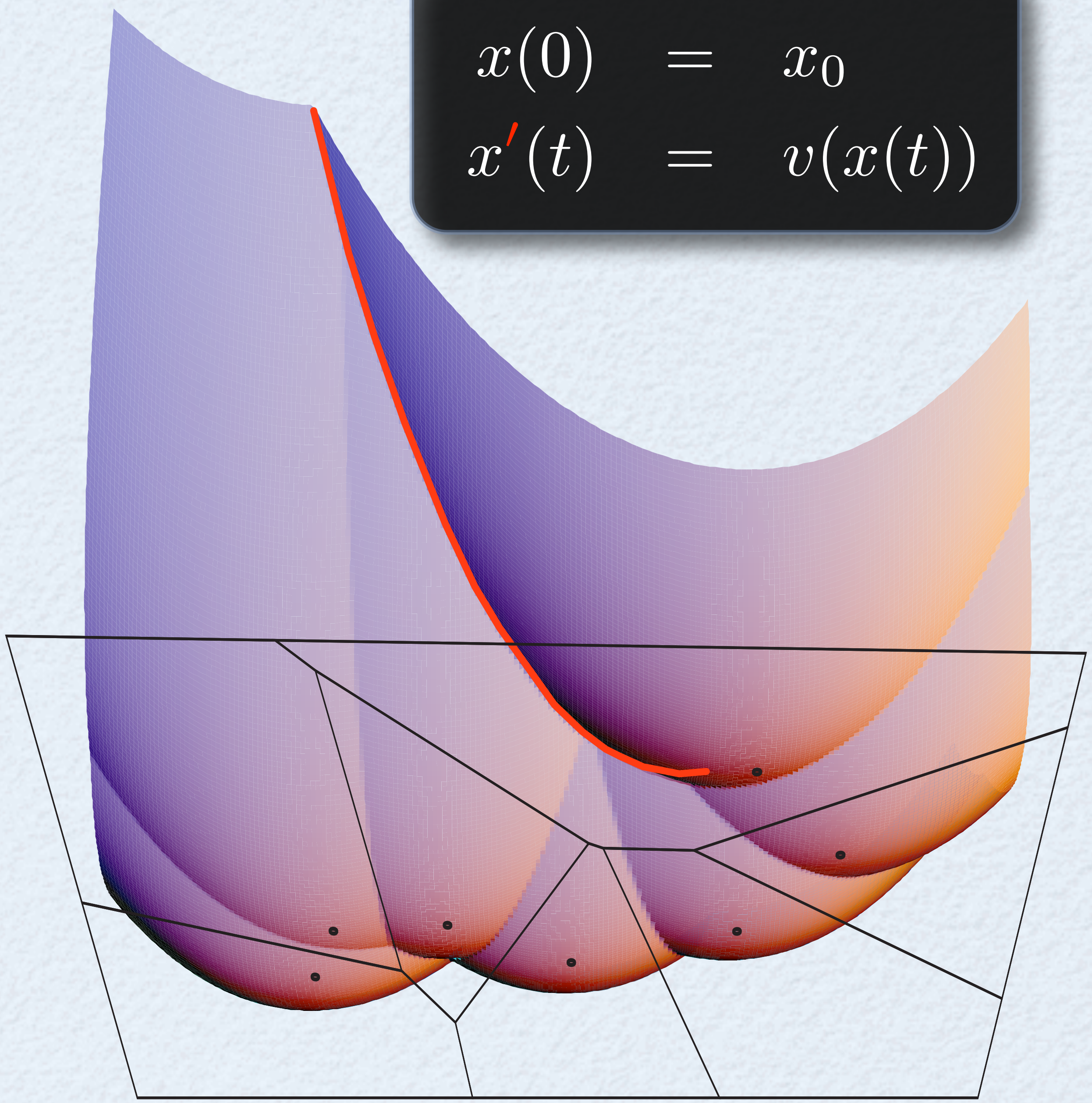

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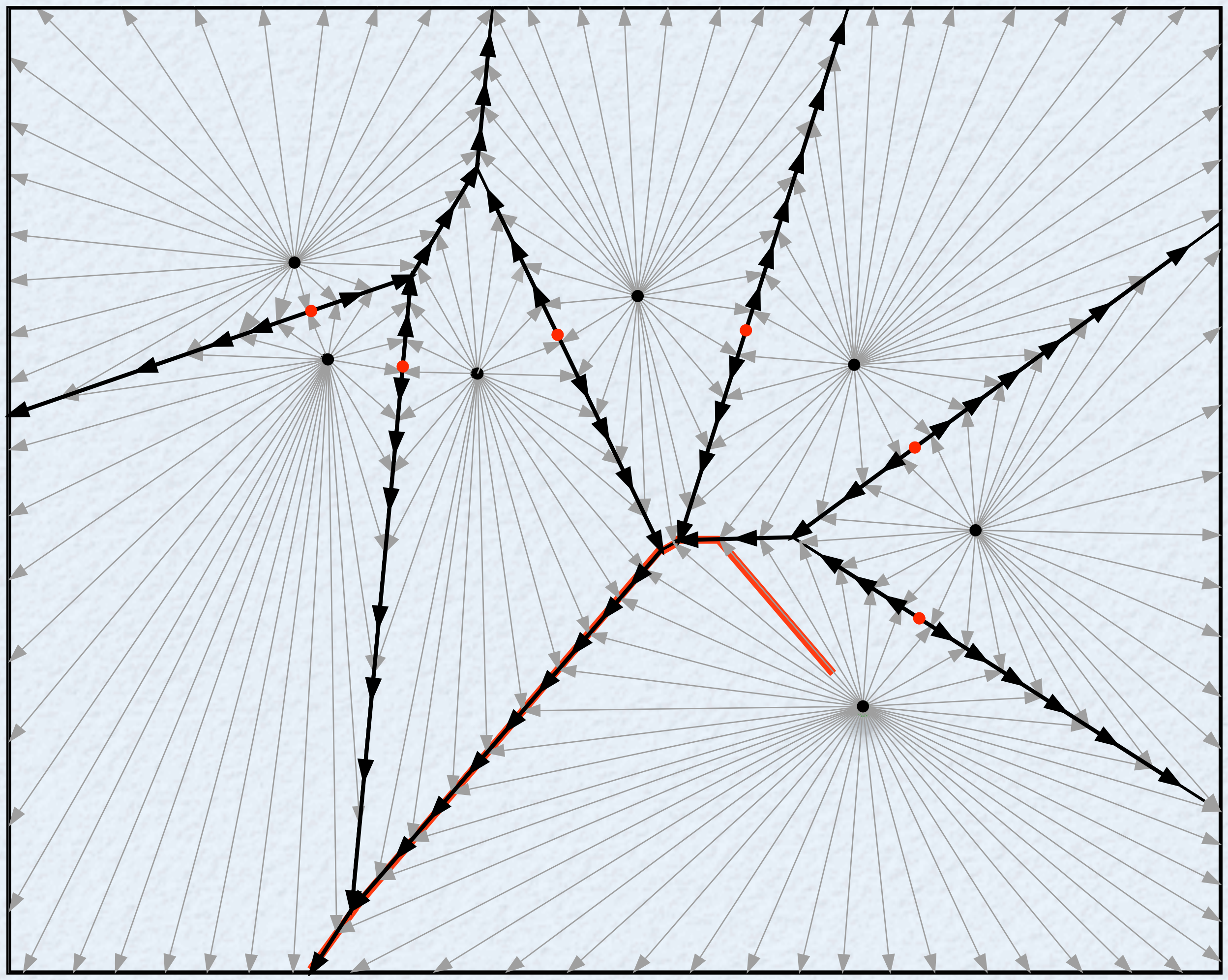

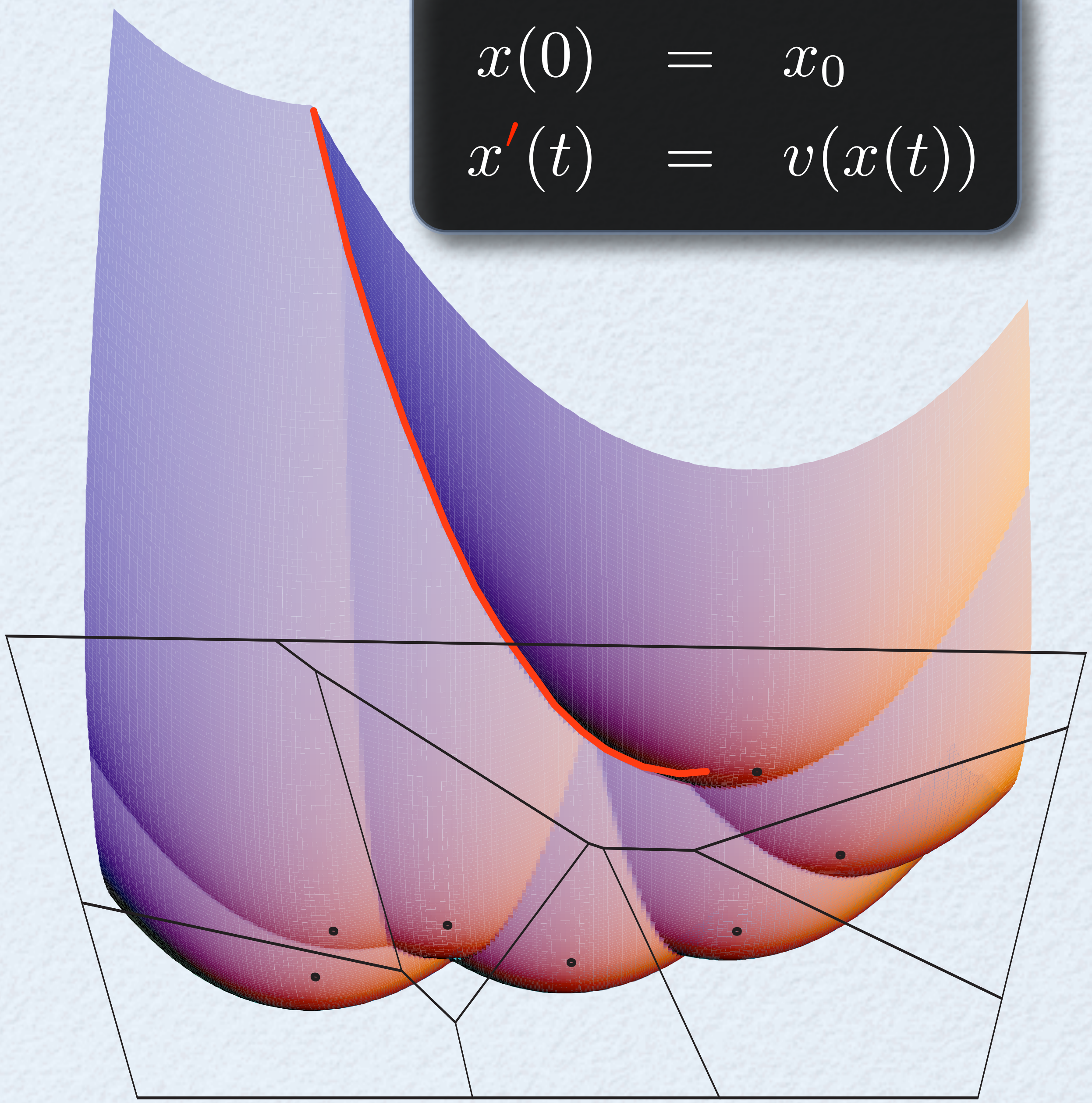

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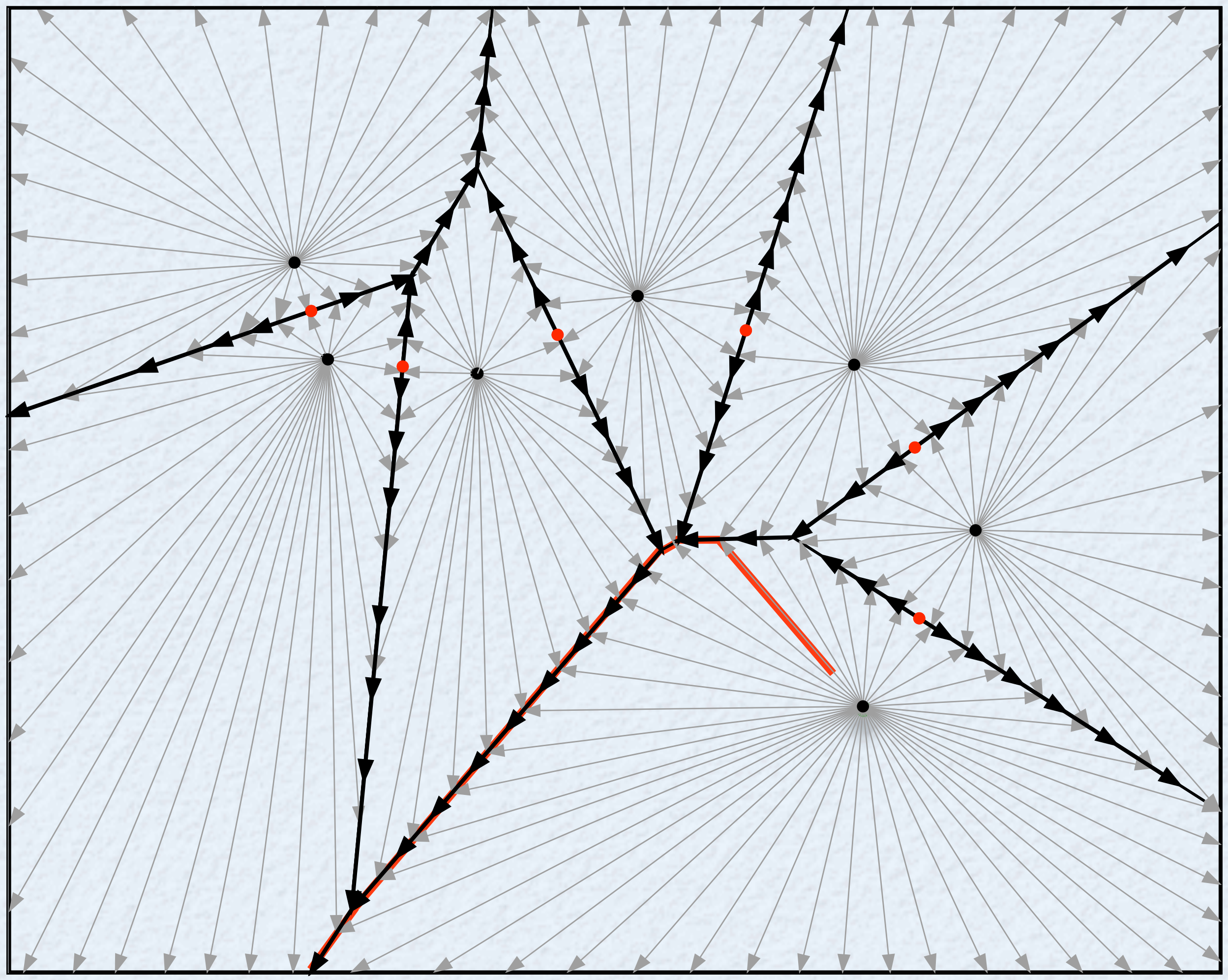

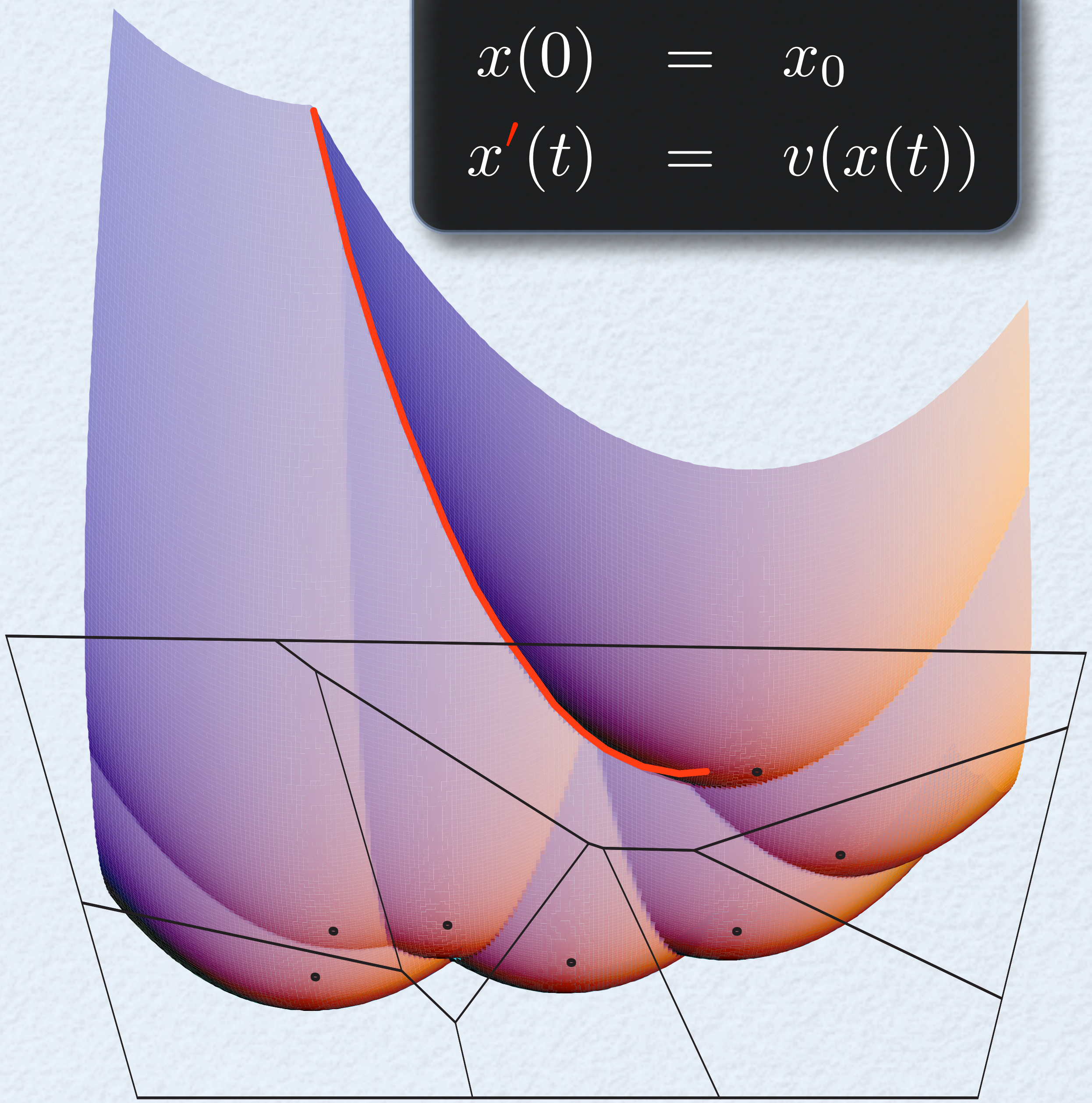

Moving at point x with speed $v(x)$ results a flow map $\phi : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$.

$\phi(t, x) = y$ means "starting at x and going for time t we reach y".

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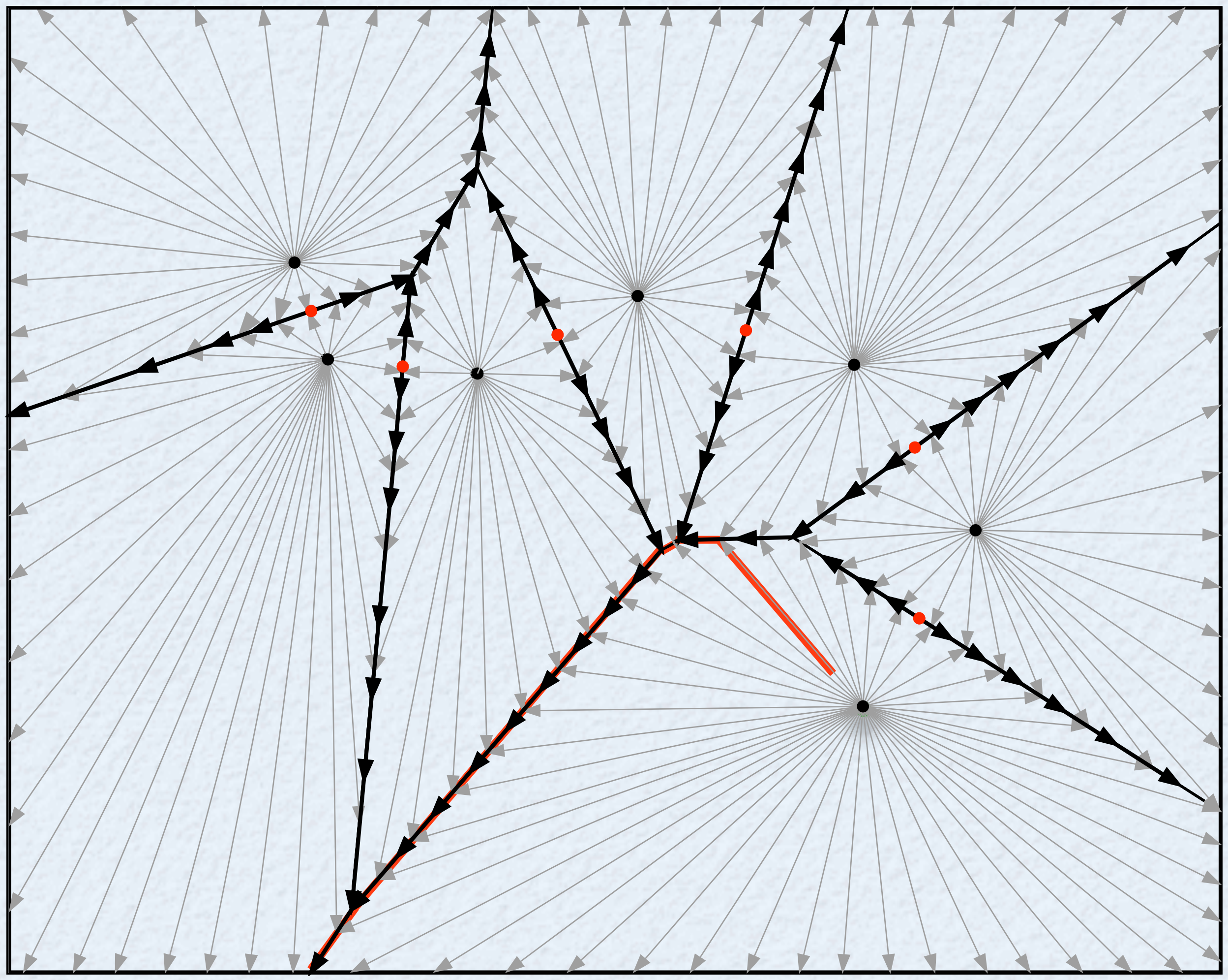

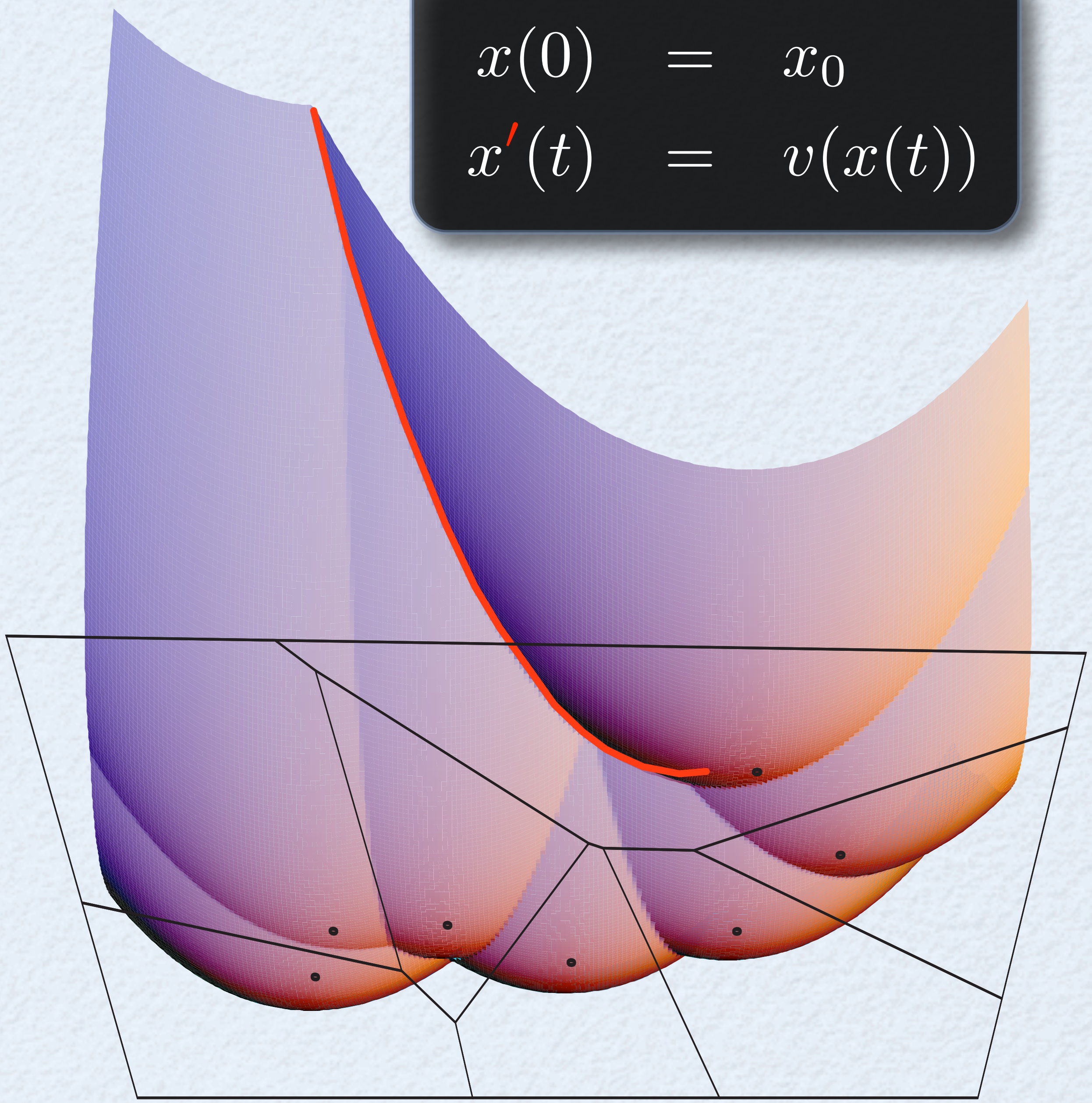

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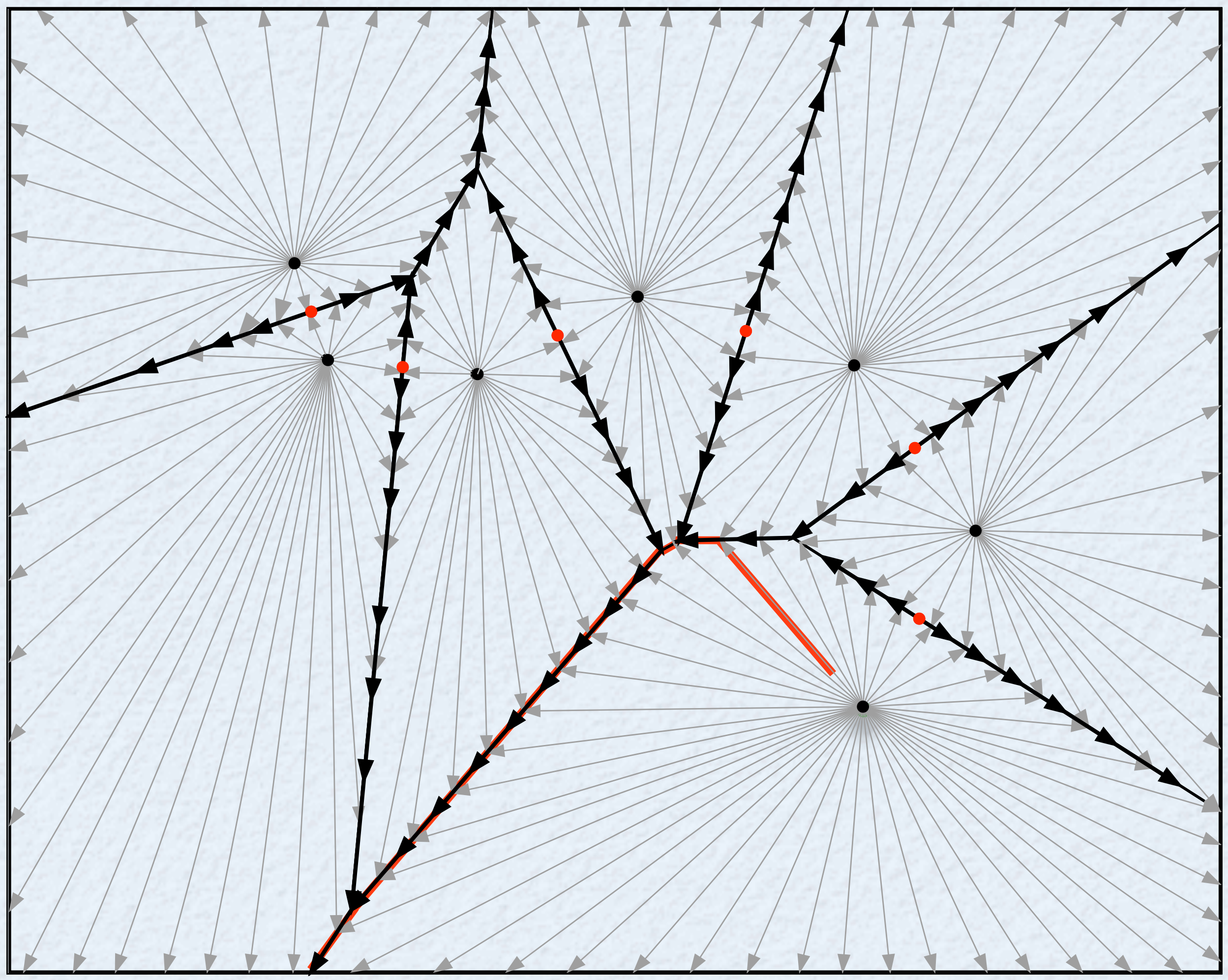

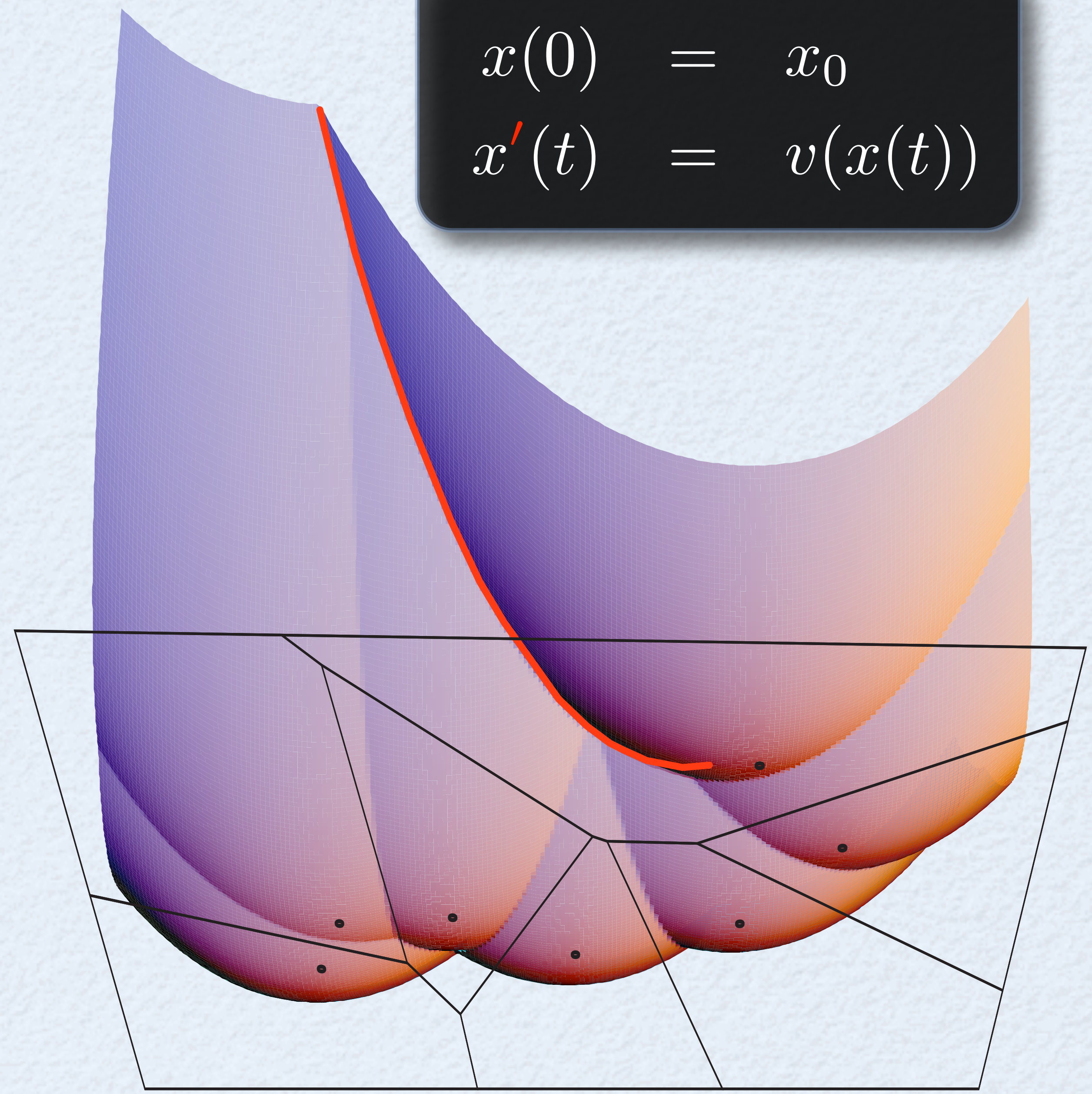

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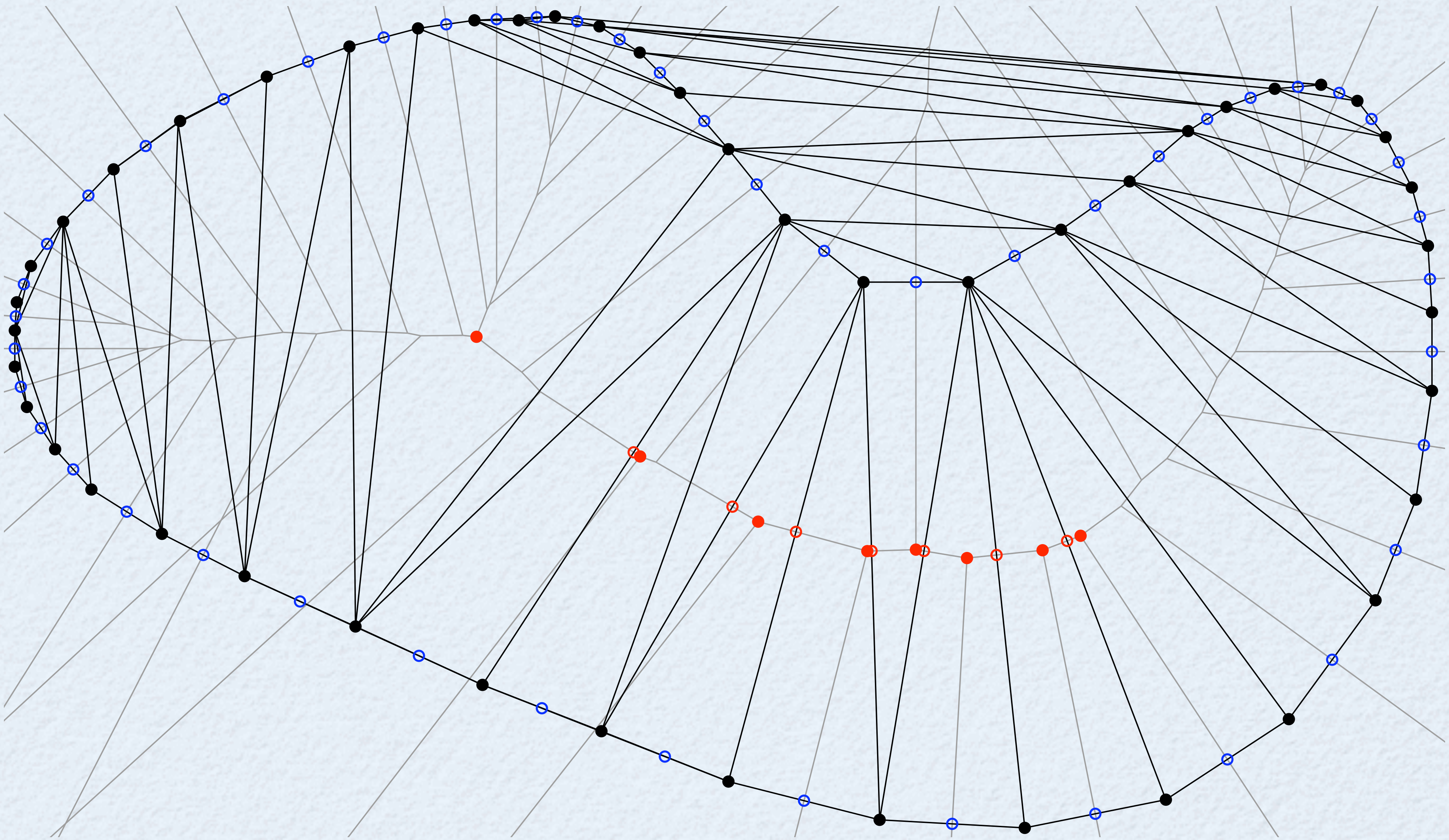
Critical Points of Distance Function

A point c with $v(c) = 0$ is called critical.



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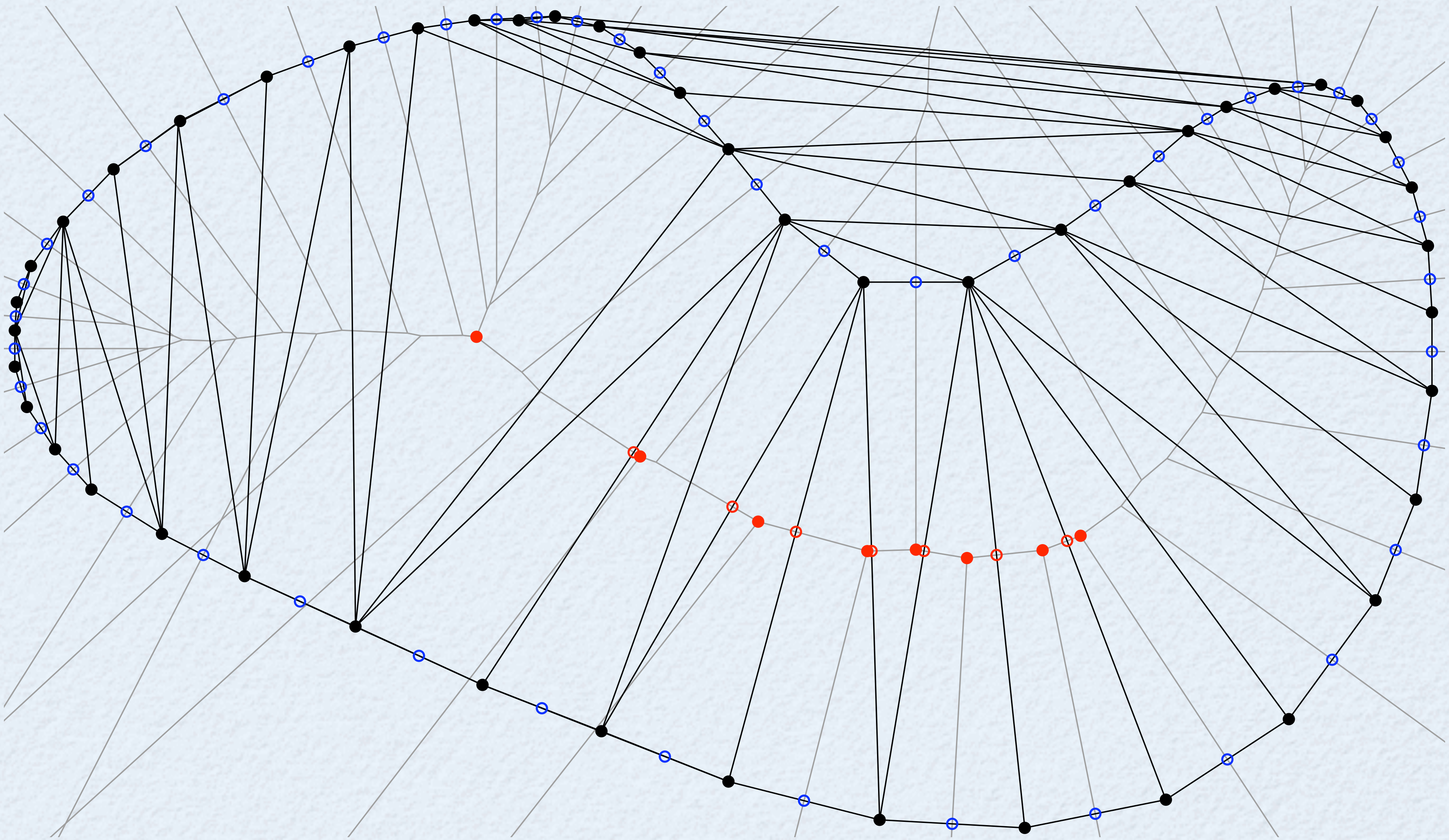

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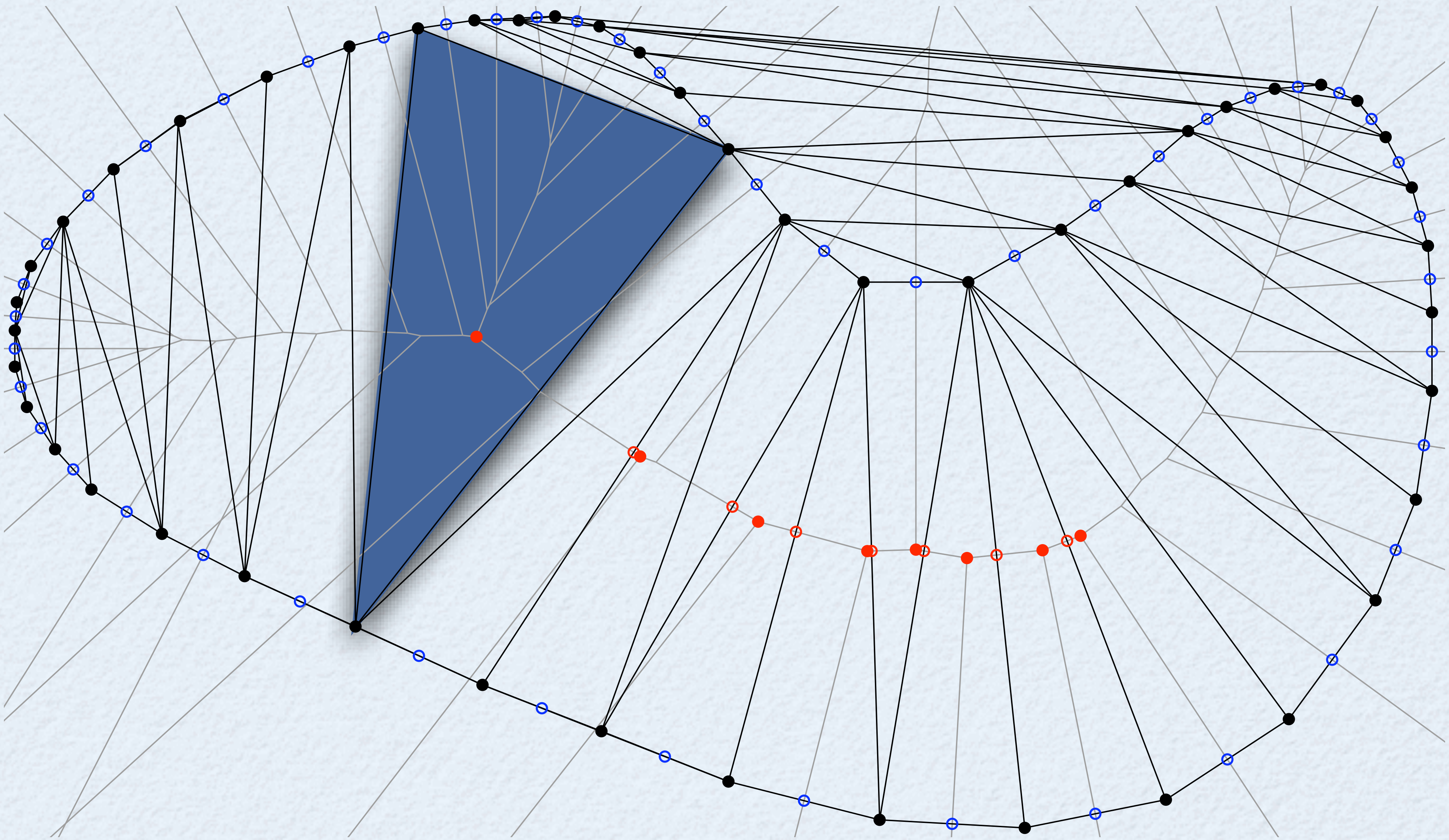

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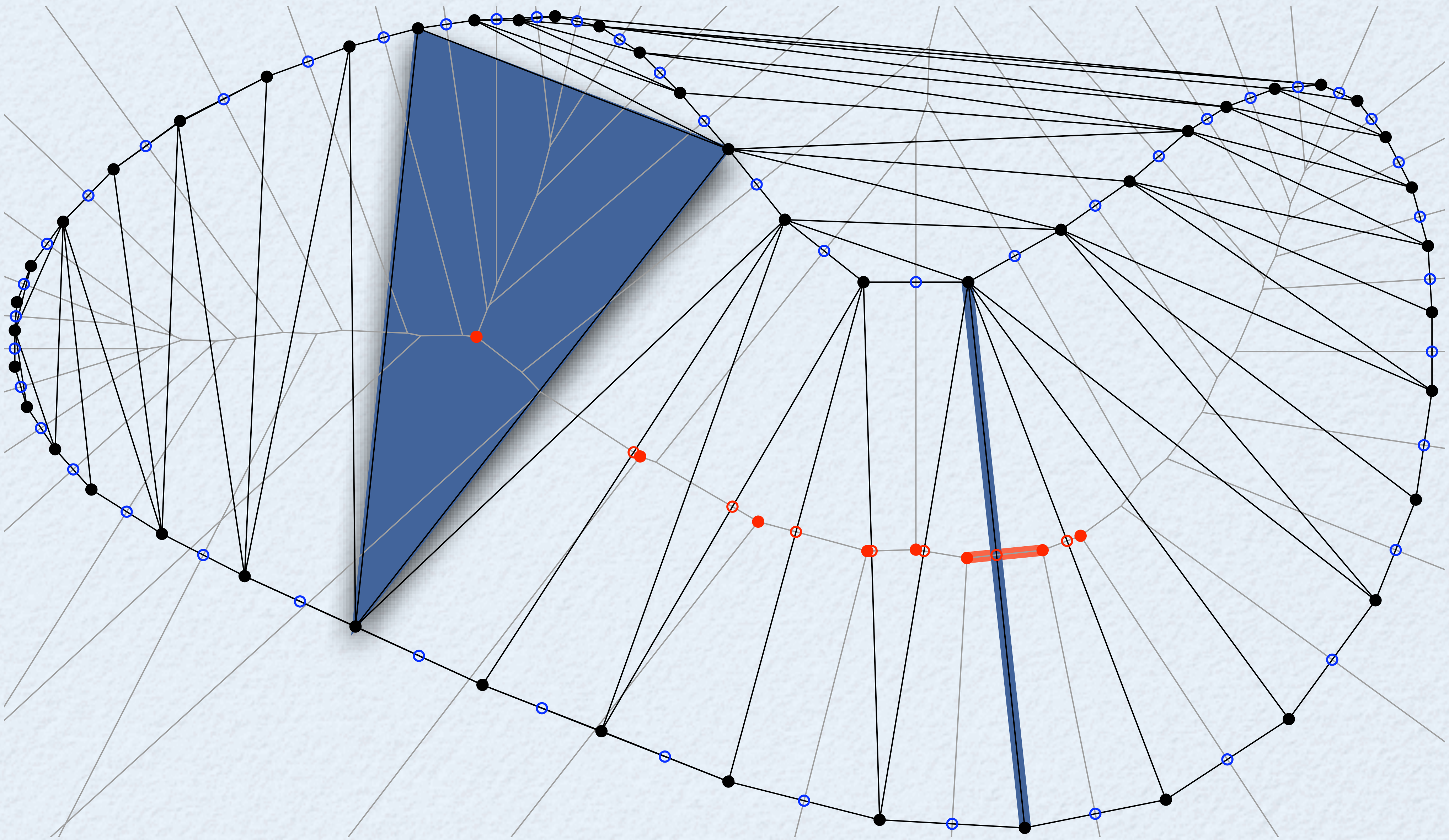

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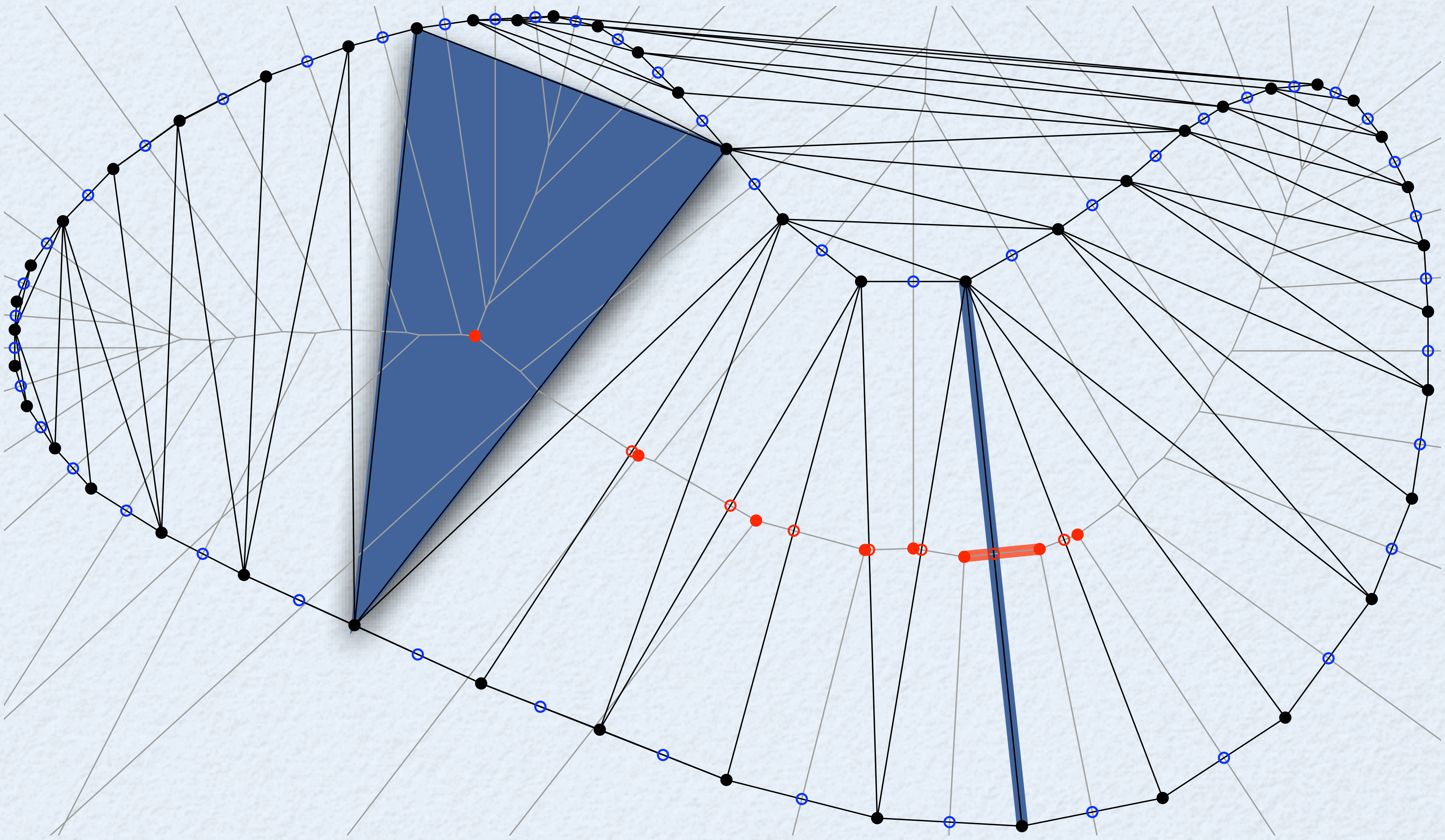
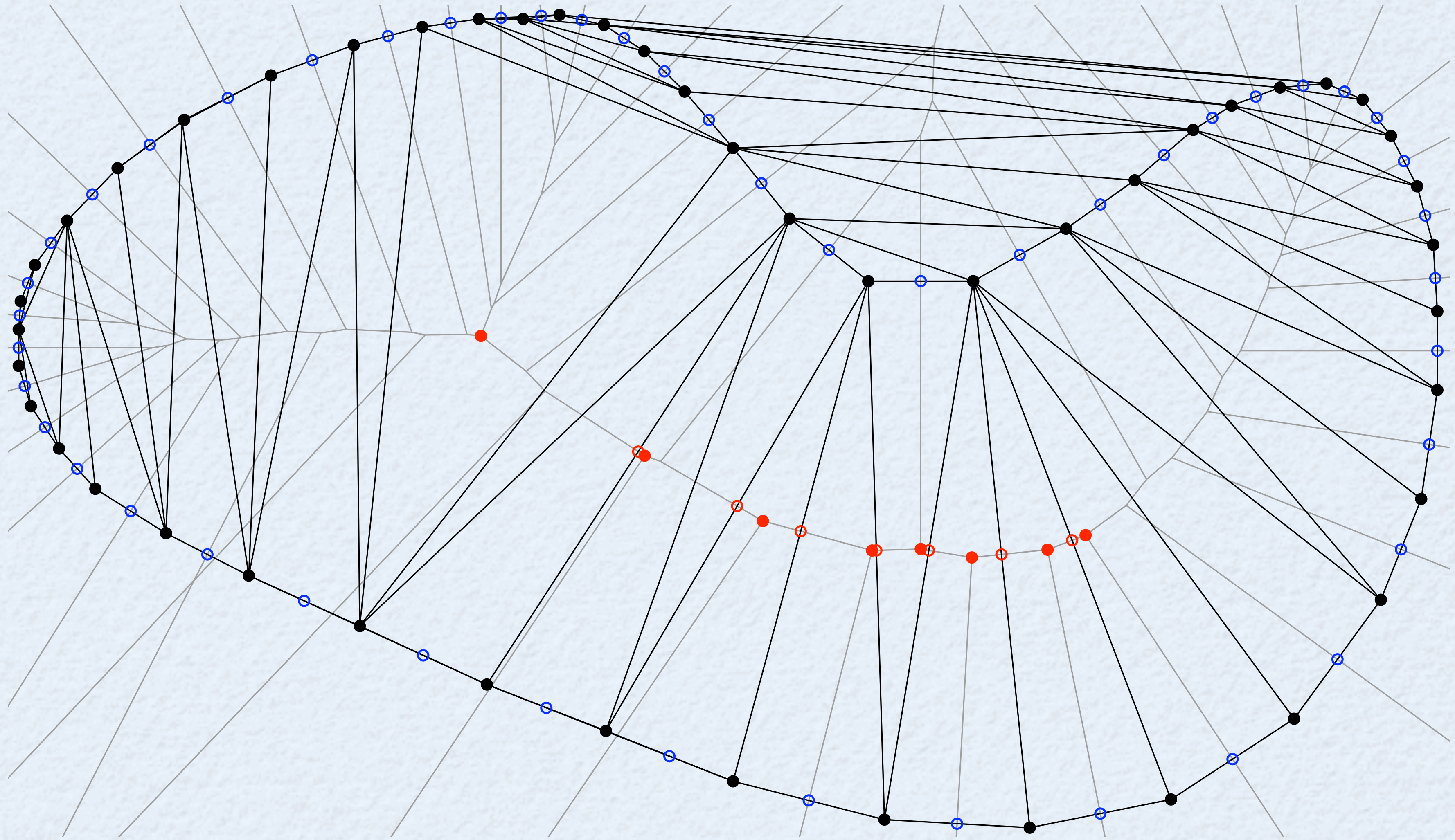


The index of c is the dimension of $D(c)$.

Stable Manifold of a Critical Point

Stable manifold of a critical point c is the set of all points that flow to c.

$$\mathsf{Sm}(c) = \{x : \phi(\infty, x) = c\}.$$



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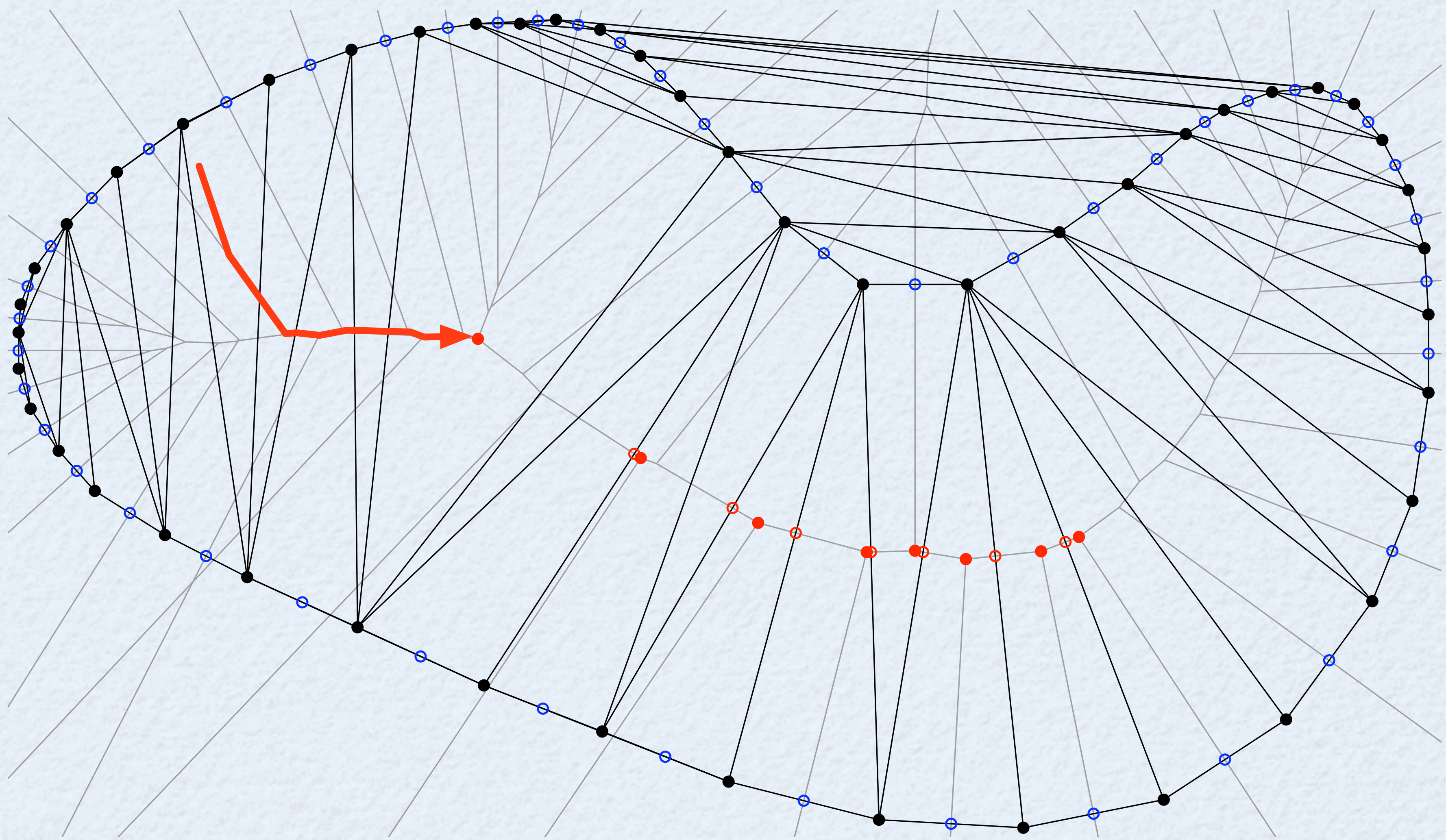

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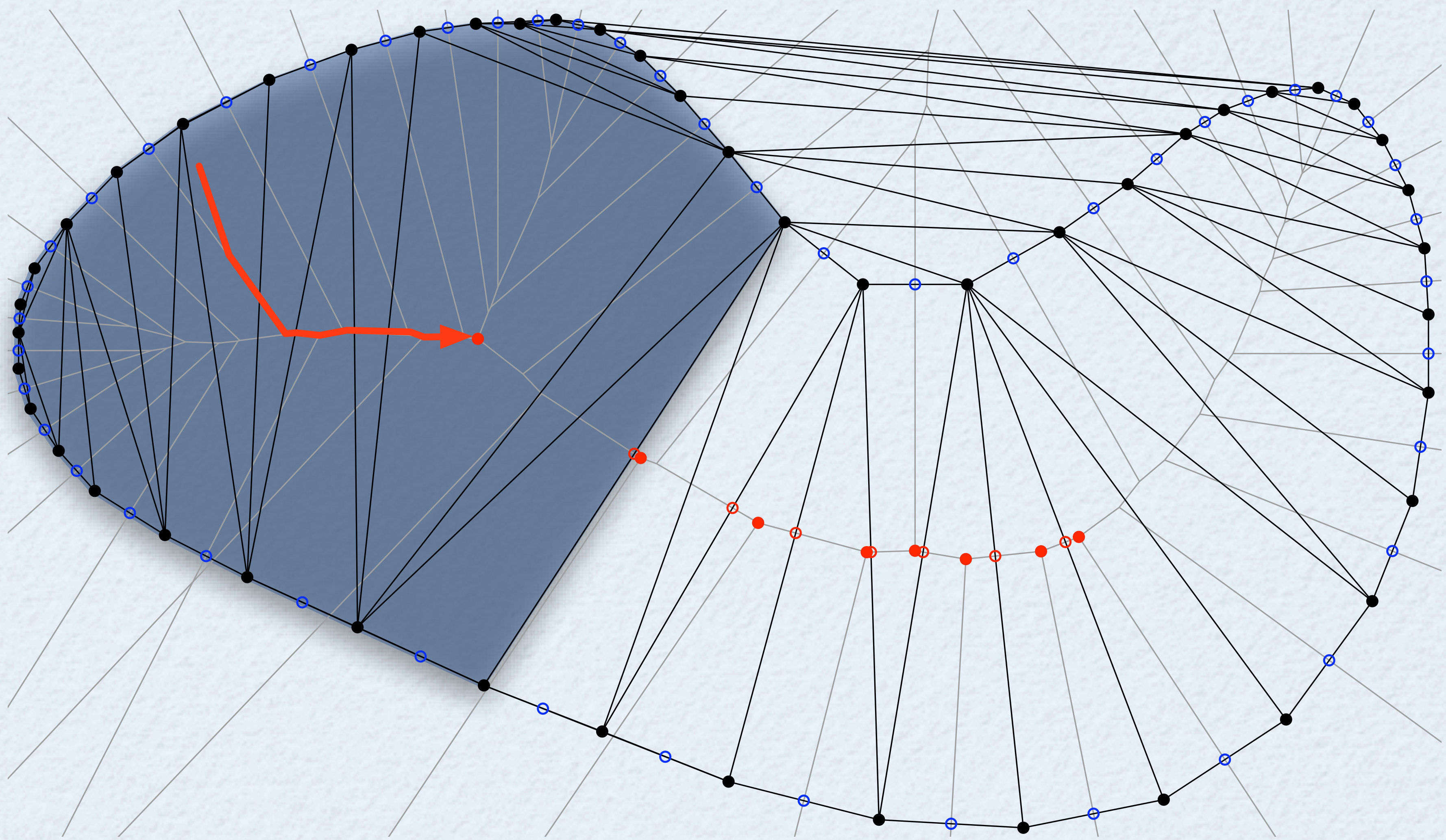

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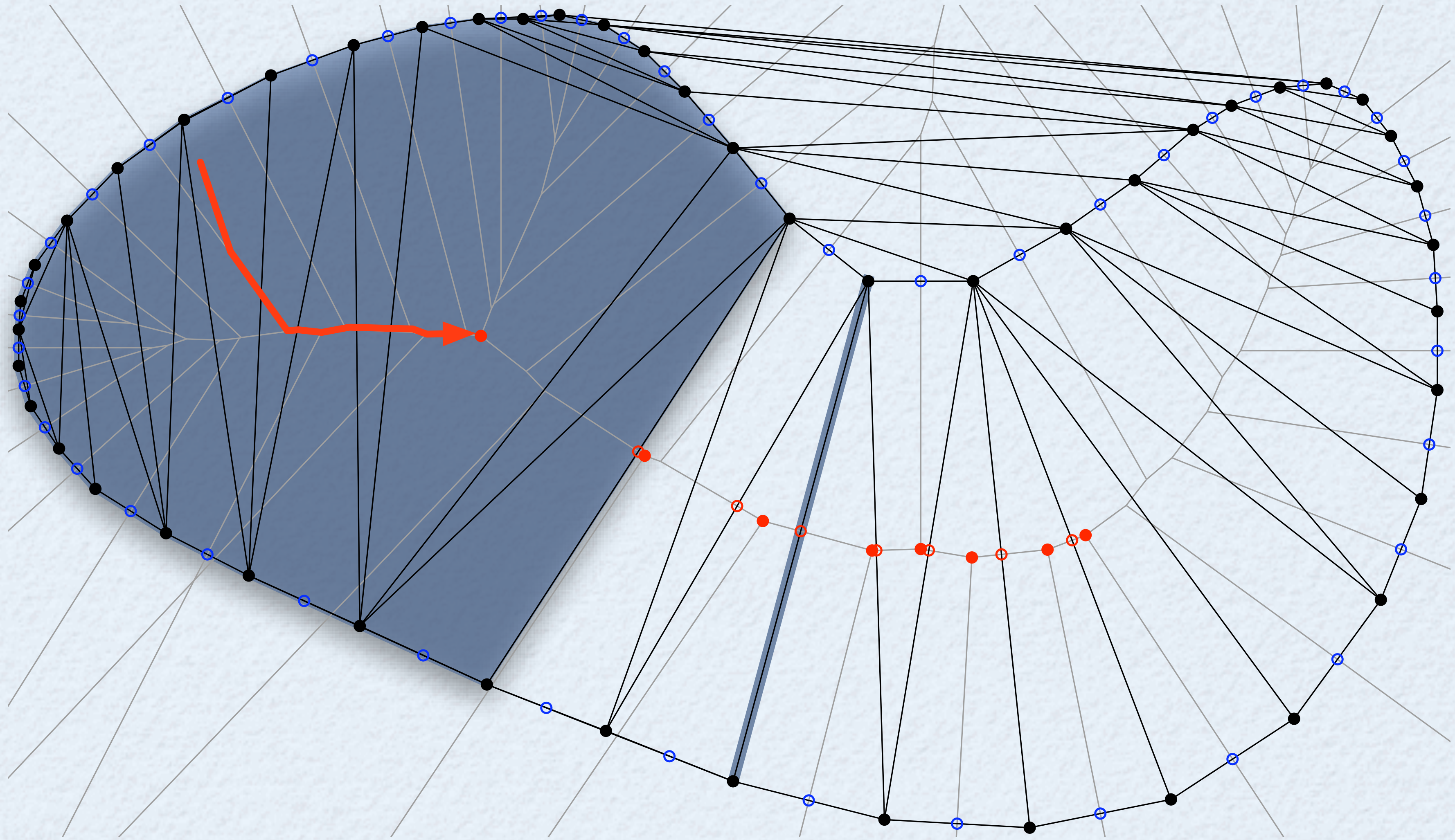

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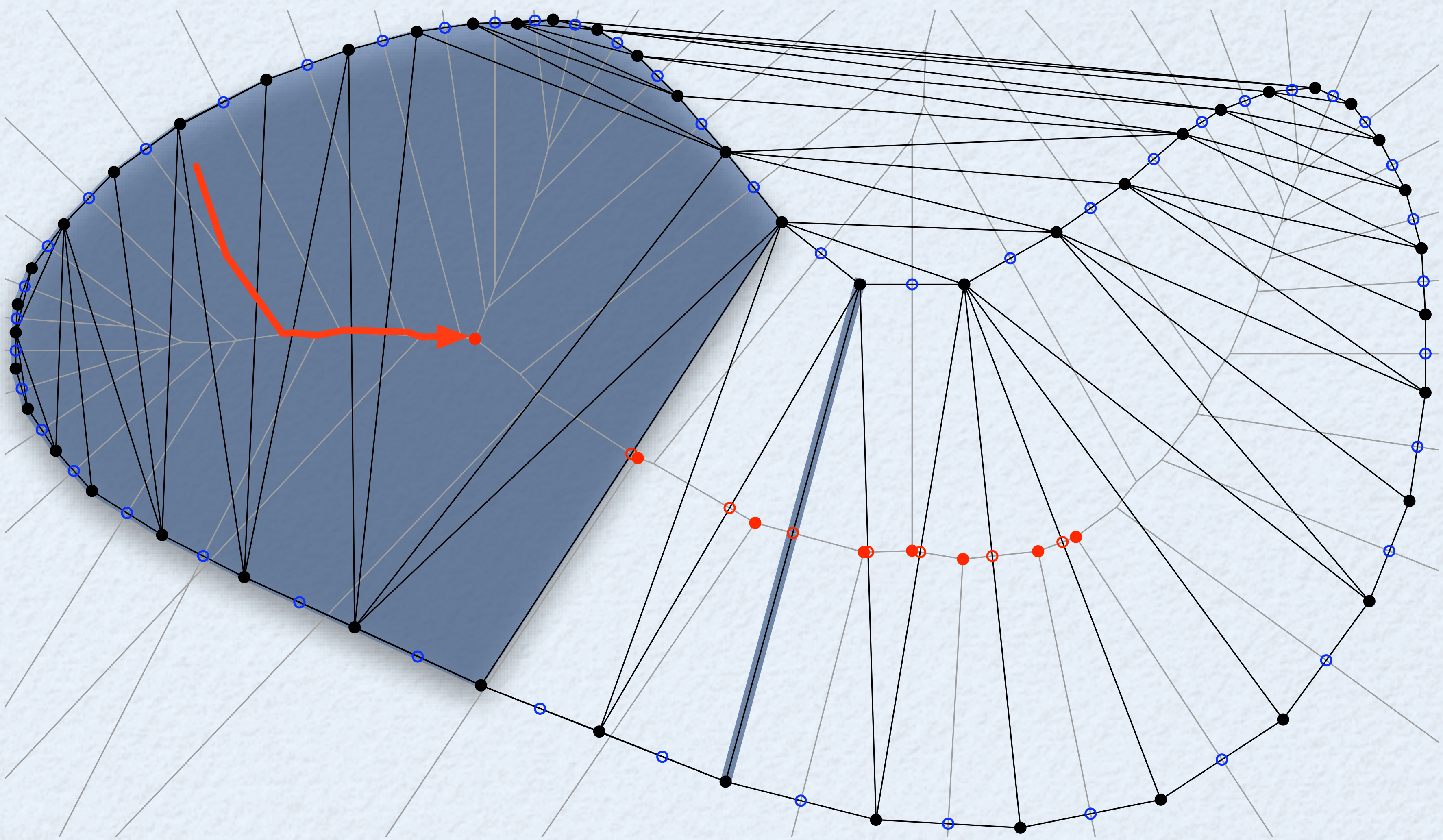

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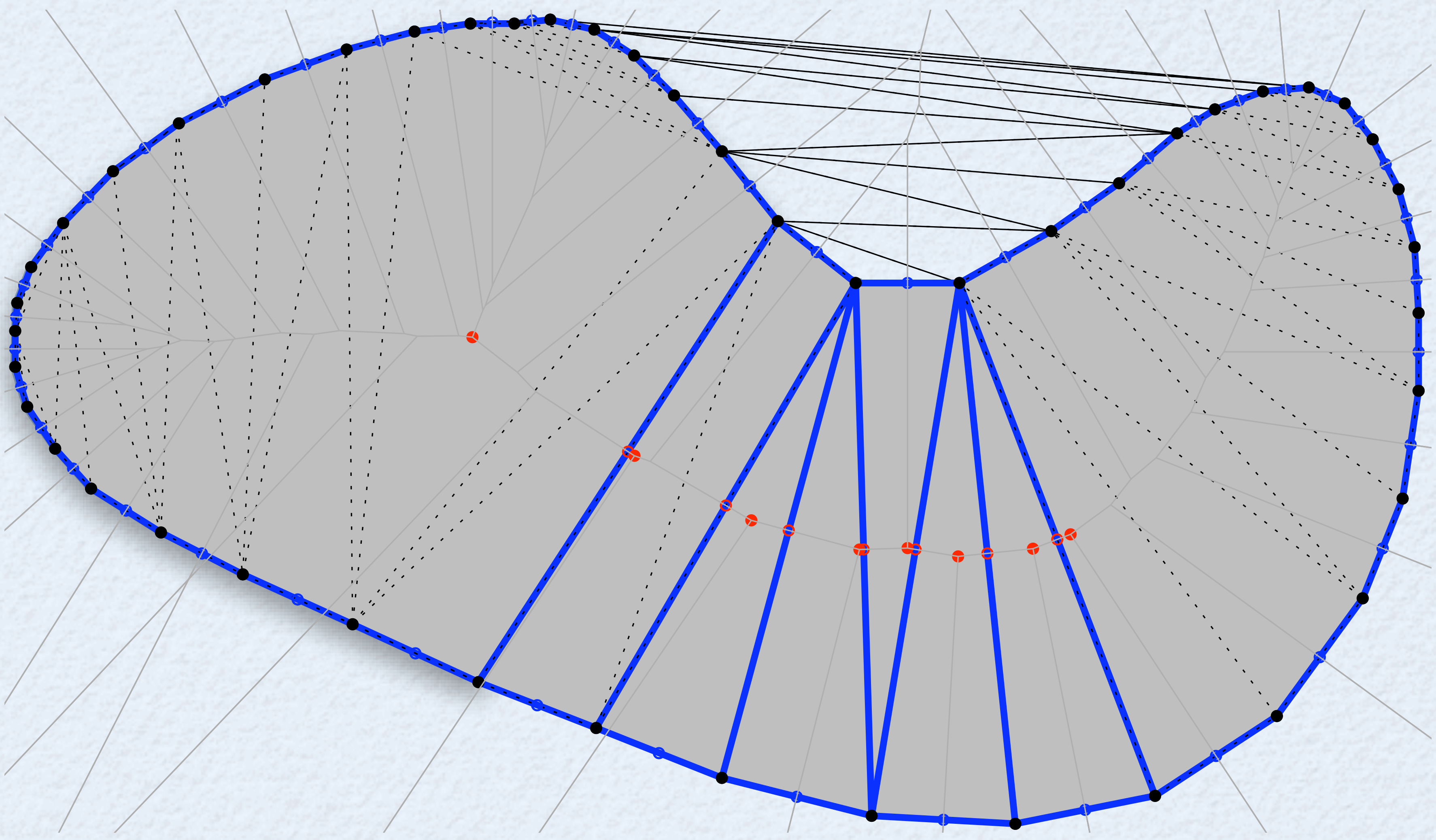

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Flow Shapes, Union of Balls, Alpha Shapes

For a point set $P \subset \mathbb{R}^n$ and $r \in \mathbb{R}$:

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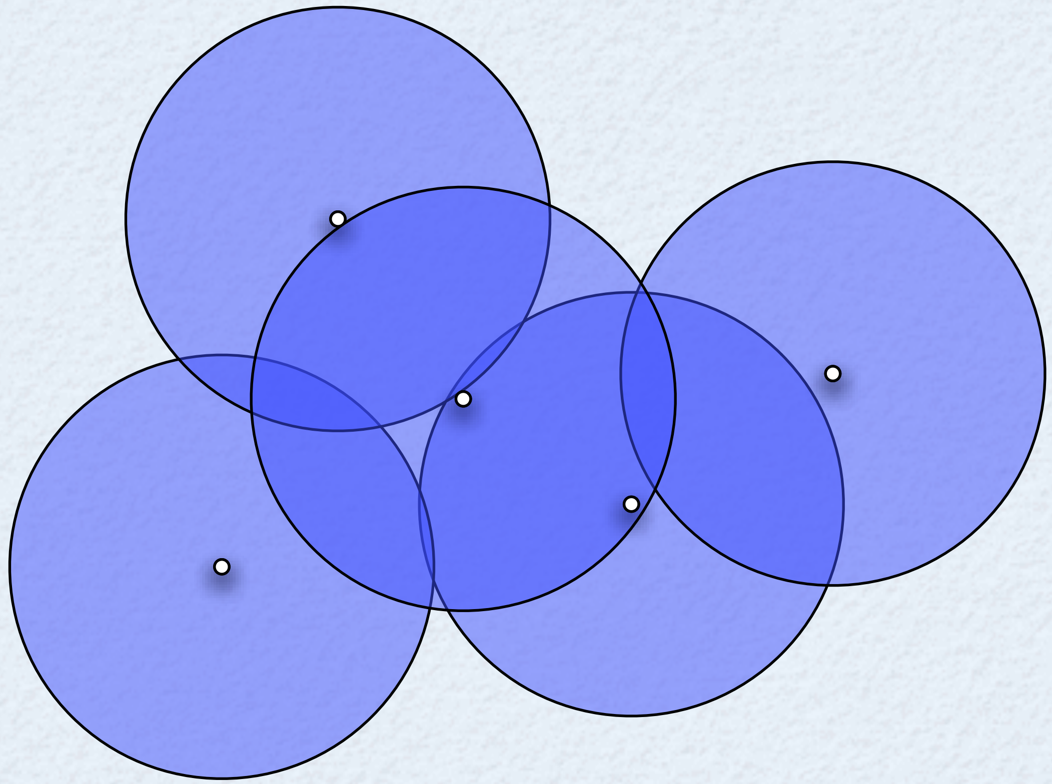

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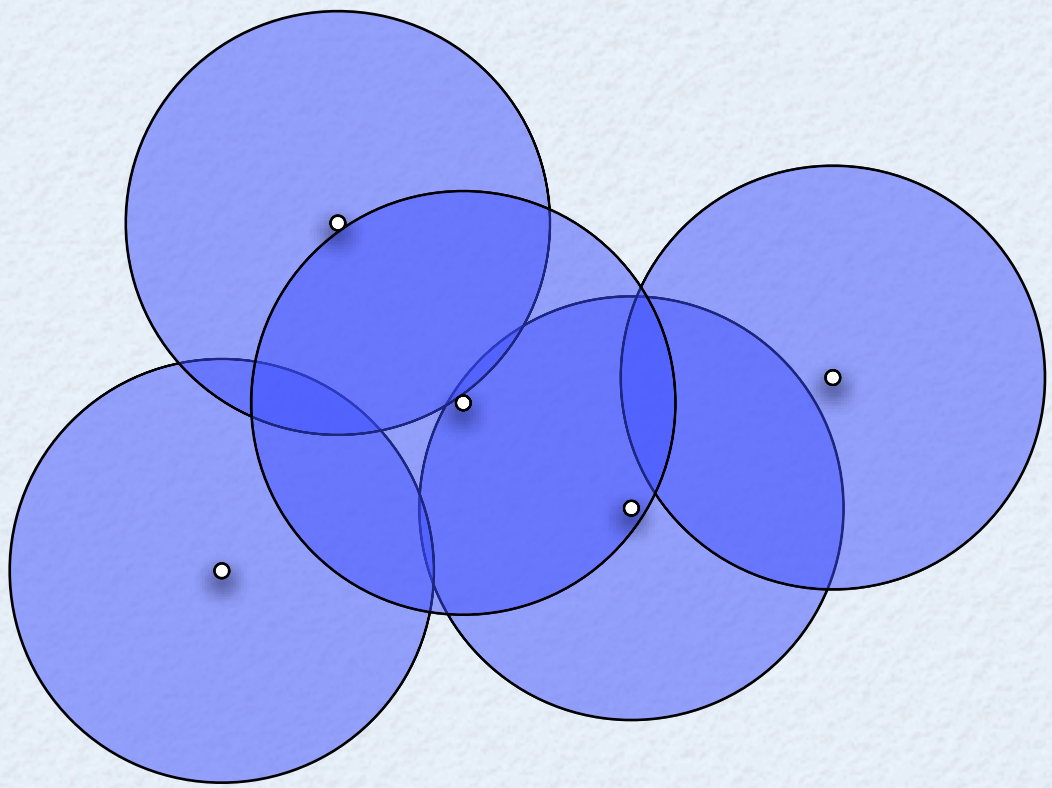

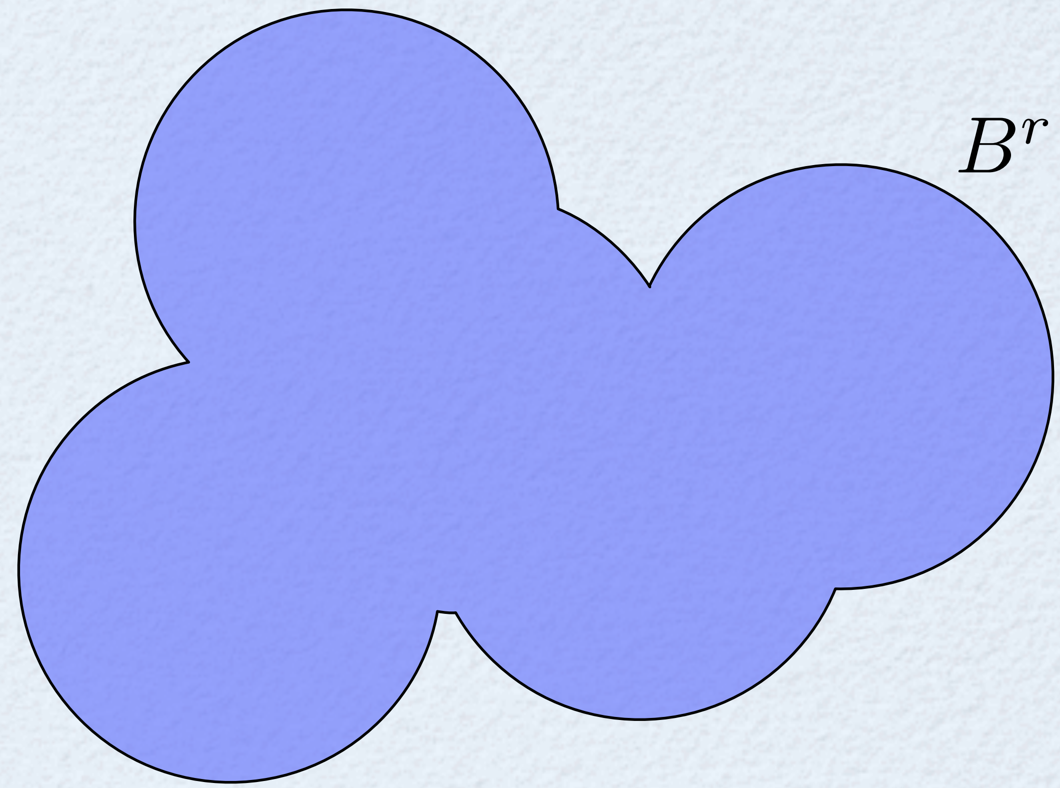

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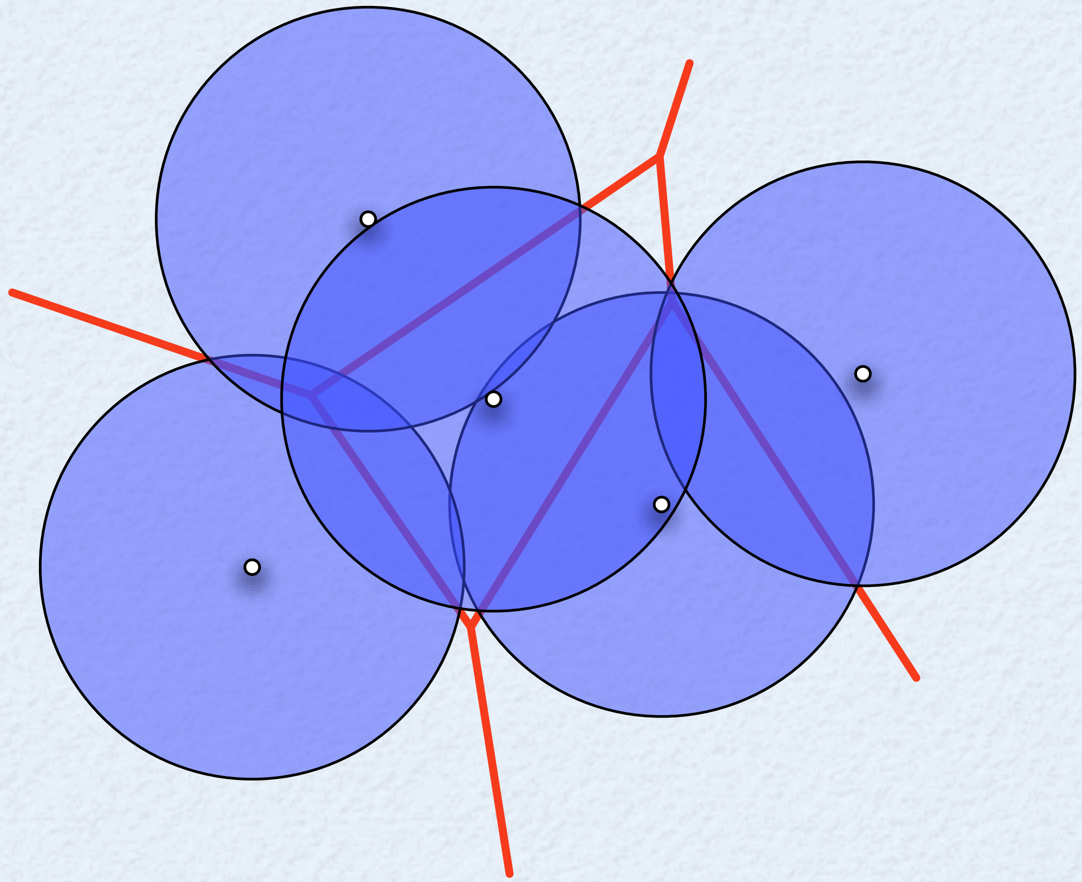

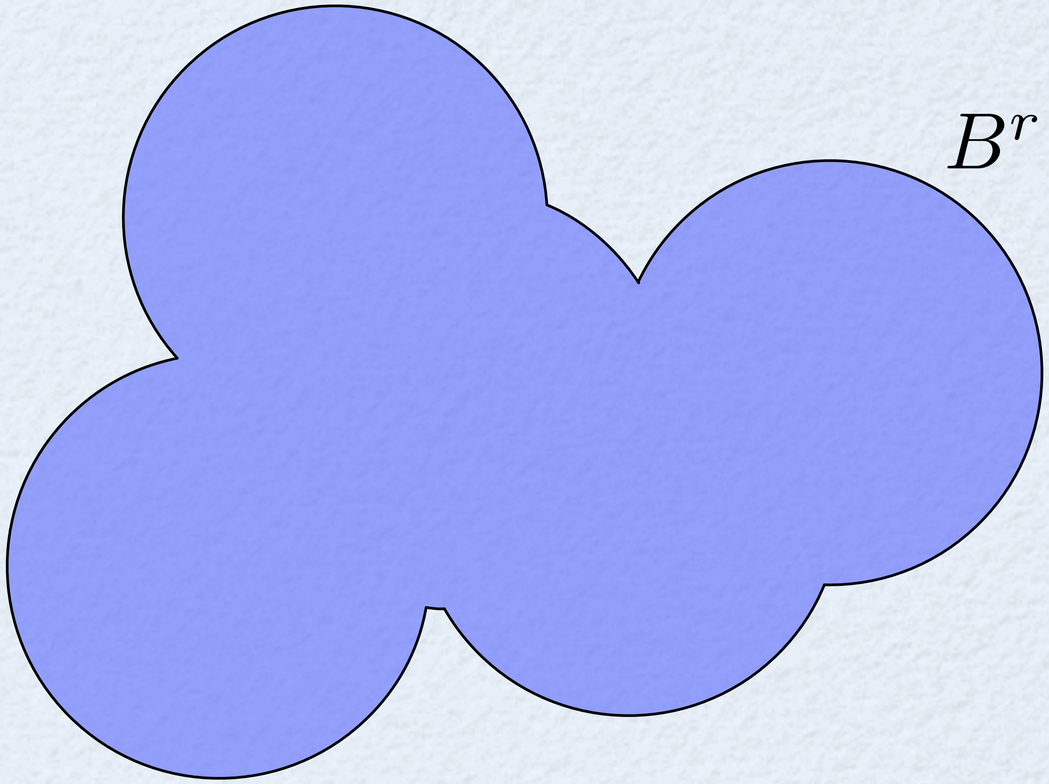

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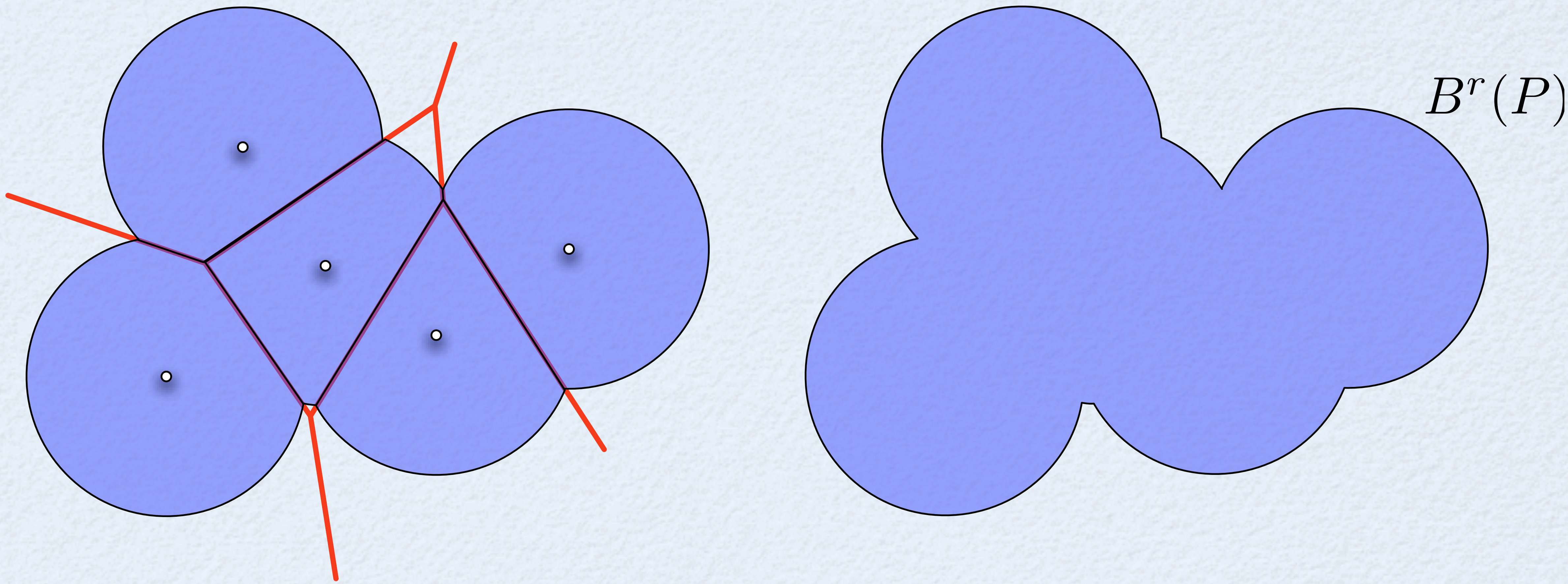

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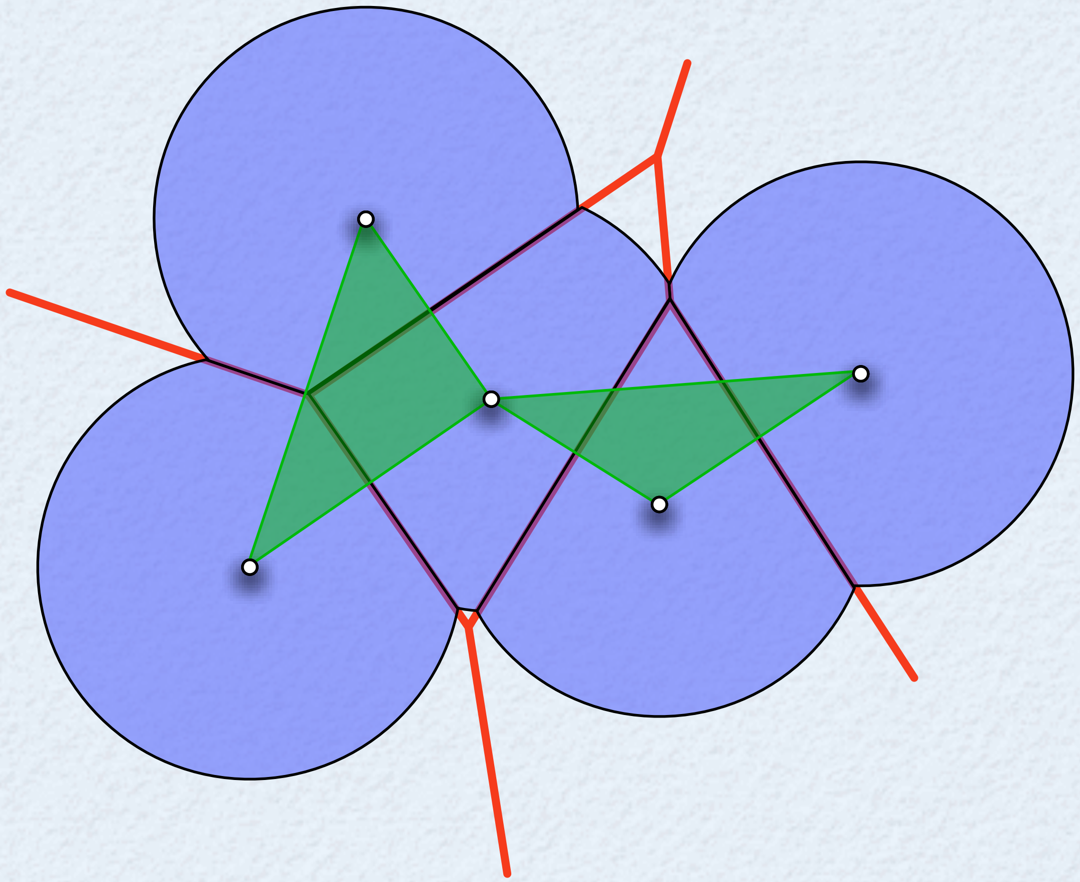

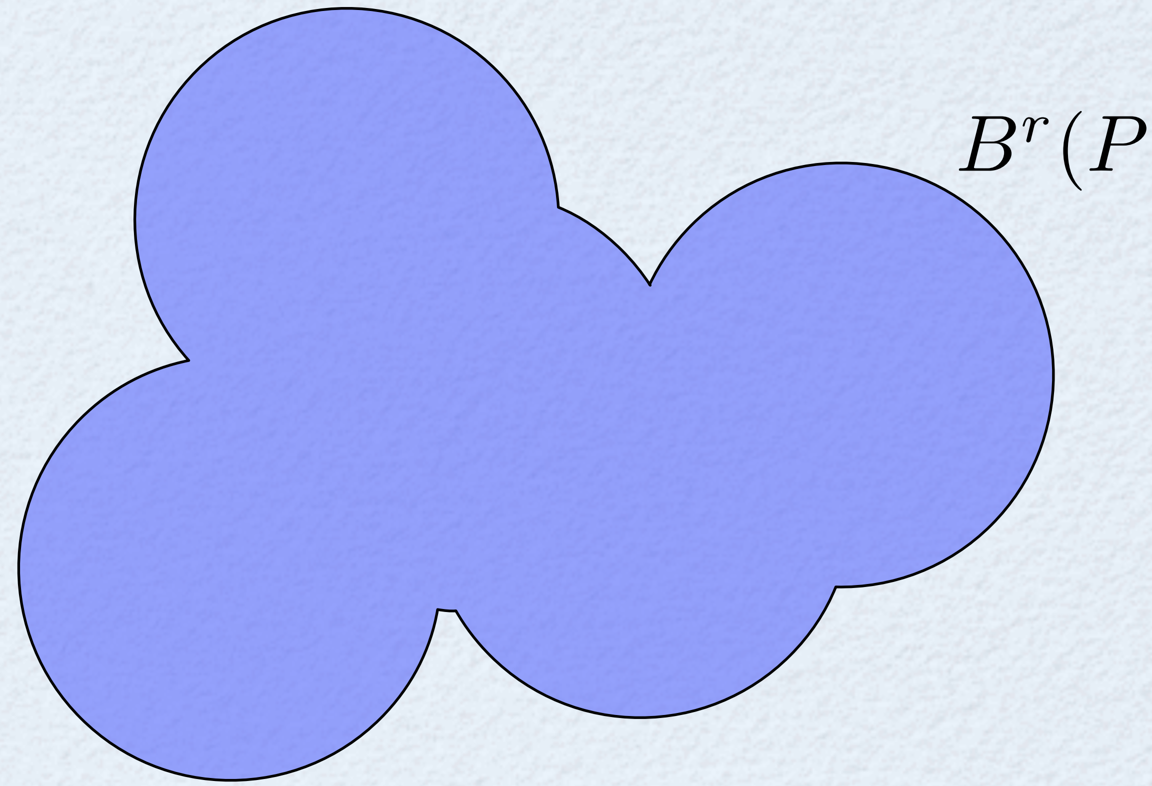

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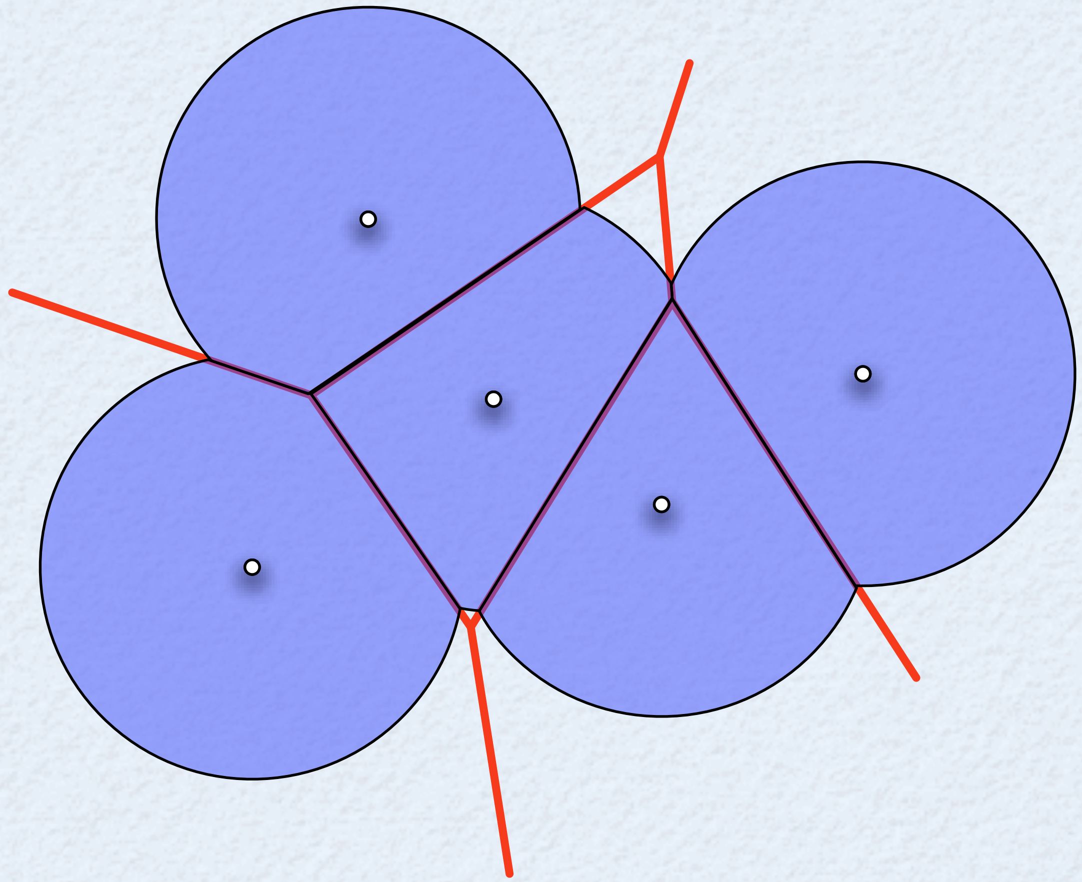
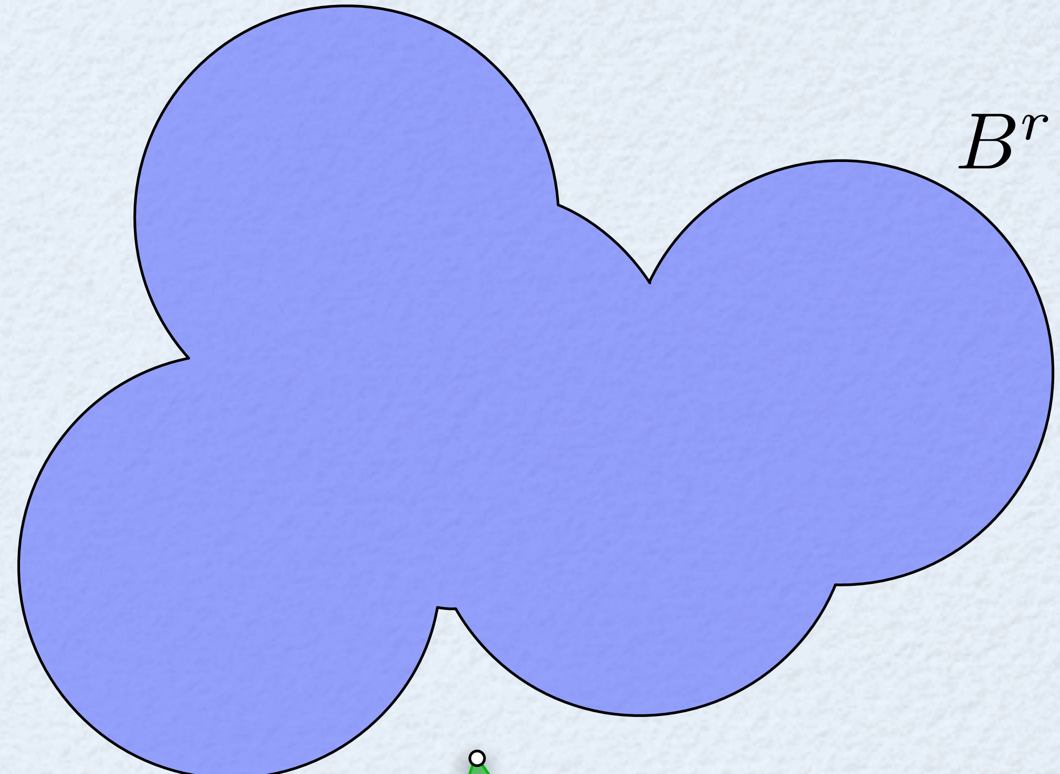
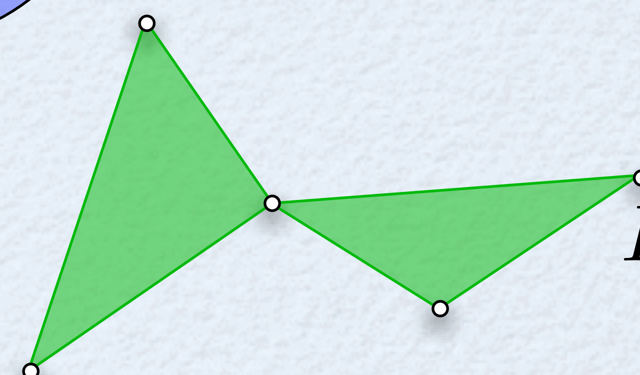




$B^r(P)$



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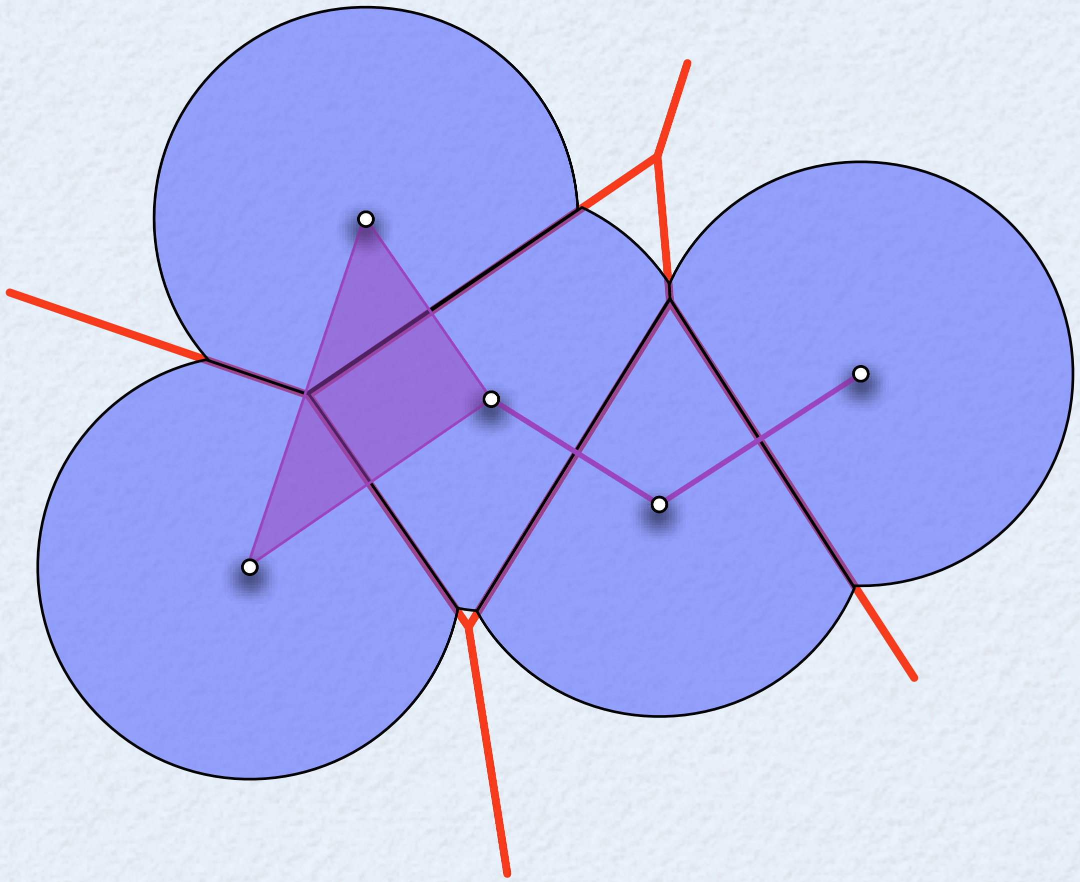

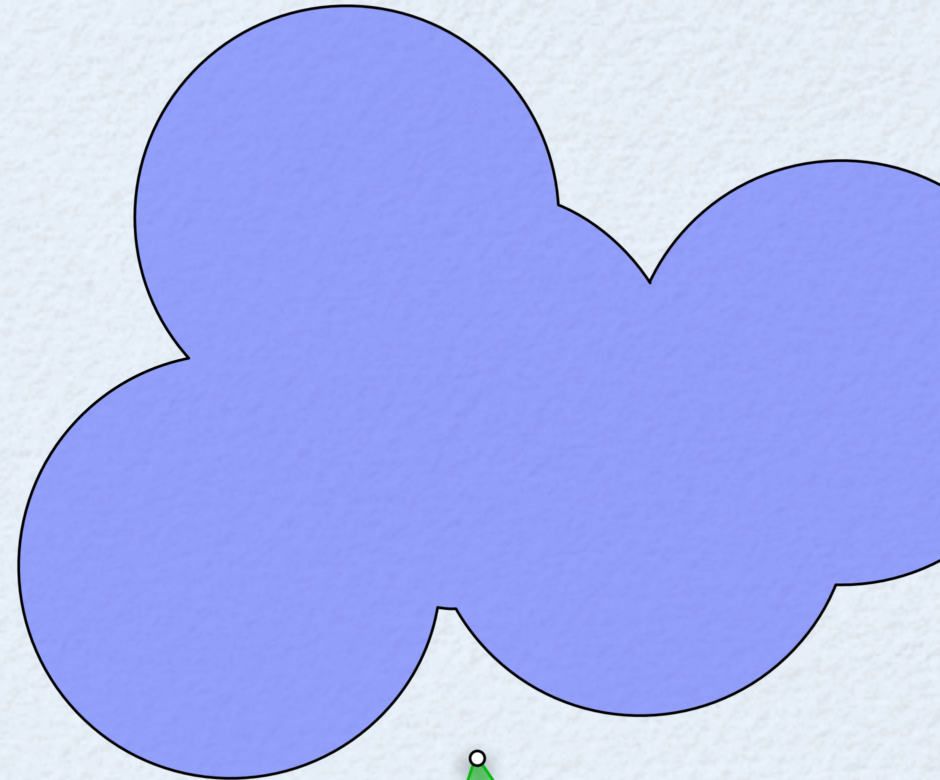

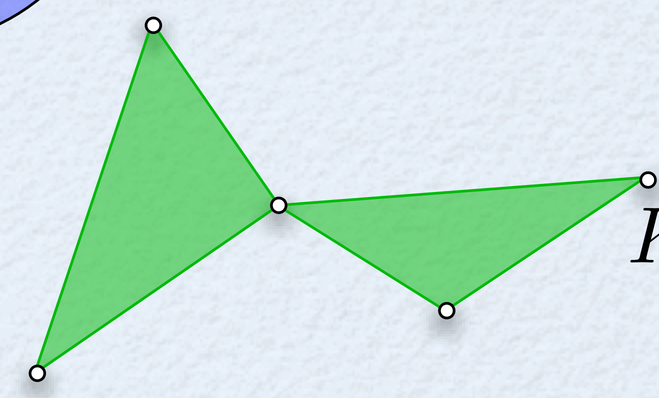

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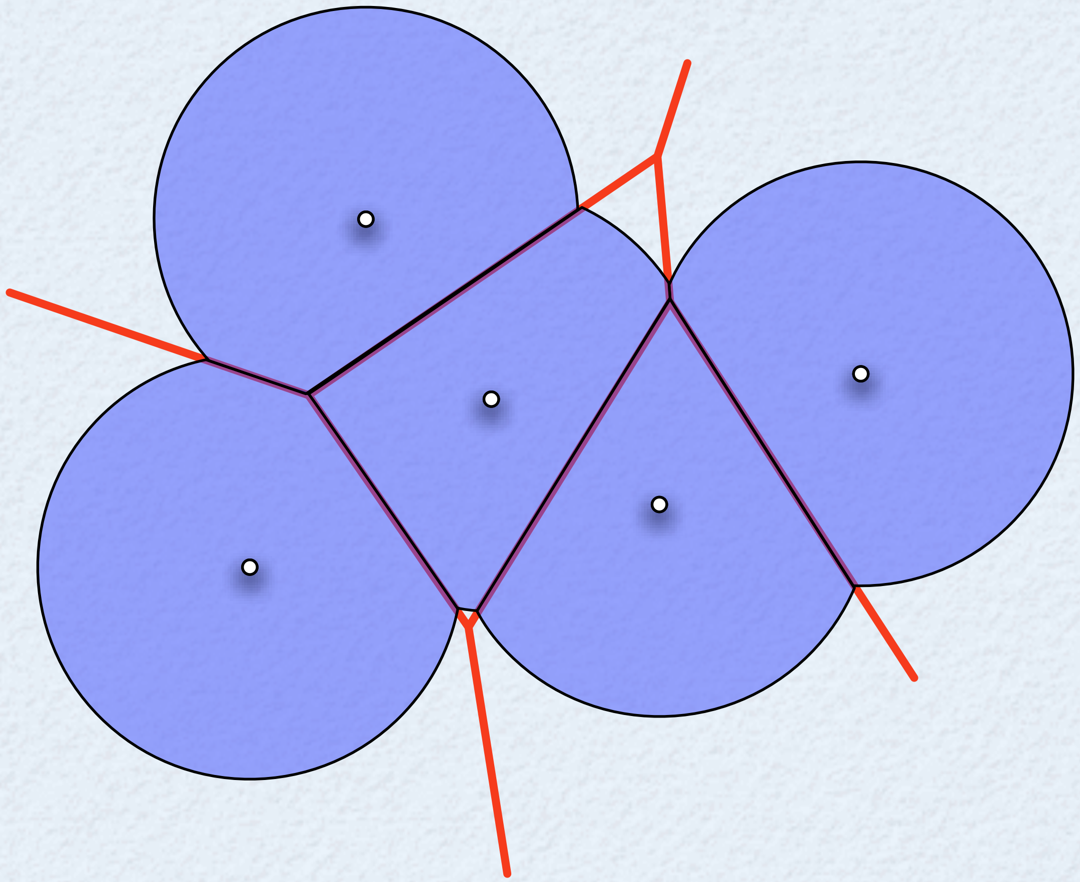

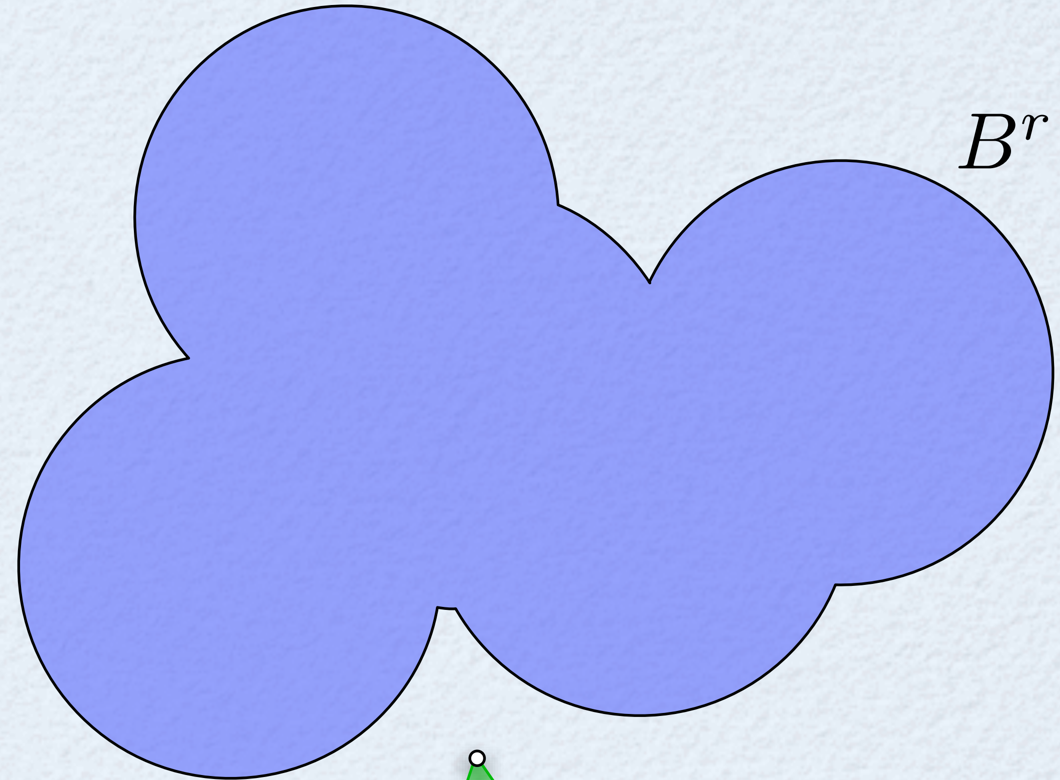

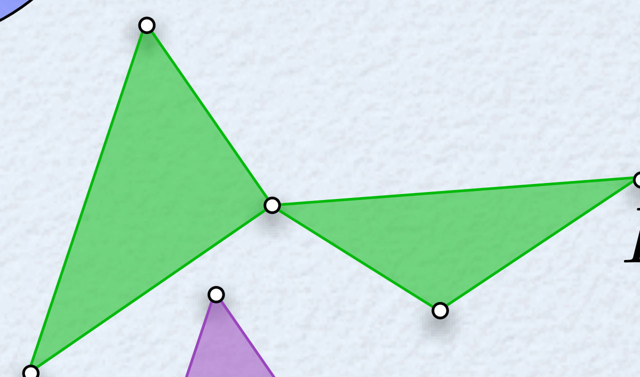

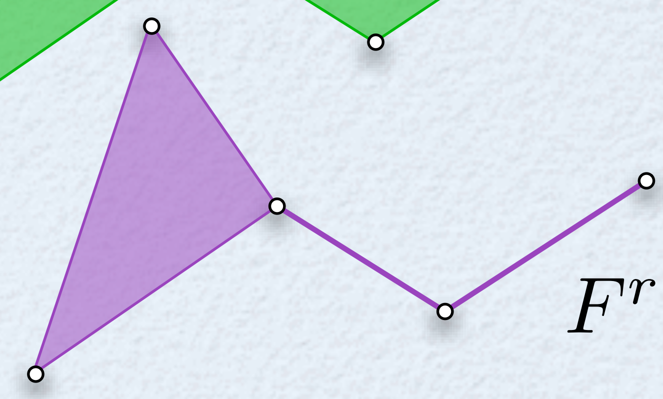

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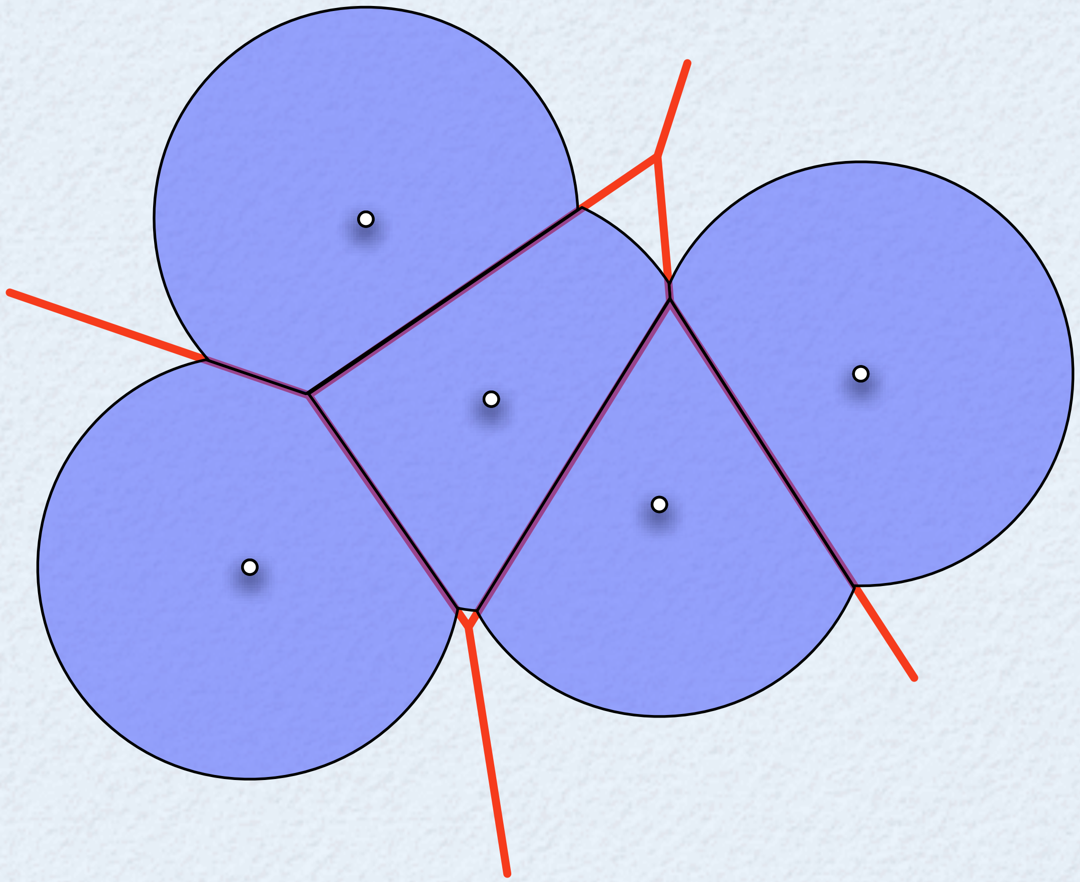

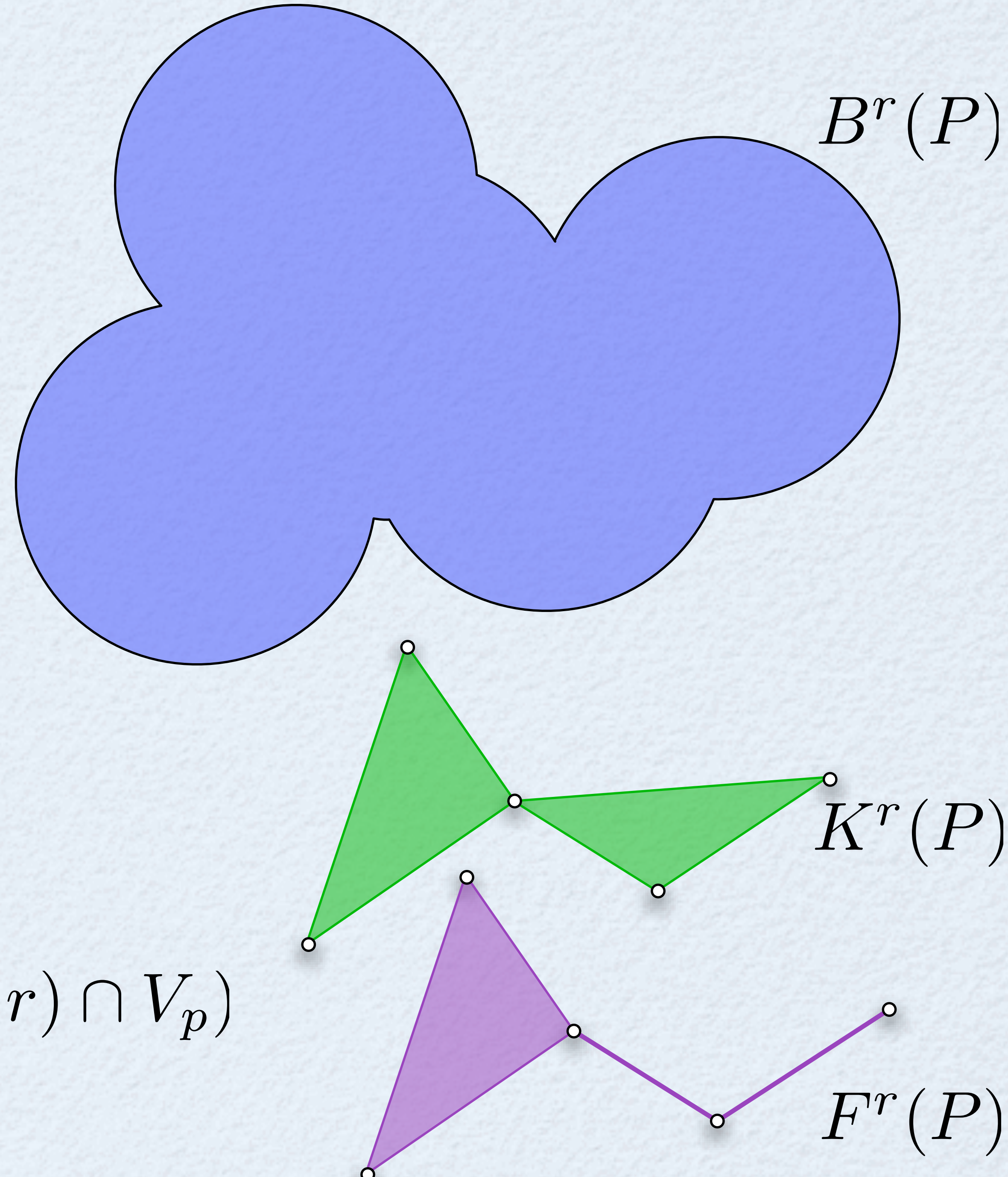

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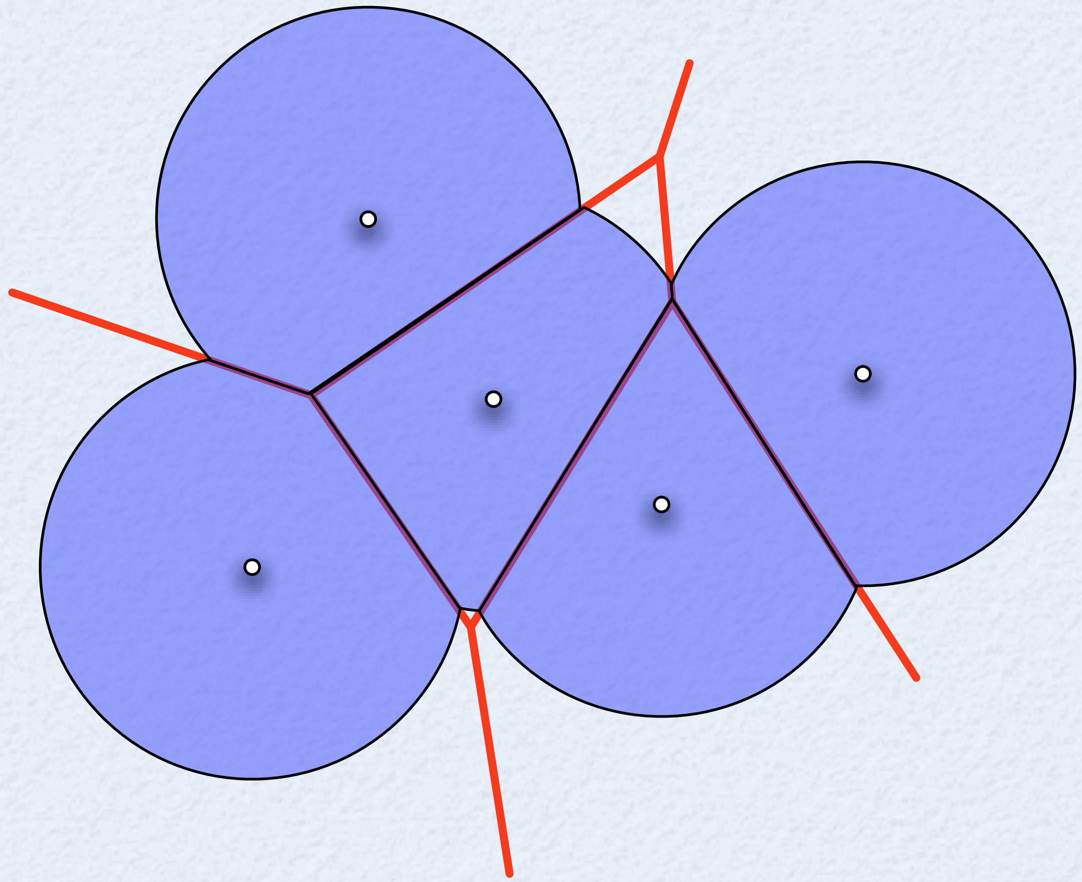
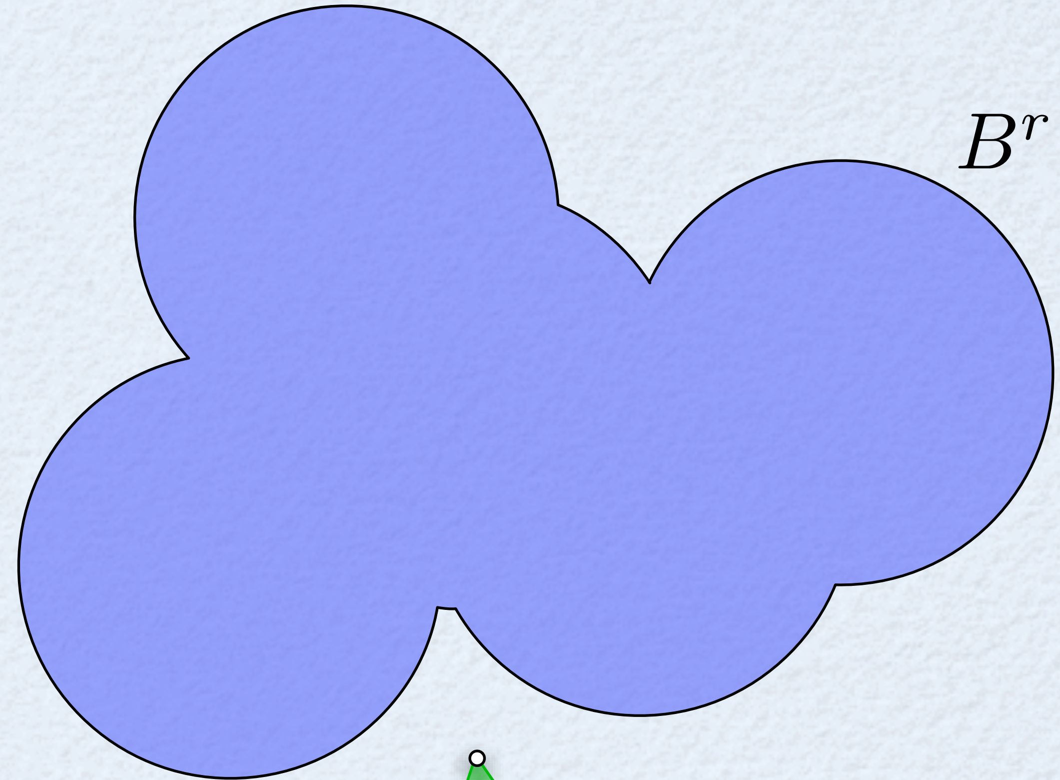
Theorem [Edel'95] $B^r(P)$ and $K^r(P)$ are homotopy equivalent.

Flow Shapes, Union of Balls, Alpha Shapes





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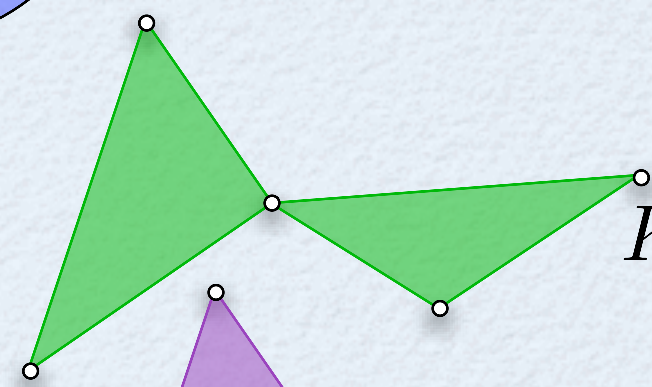

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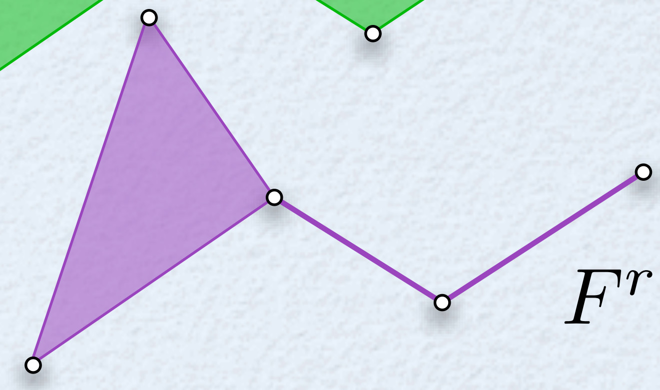

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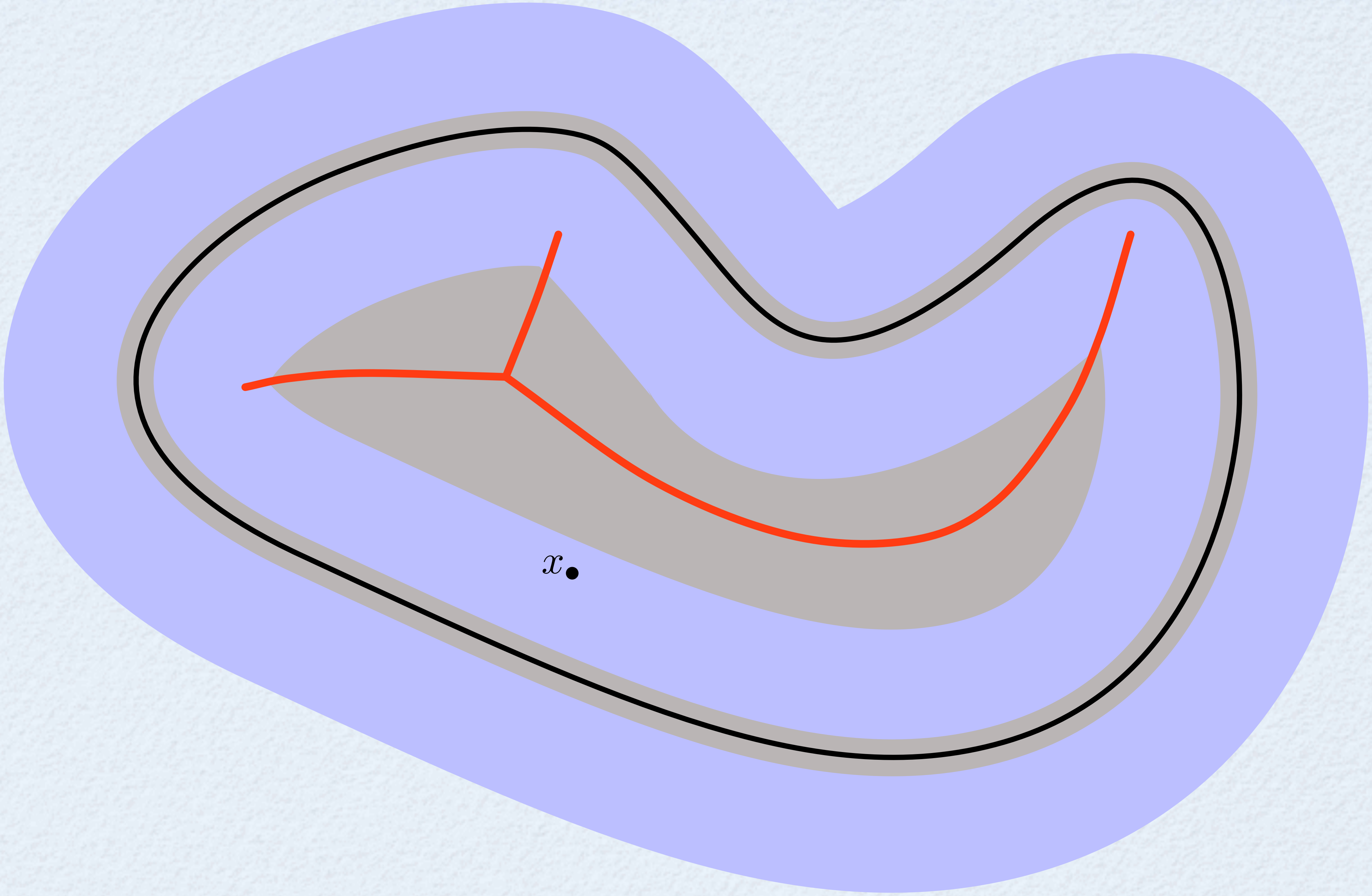
Theorem [DGJ'03, BG'05] $F^r(P)$ and $K^r(P)$ are homotopy equivalent.

Separation of Critical Points



Theorem [DGRS'05] If P is a uniform ε-sample of Σ with $\varepsilon < 1/\sqrt{3}$, then any critical point c of h is either shallow, i.e. $\text{dist}(c, \Sigma) \leq \varepsilon^2 \cdot \tau$ or is deep, i.e. $\text{dist}(c, \Sigma) \geq (1 - 2\varepsilon^2)\tau$, where $\tau = \text{reach}(\Sigma)$.

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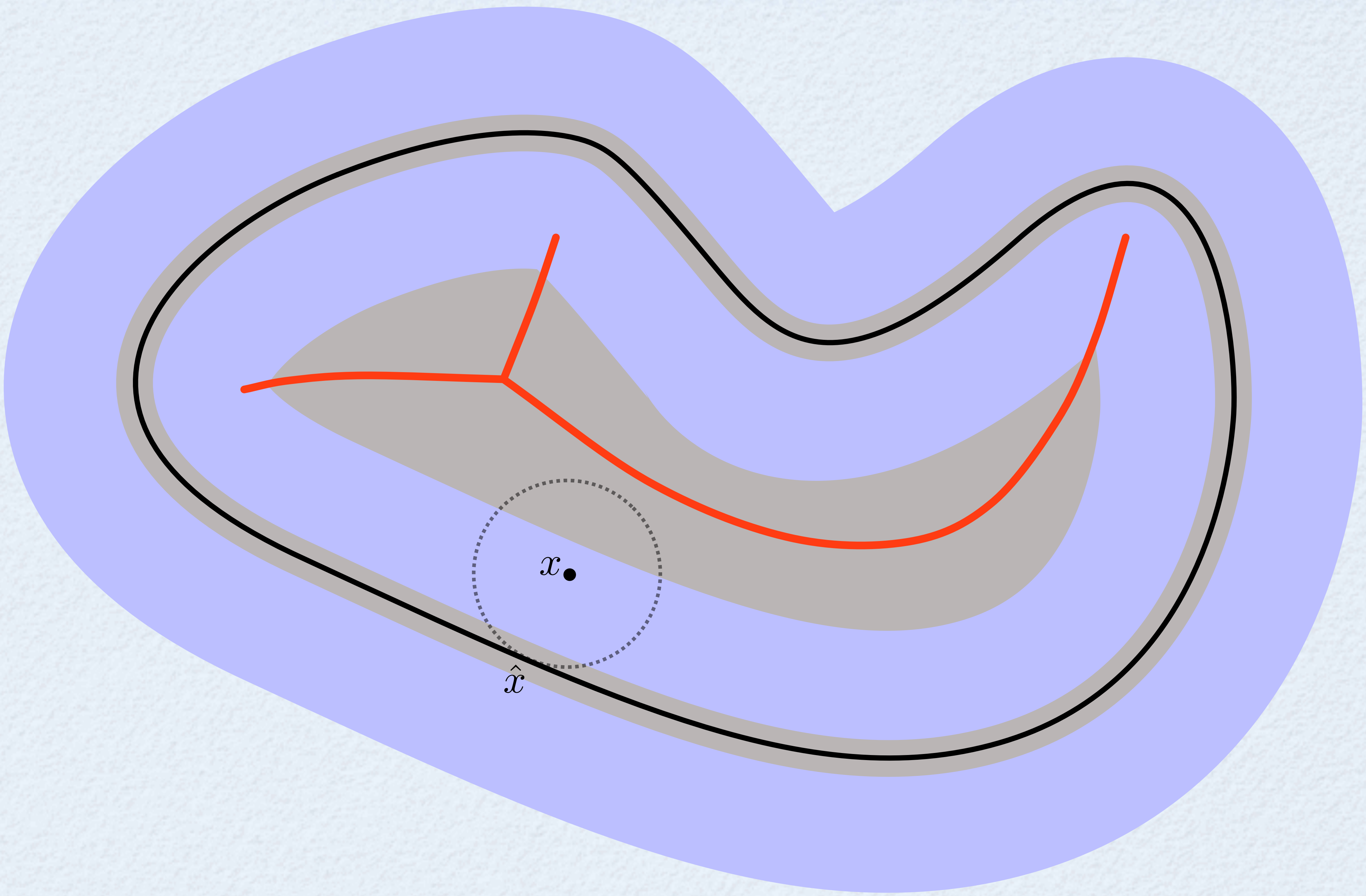

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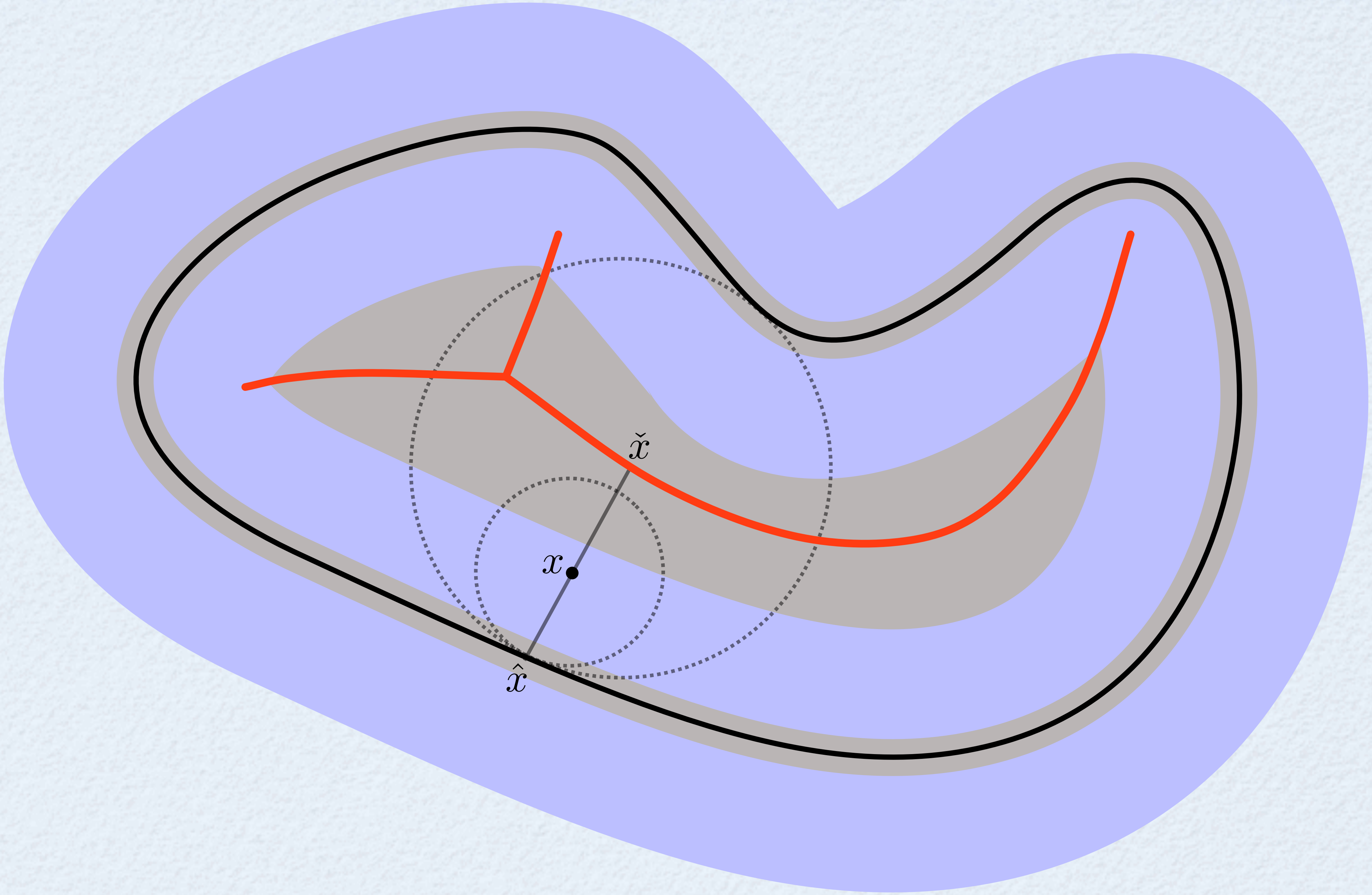

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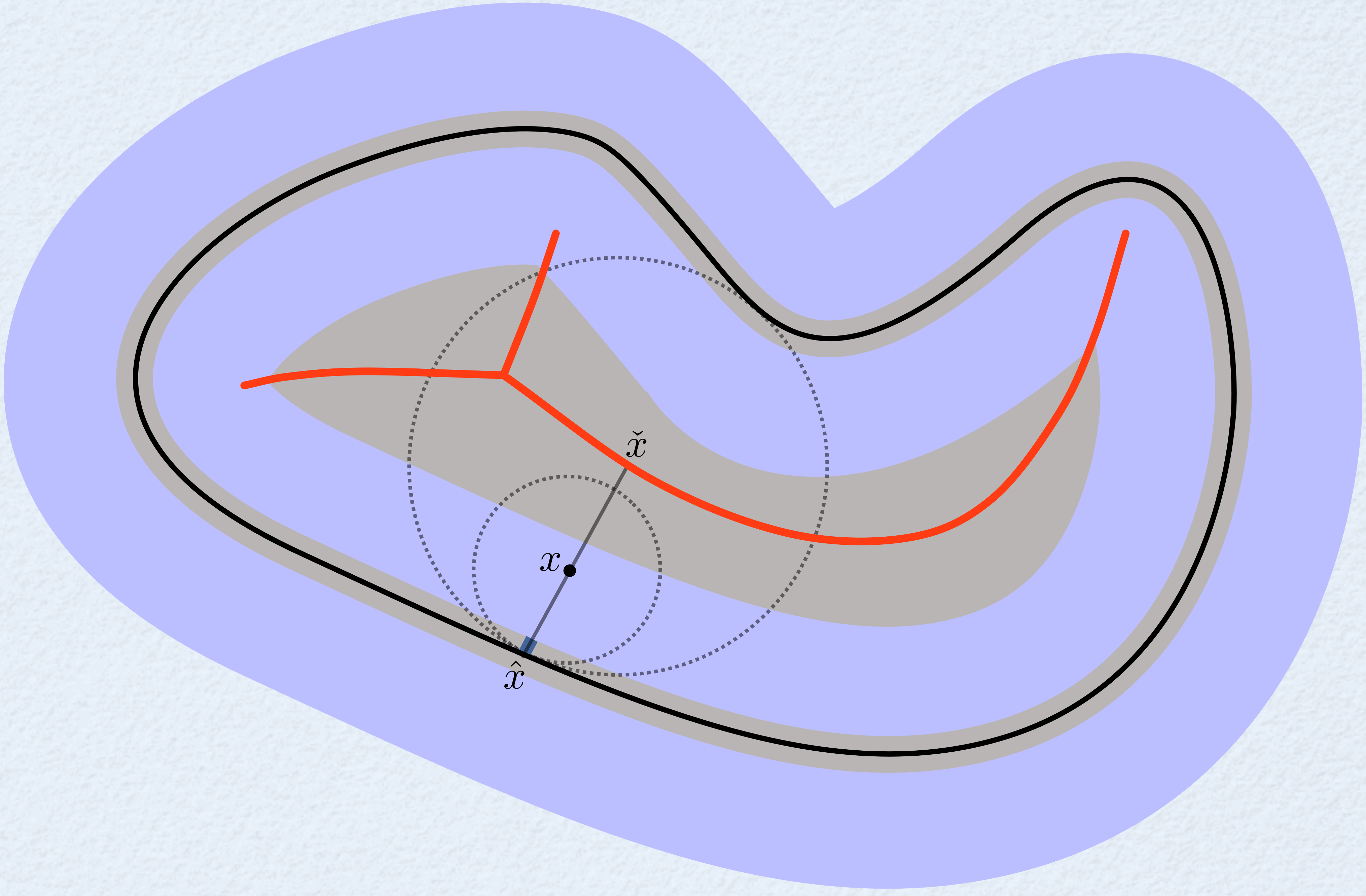

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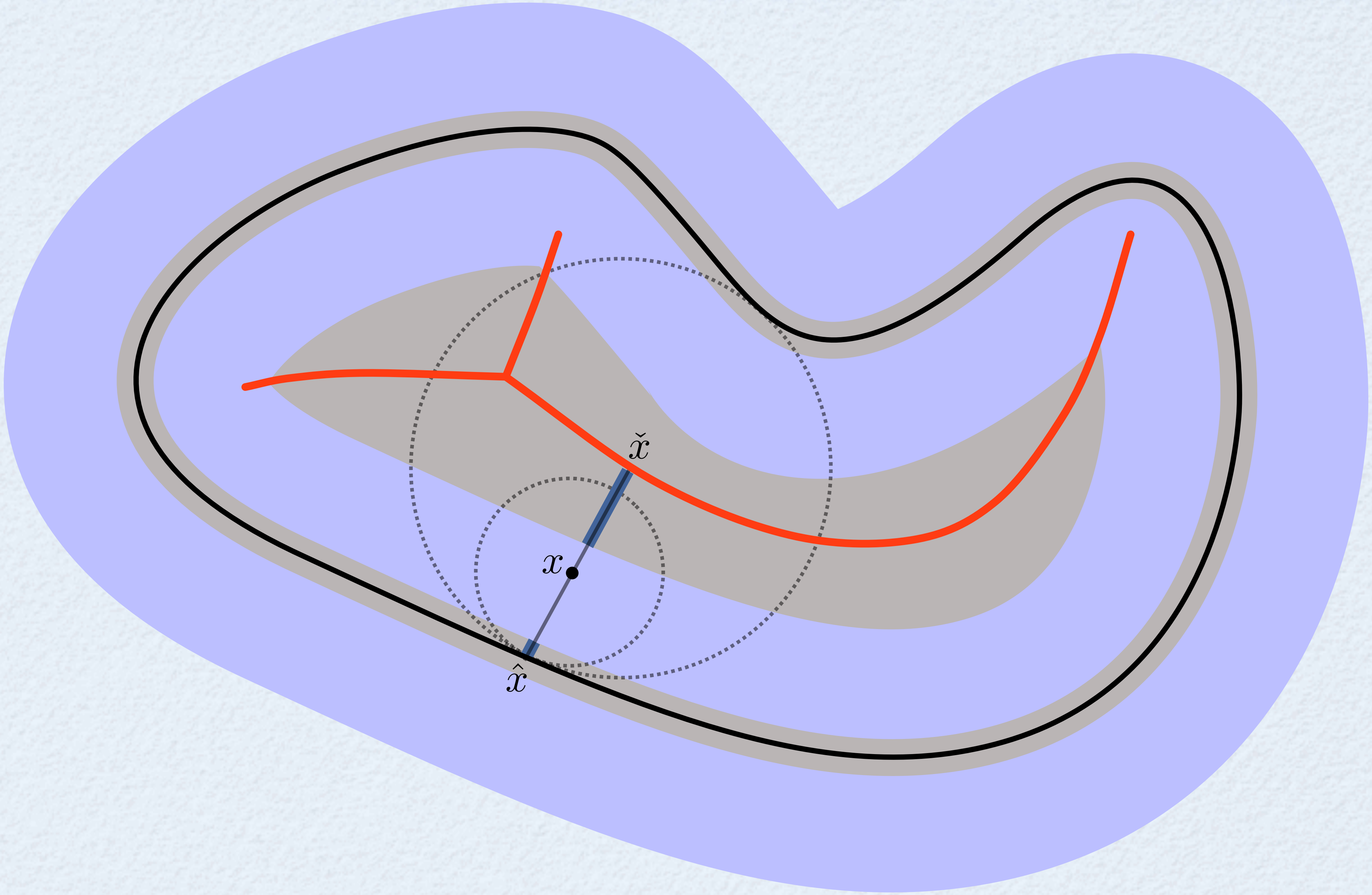

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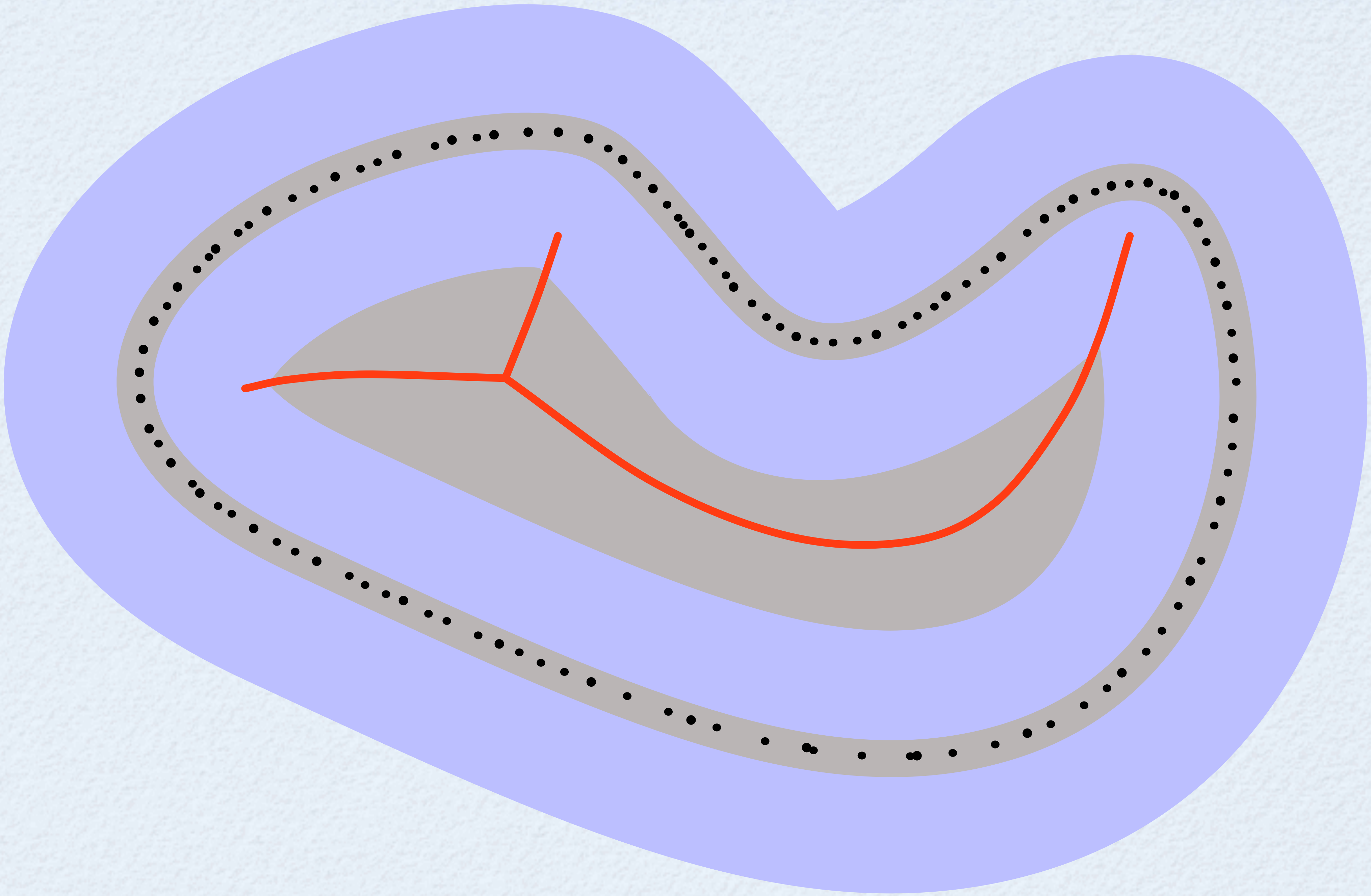
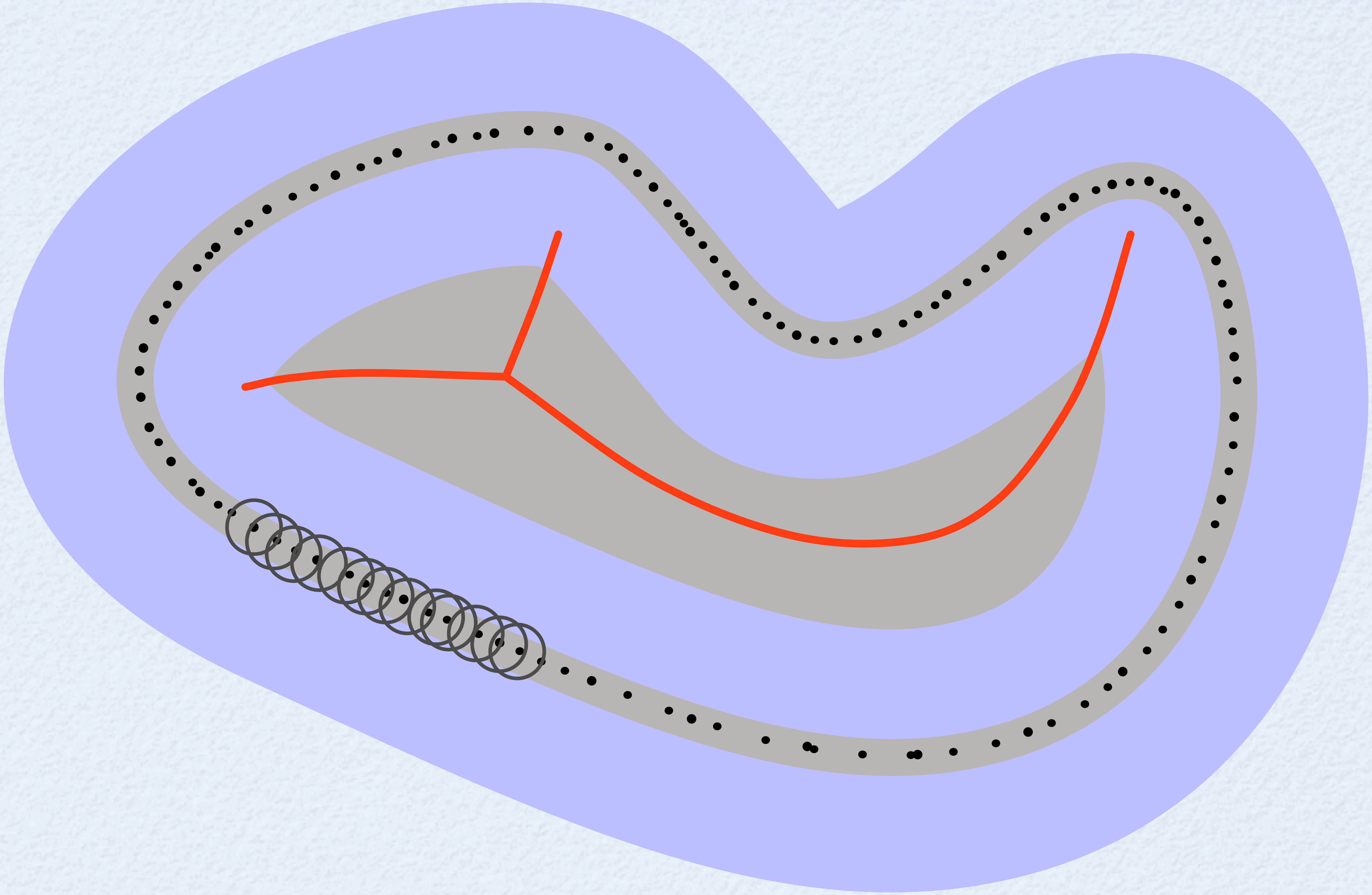
Separation of Critical Points (uniform)



Theorem. If P is a uniform ε-sample, then for a shallow c, $\mathrm{dist}(c, P) < \sqrt{5/3}\varepsilon\tau$ and for a deep c, $\mathrm{dist}(c, P) > (1 - 2\varepsilon^2)\tau$.

Separation of Critical Points (uniform)



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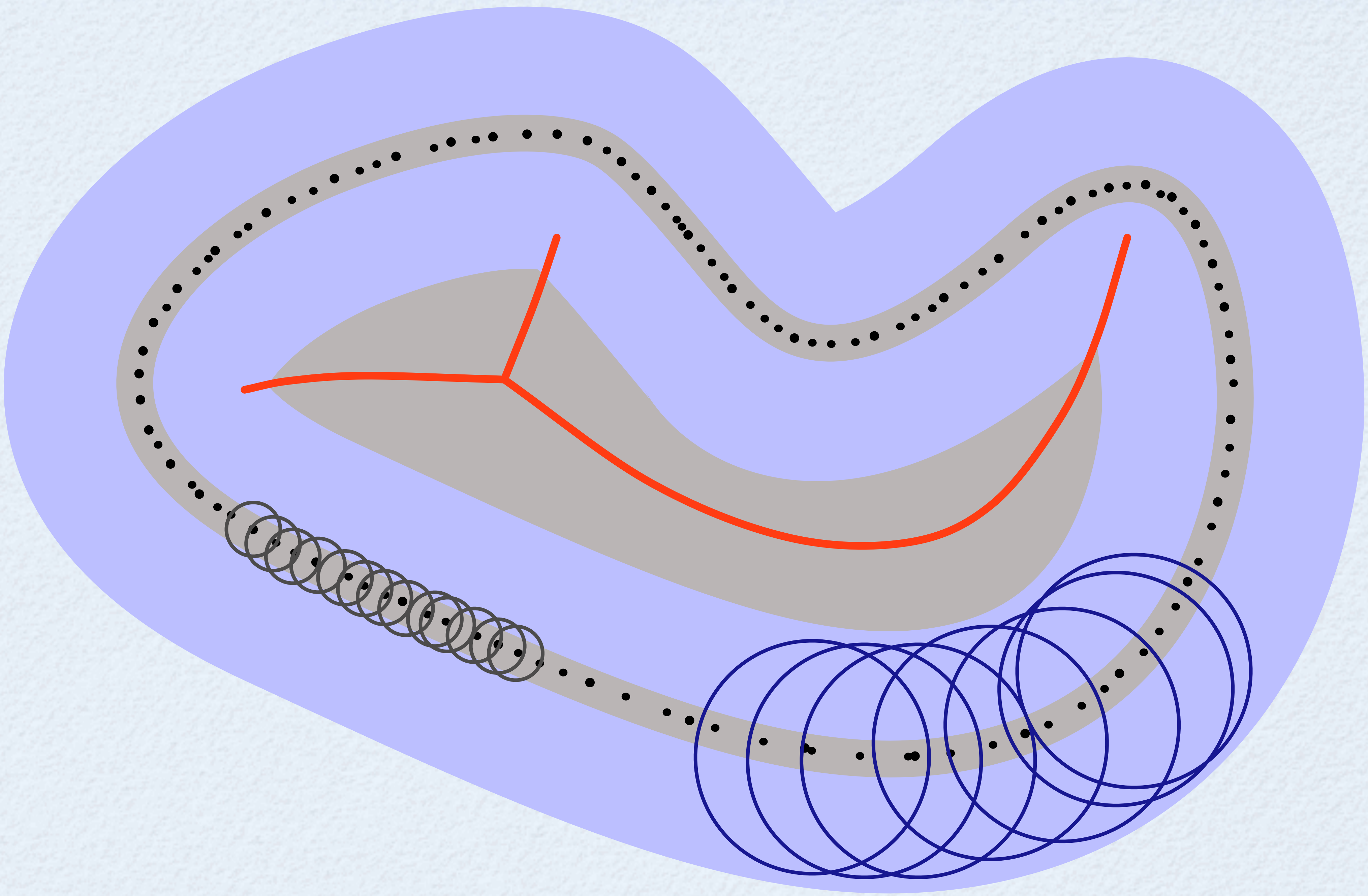

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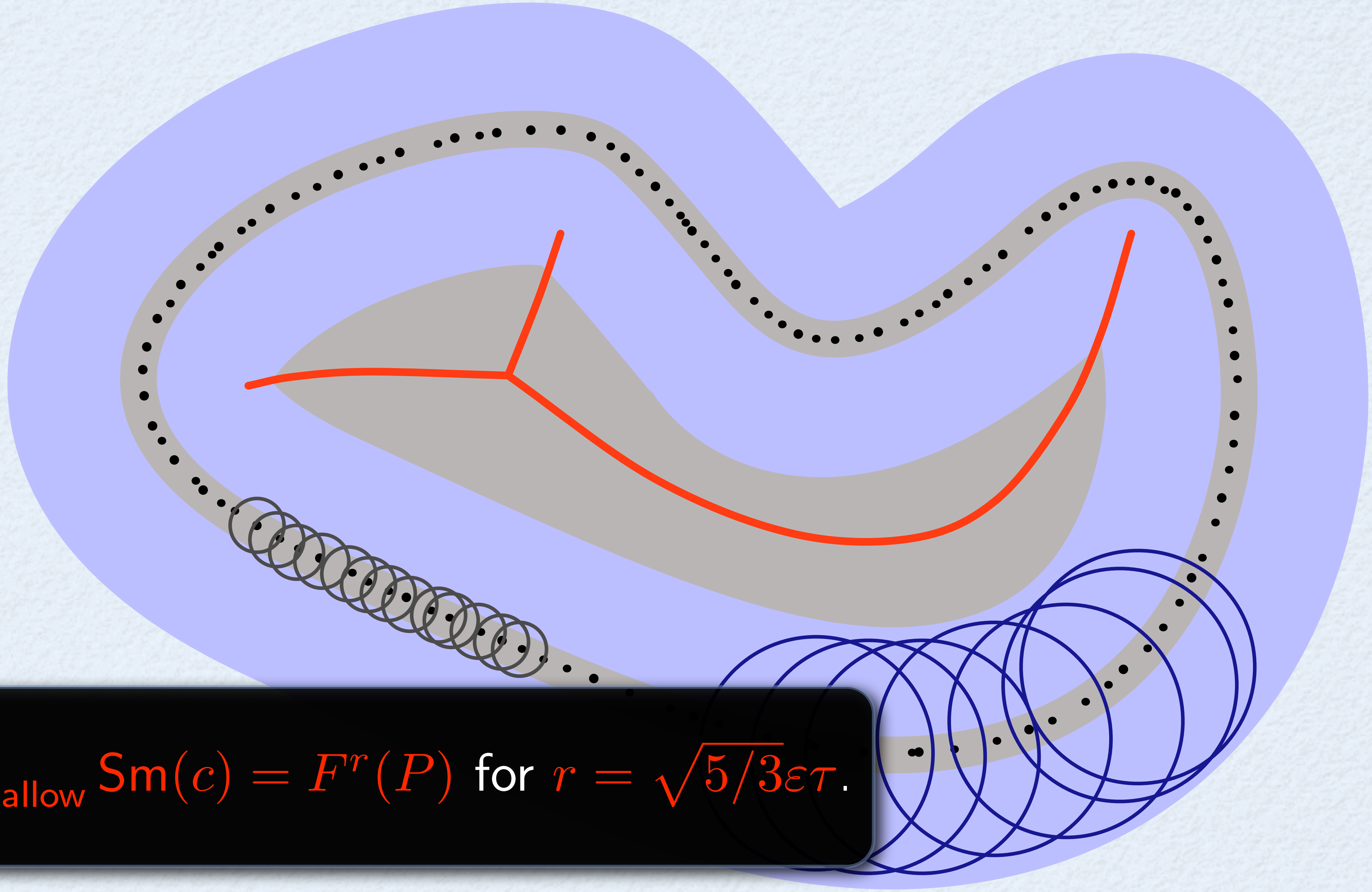

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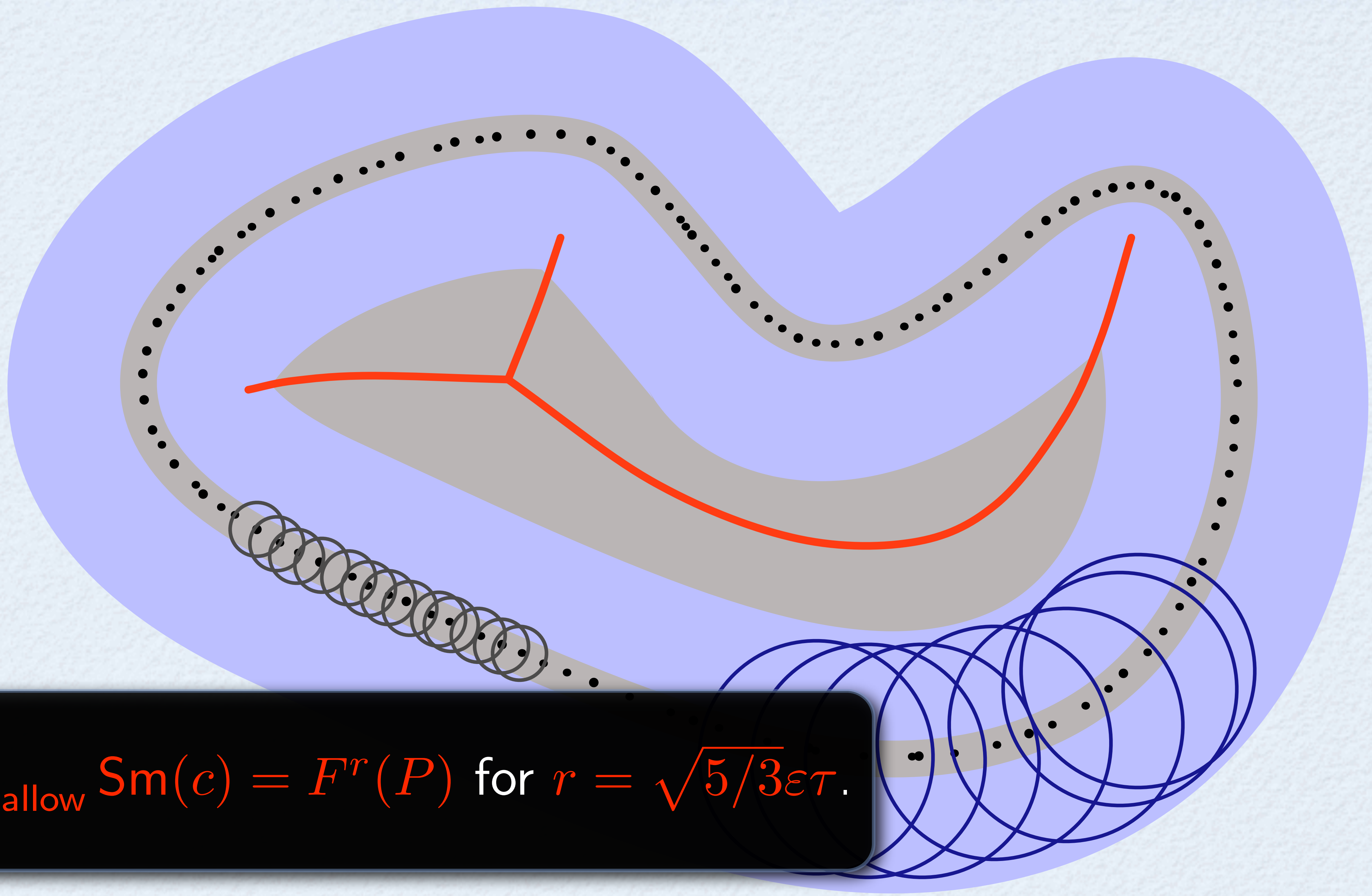

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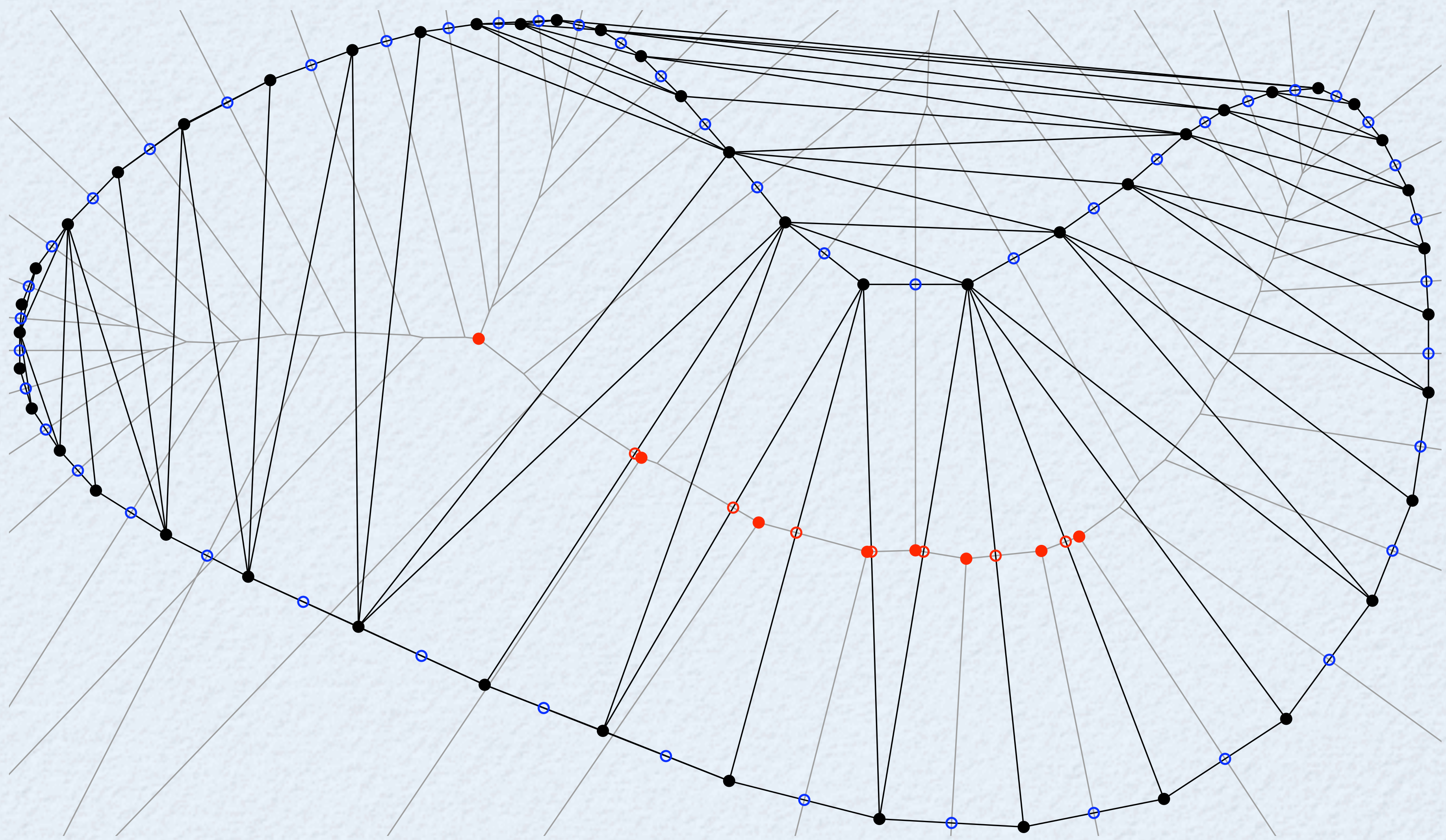
Theorem. [NSW'06] If P is a uniform ε-sample of Σ then $B^{(r)}(P)$ is homotopy equivalent to Σ (when r and ε are in the right range).

Unstable Manifold of a Critical Point

Unstable manifold of c is intersection of flows all neighborhoods of c.

$$\mathsf{Um}(c) = \bigcap_{\varepsilon>0} \phi(B(x,\varepsilon)) = \phi(V(c)).$$



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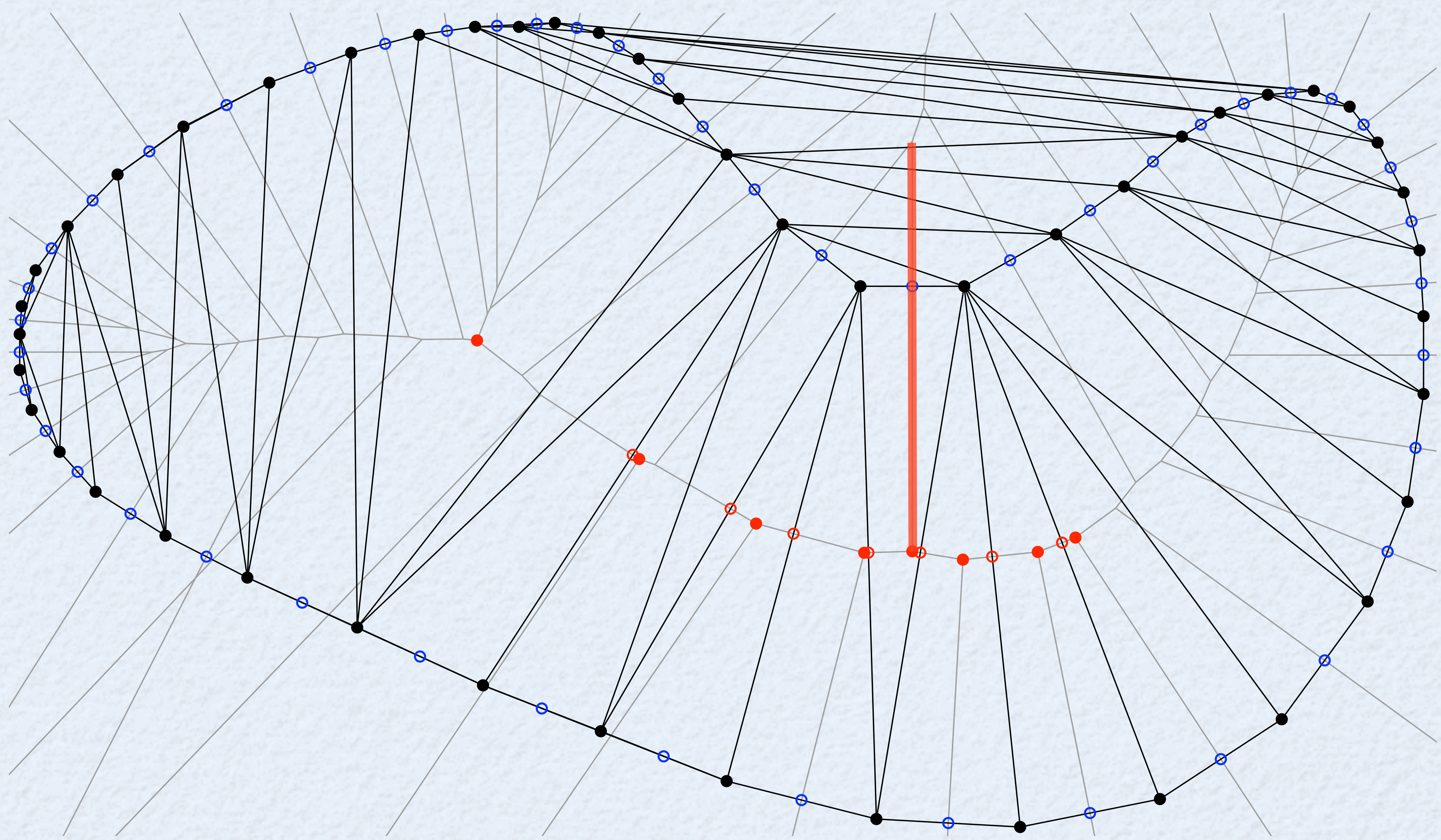

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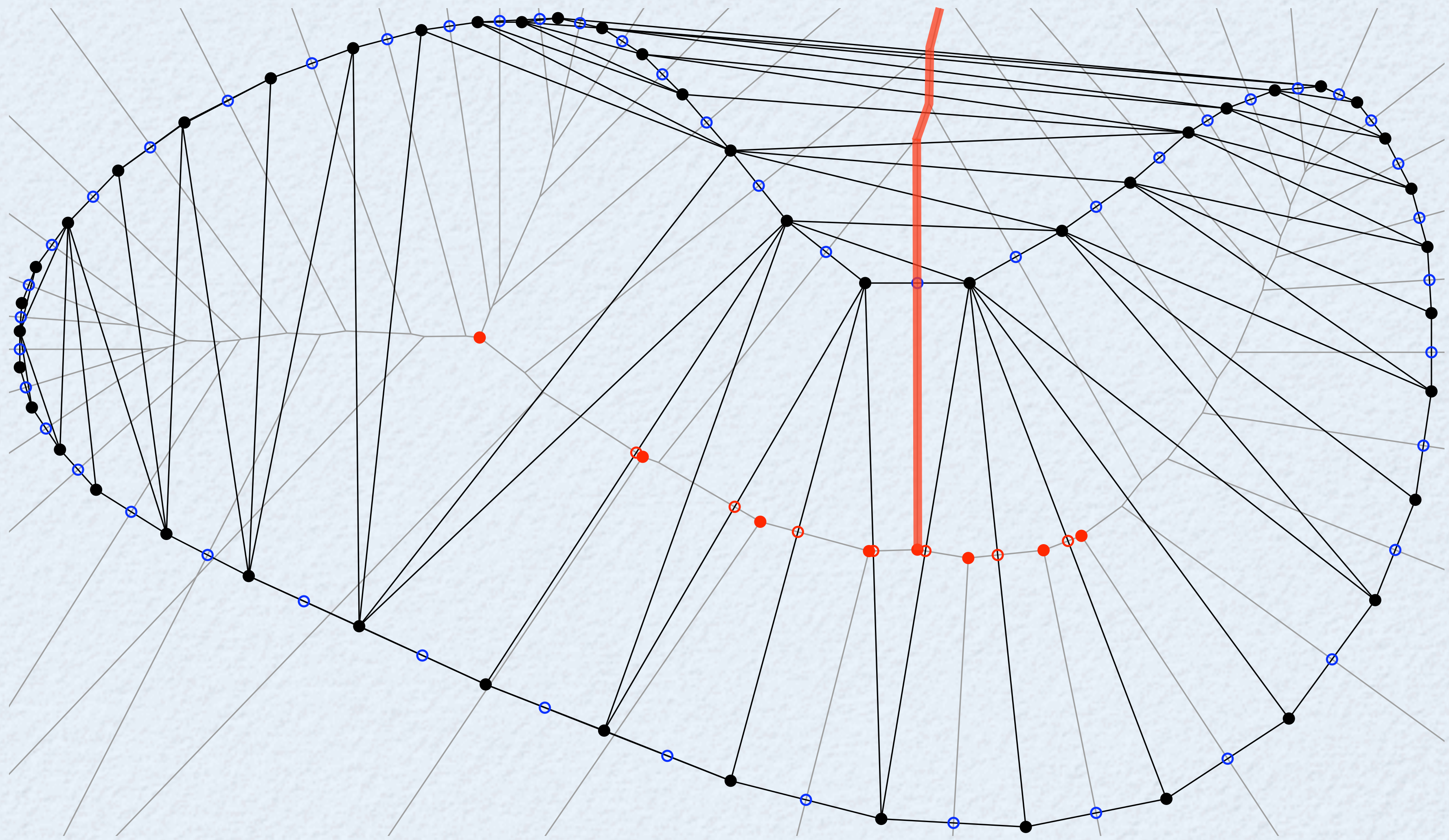

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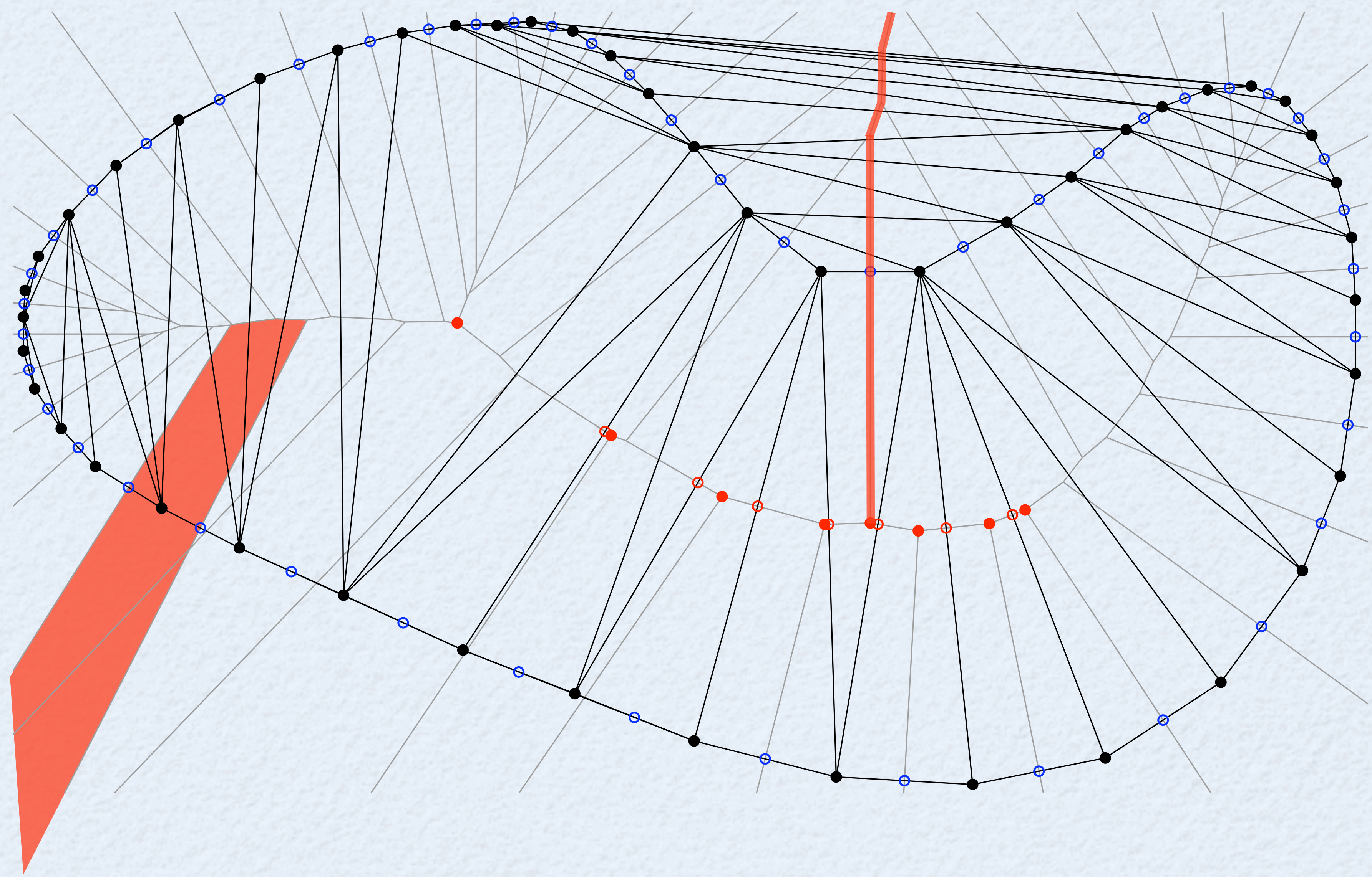

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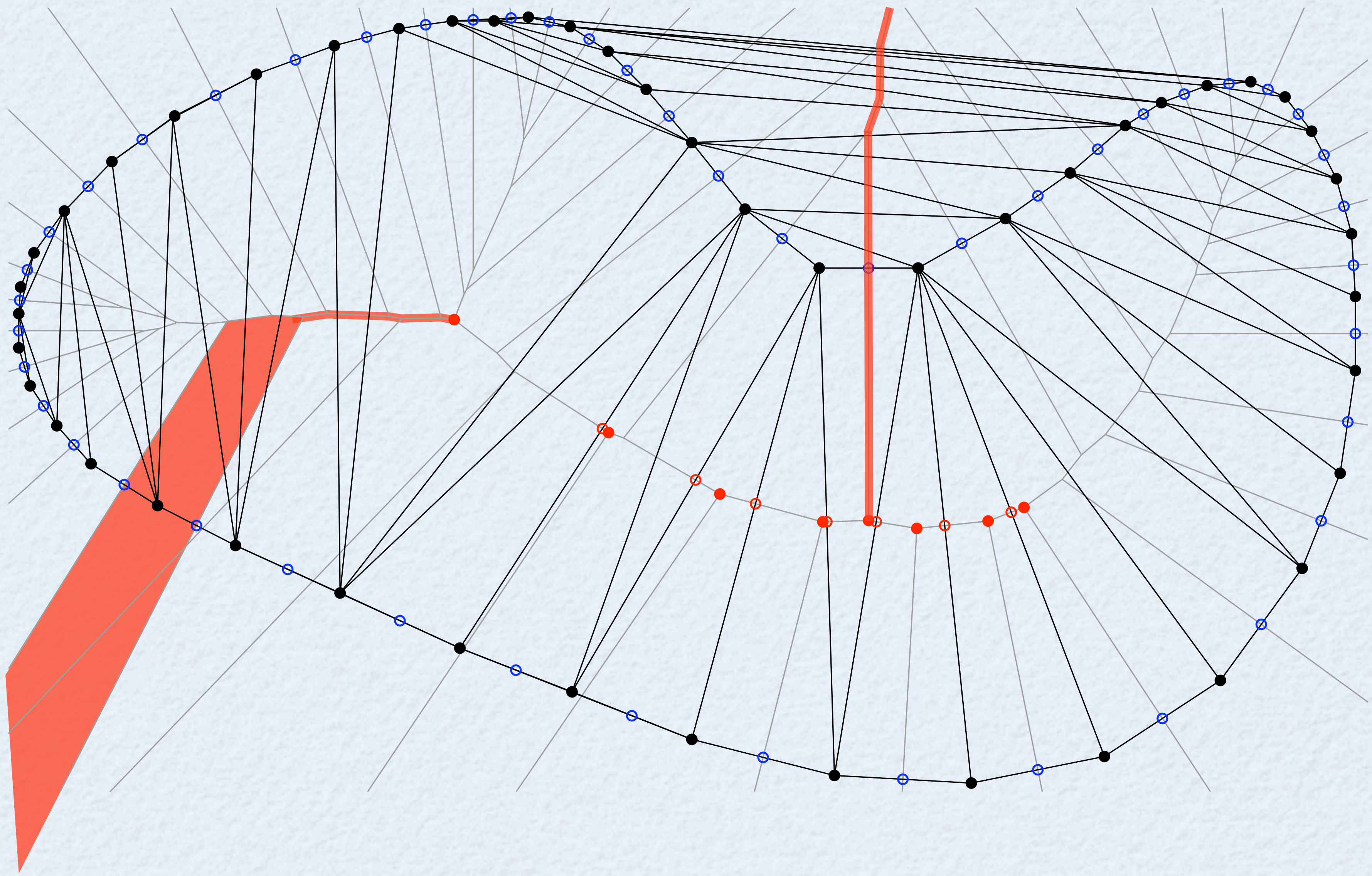

Using Flow to Prove Homotopy Equivalence

Lemma [Lieutier'04]. If $Y \subset X$ are bounded and

1. $\phi(X) = X$ and $\phi(Y) = Y$, and
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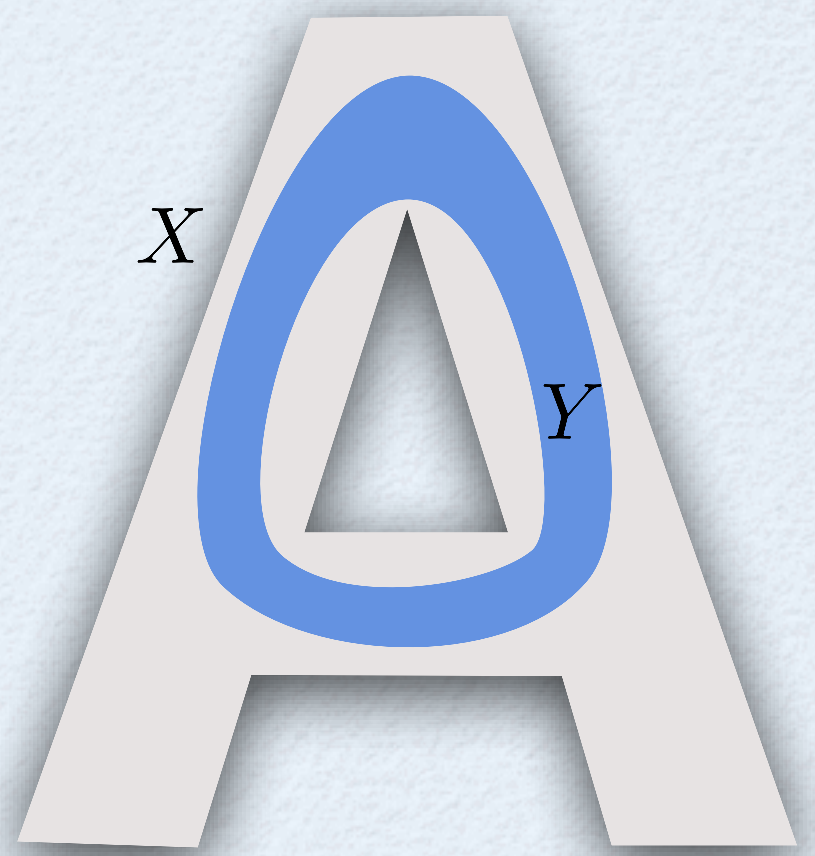
then X and Y are homotopy equivalent.



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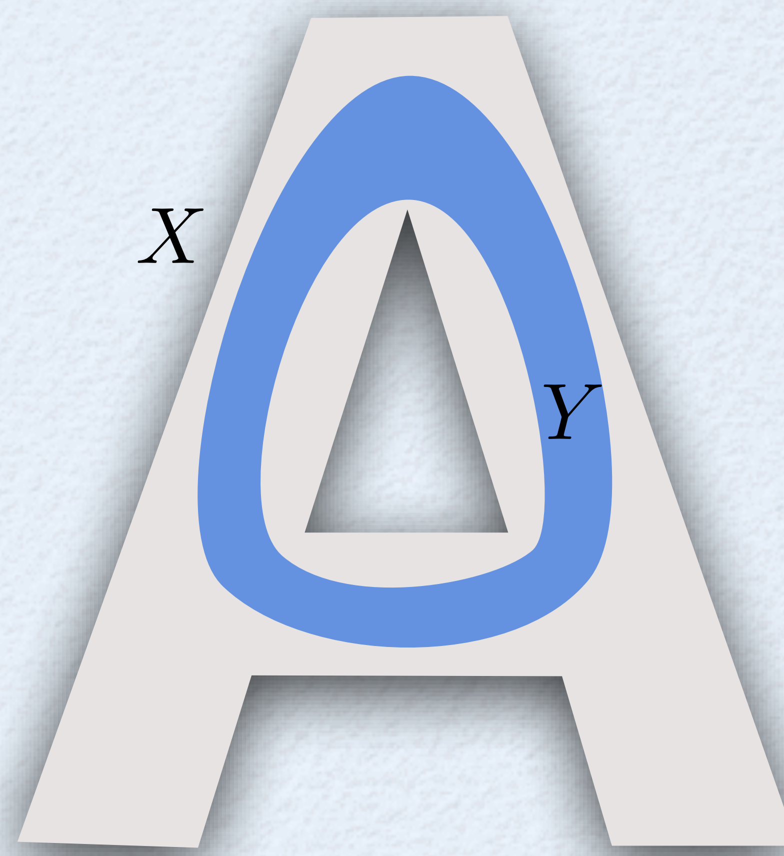

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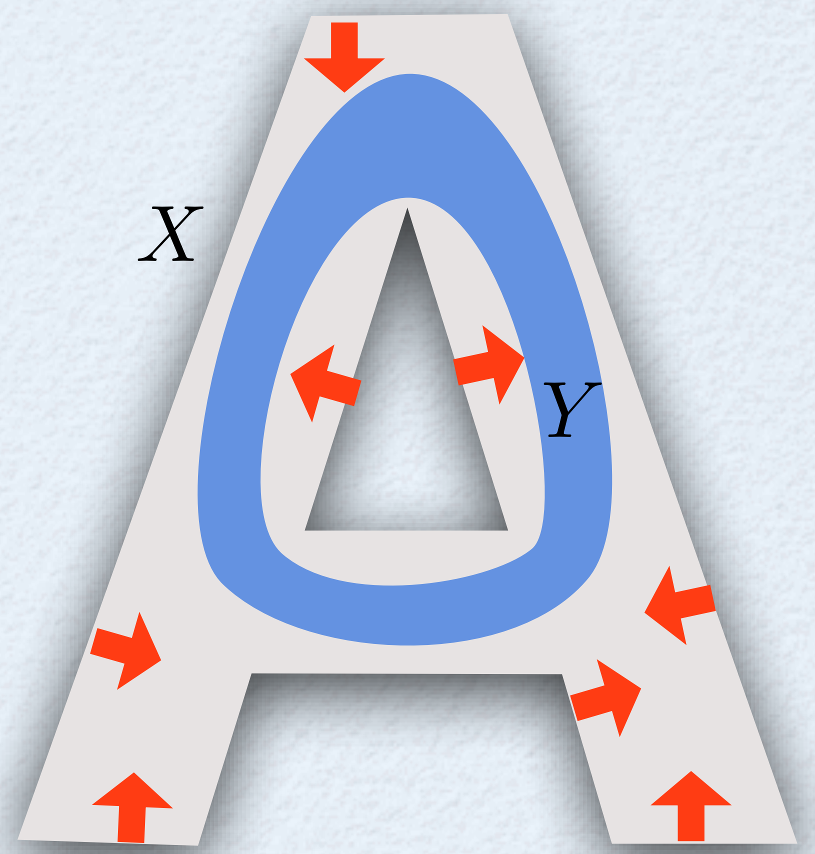

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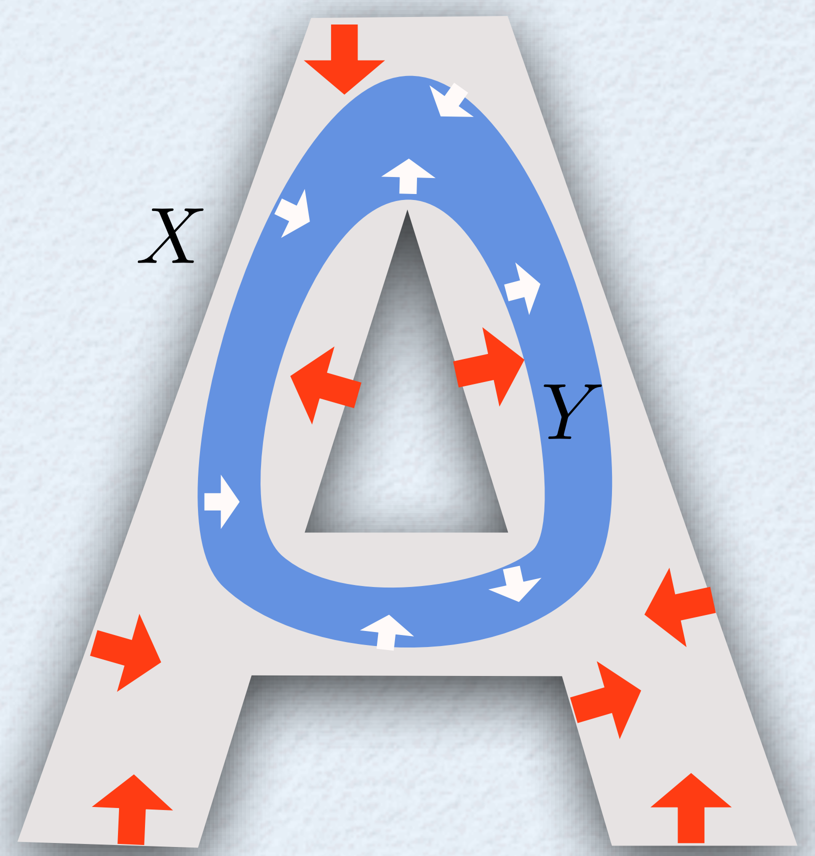

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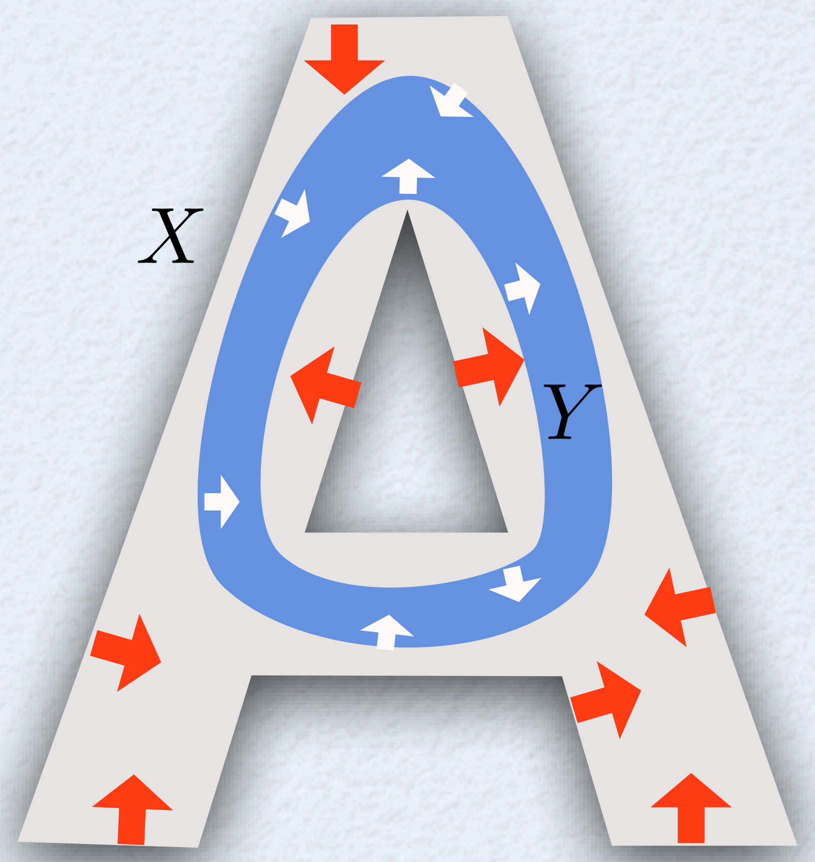
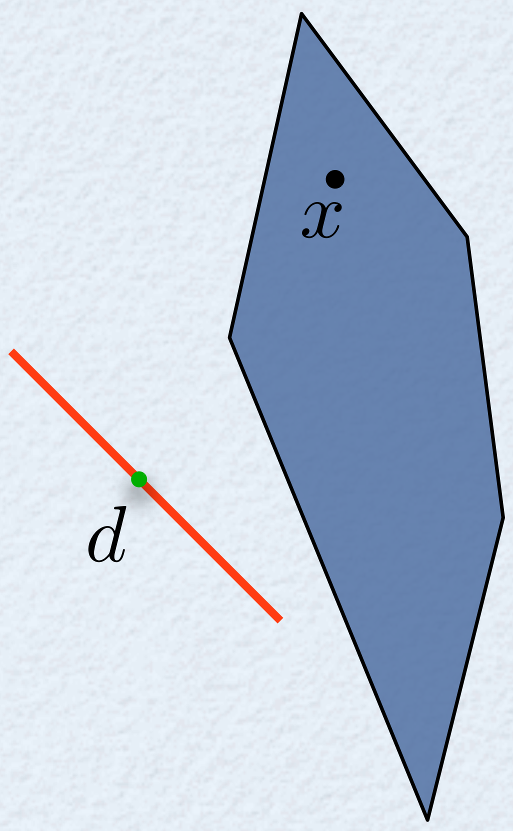
So, we "push X into Y" at speed > 0.



Idea to Lower Bound the Speed

If $V(x) \cap D(x) = \emptyset$ then

$$\begin{aligned} \|v(x)\| &= 2 \cdot \|x - d(x)\| \\ &\geq 2 \cdot \mathsf{dist}(V(x), D(x)). \end{aligned}$$



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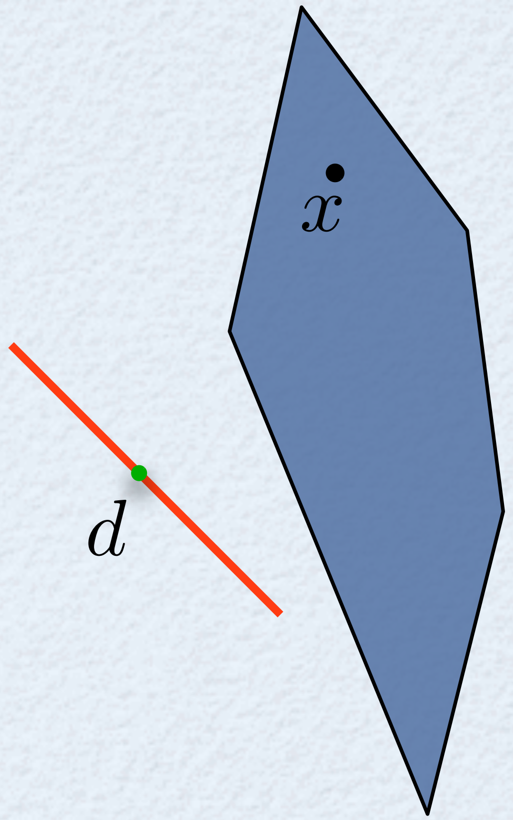
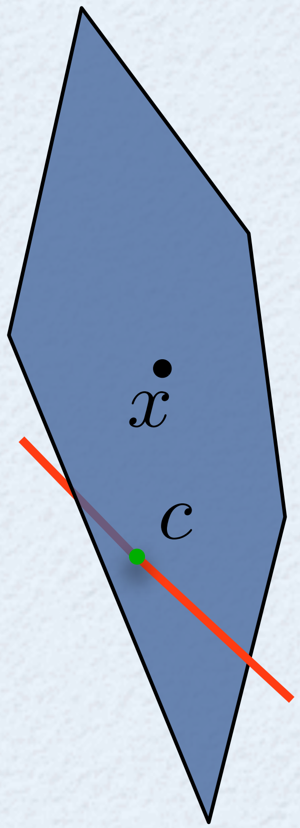


If $V(x) \cap D(x) = \{c\}$ then $x \in \mathsf{Um}(c)$.



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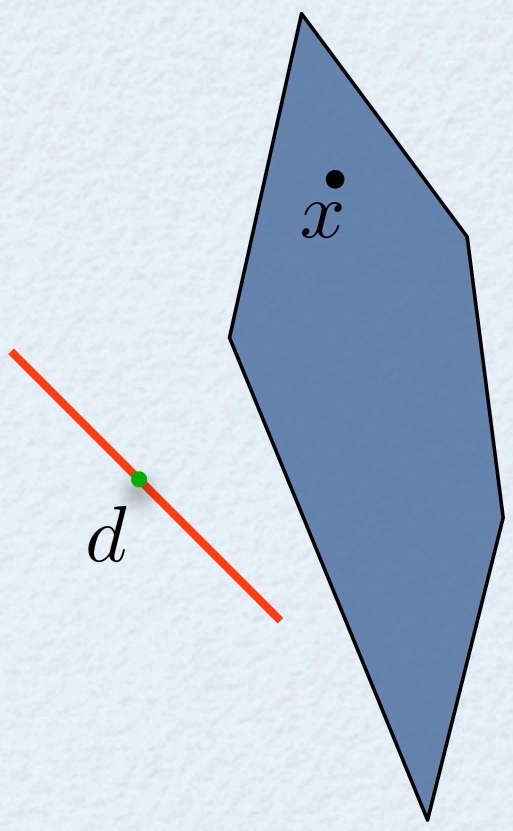

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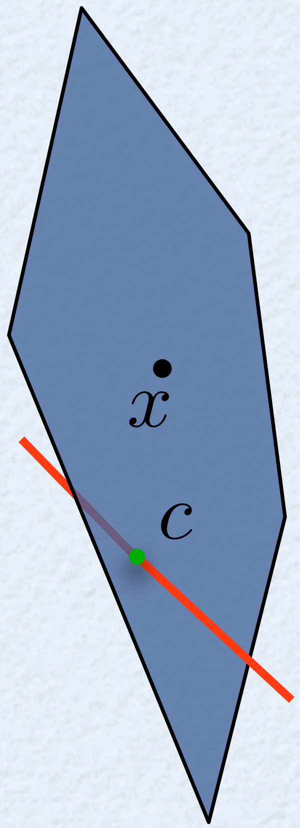
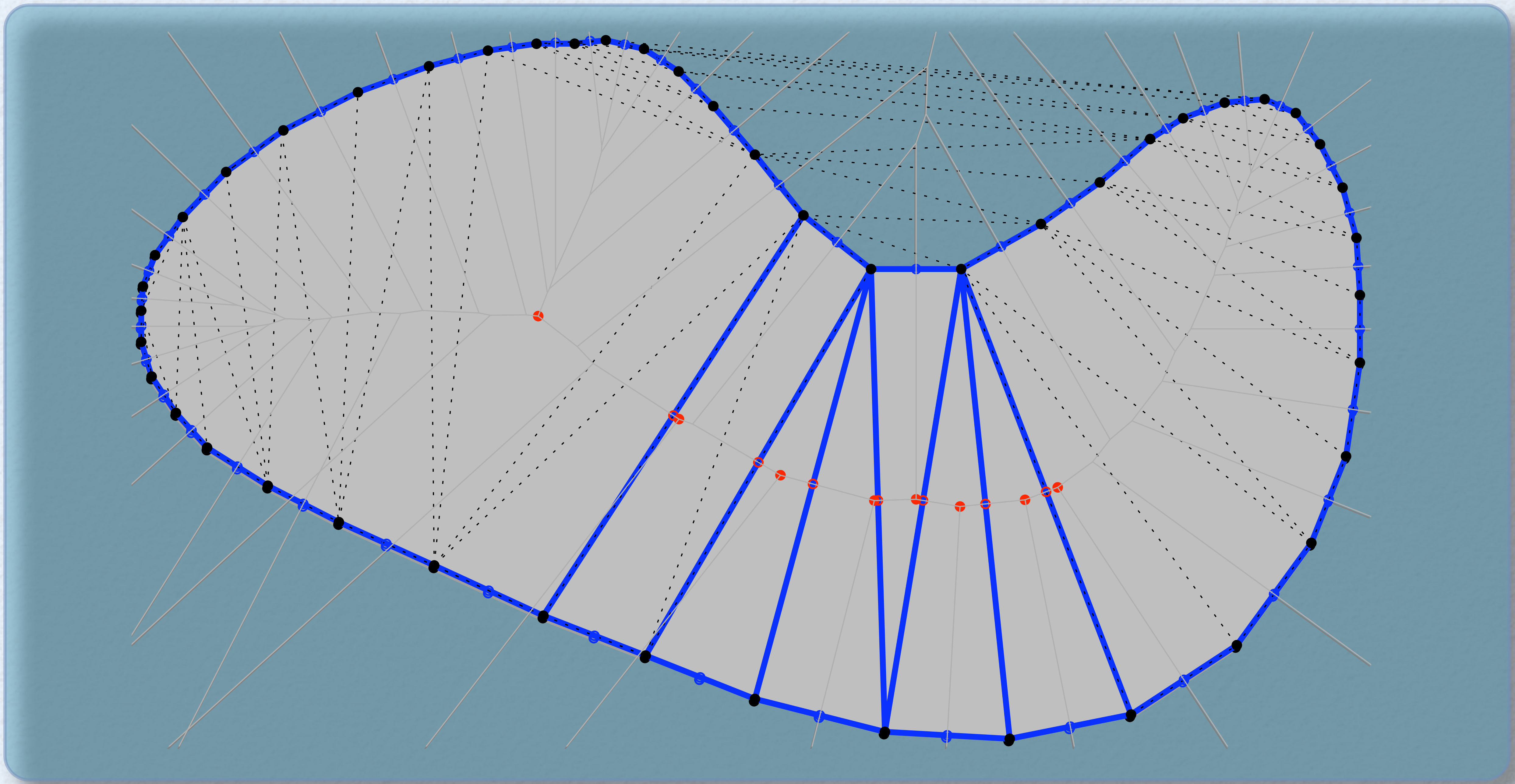


So, if $\mathsf{Um}(c) \subset Y$ we are fine!

Homotopy Type of the Complement

Theorem. $U := \bigcup_{c:\mathsf{deep}} \mathsf{Sm}(c)$ is homotopy equivalent to Σ^c.



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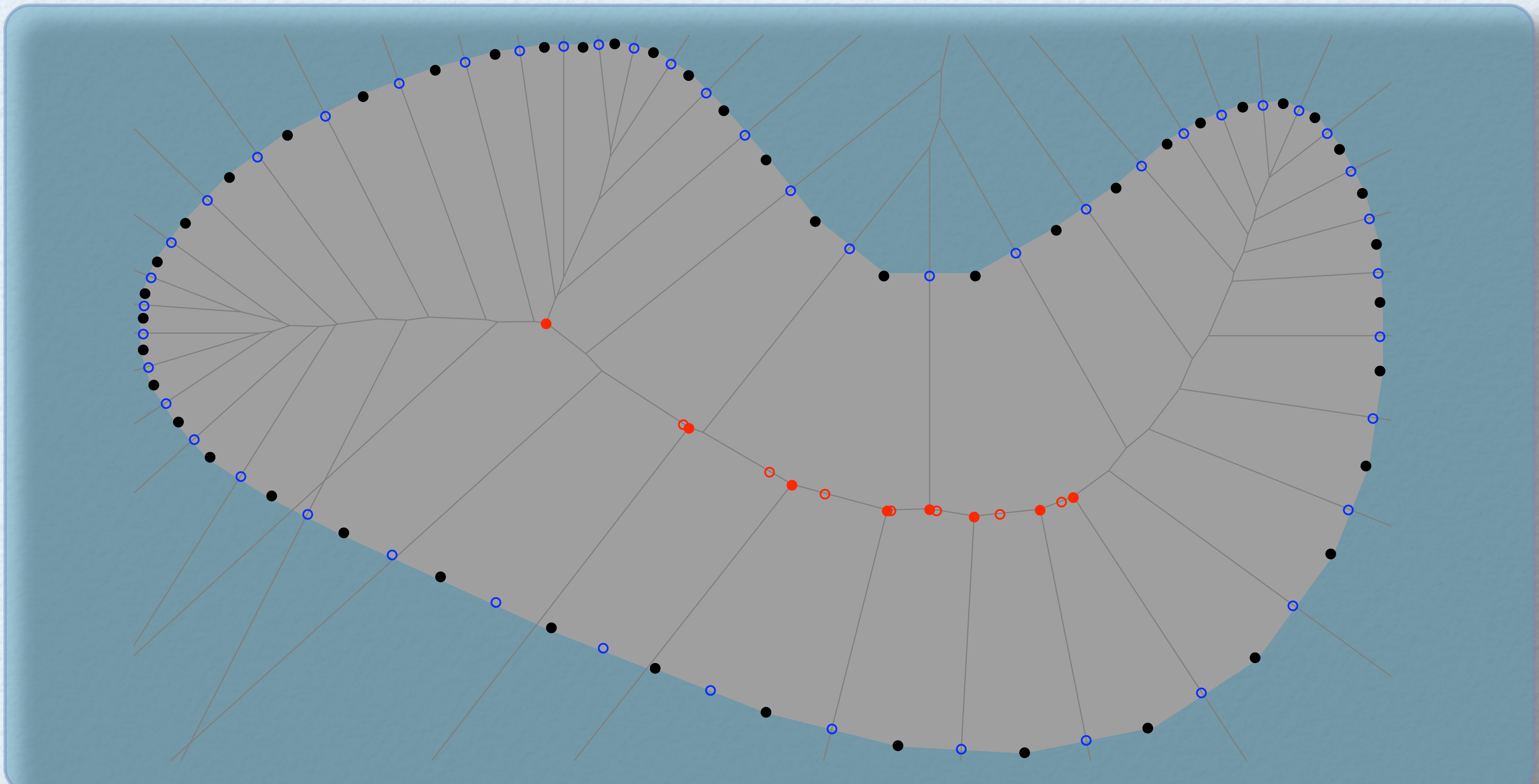

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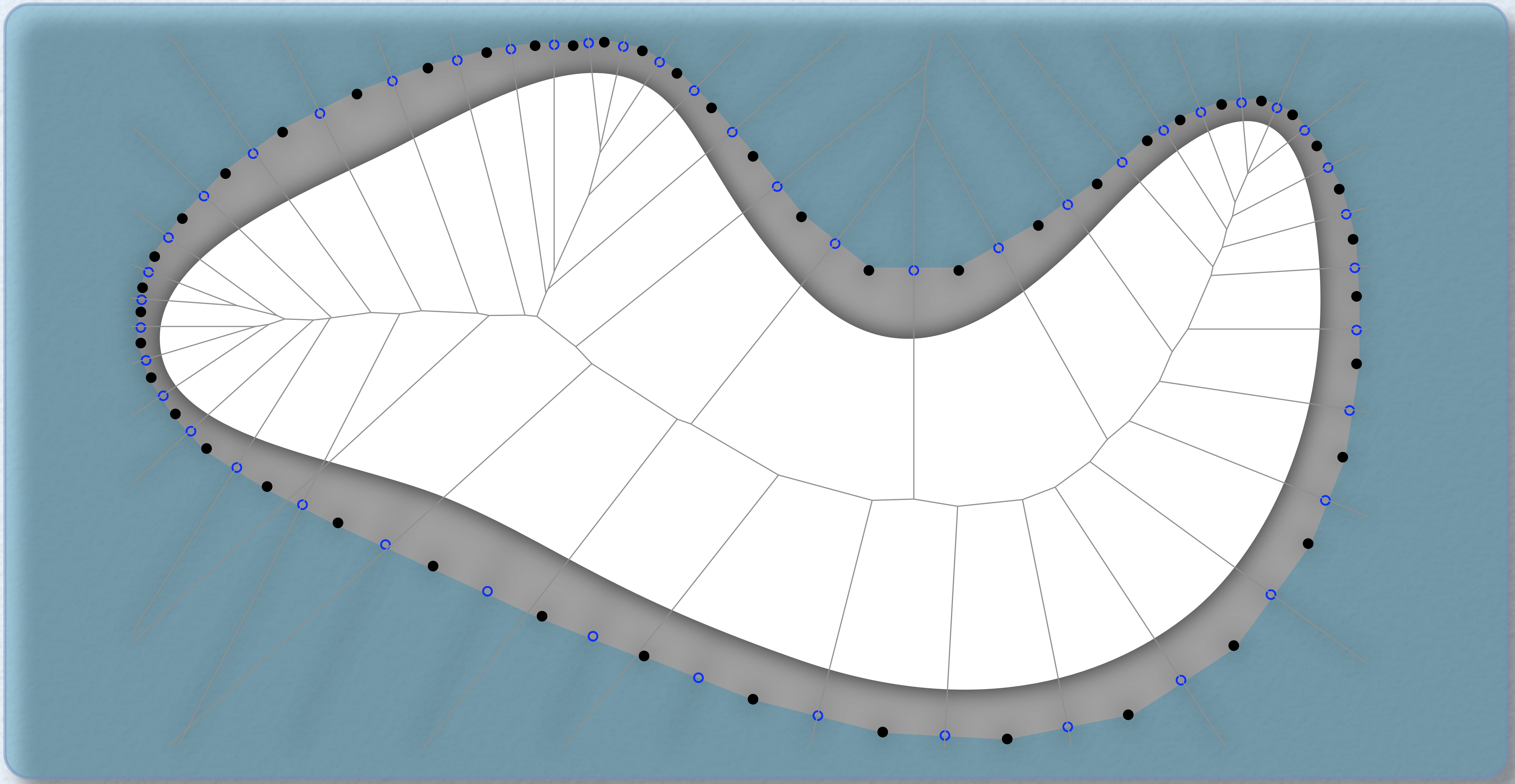

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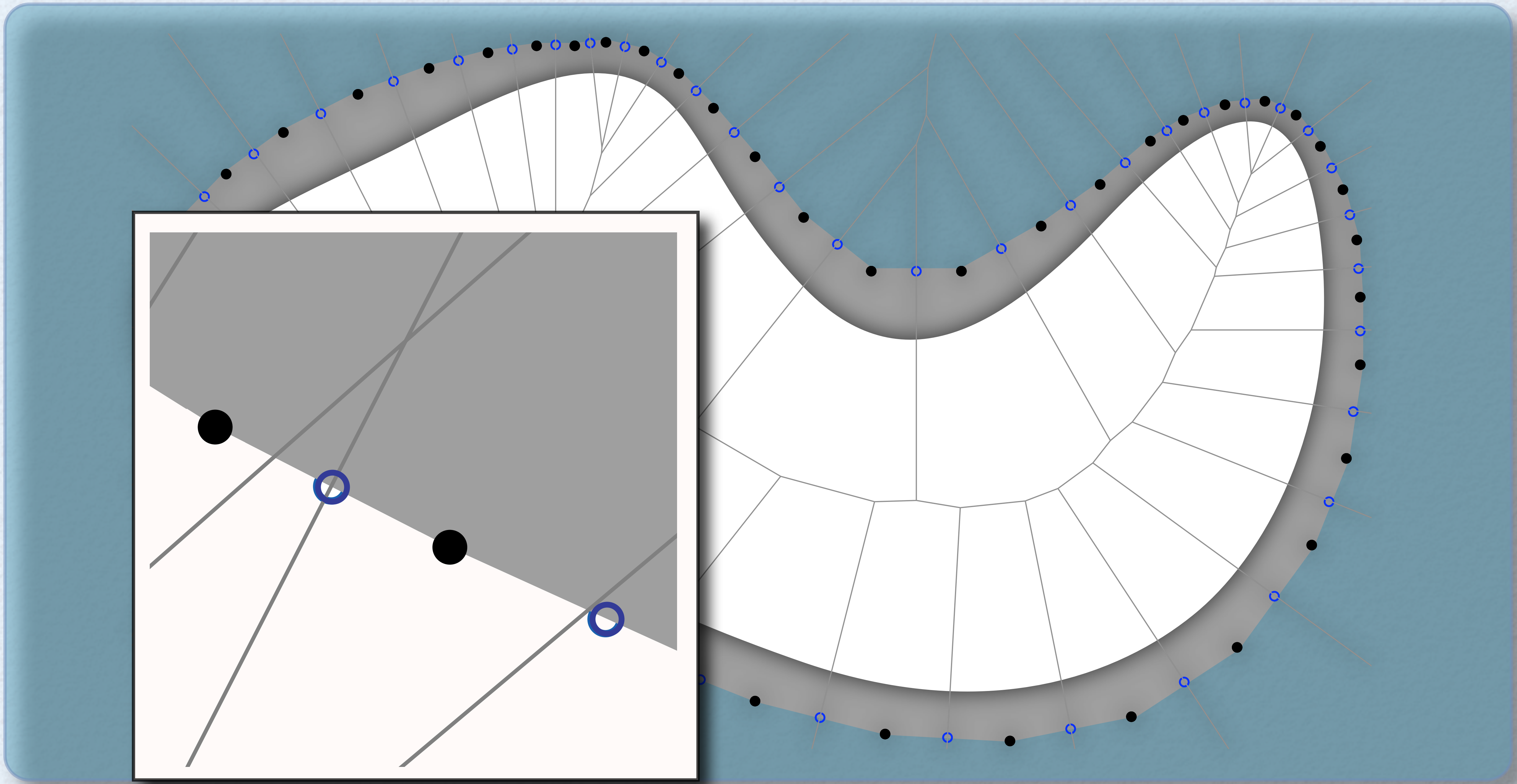

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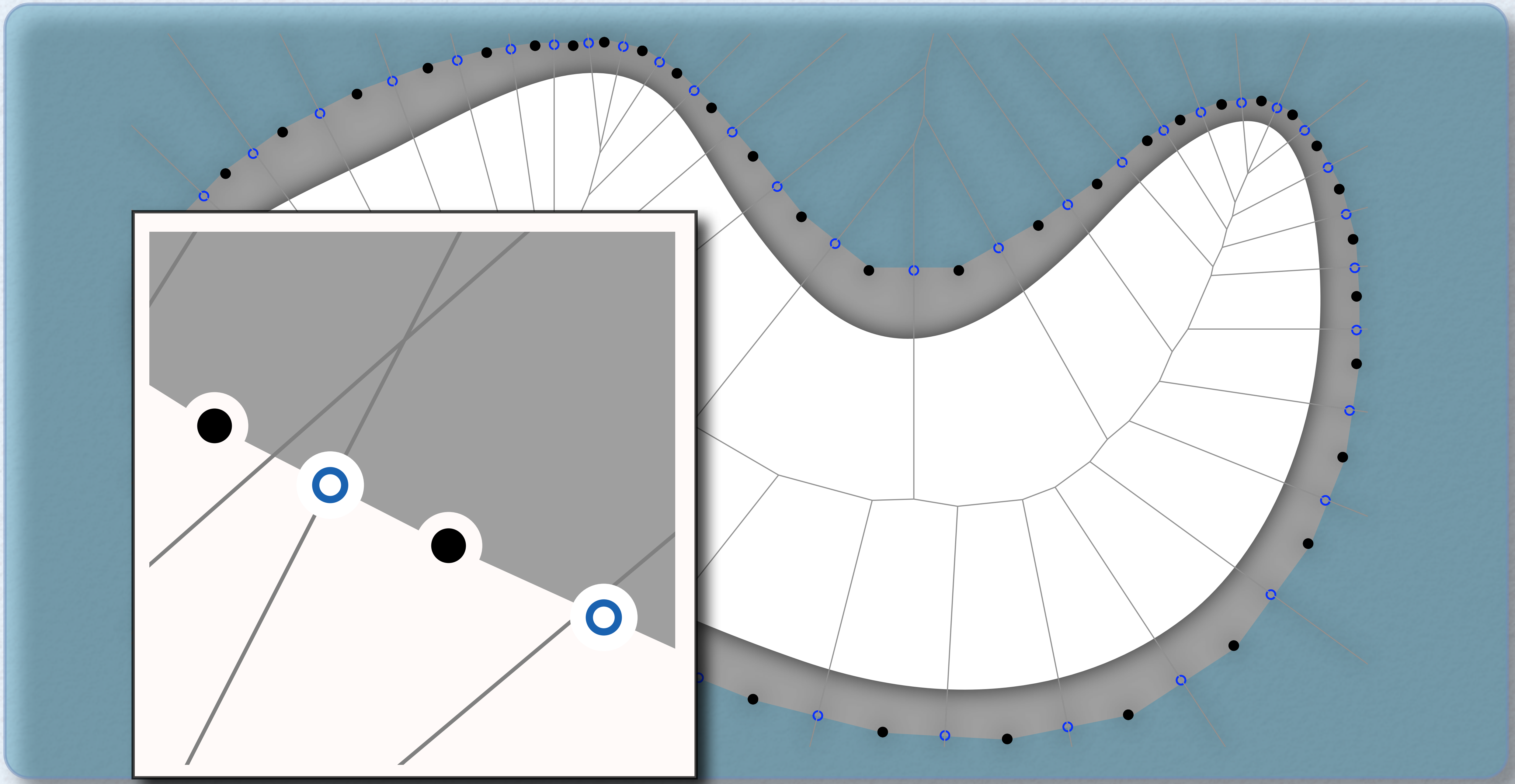

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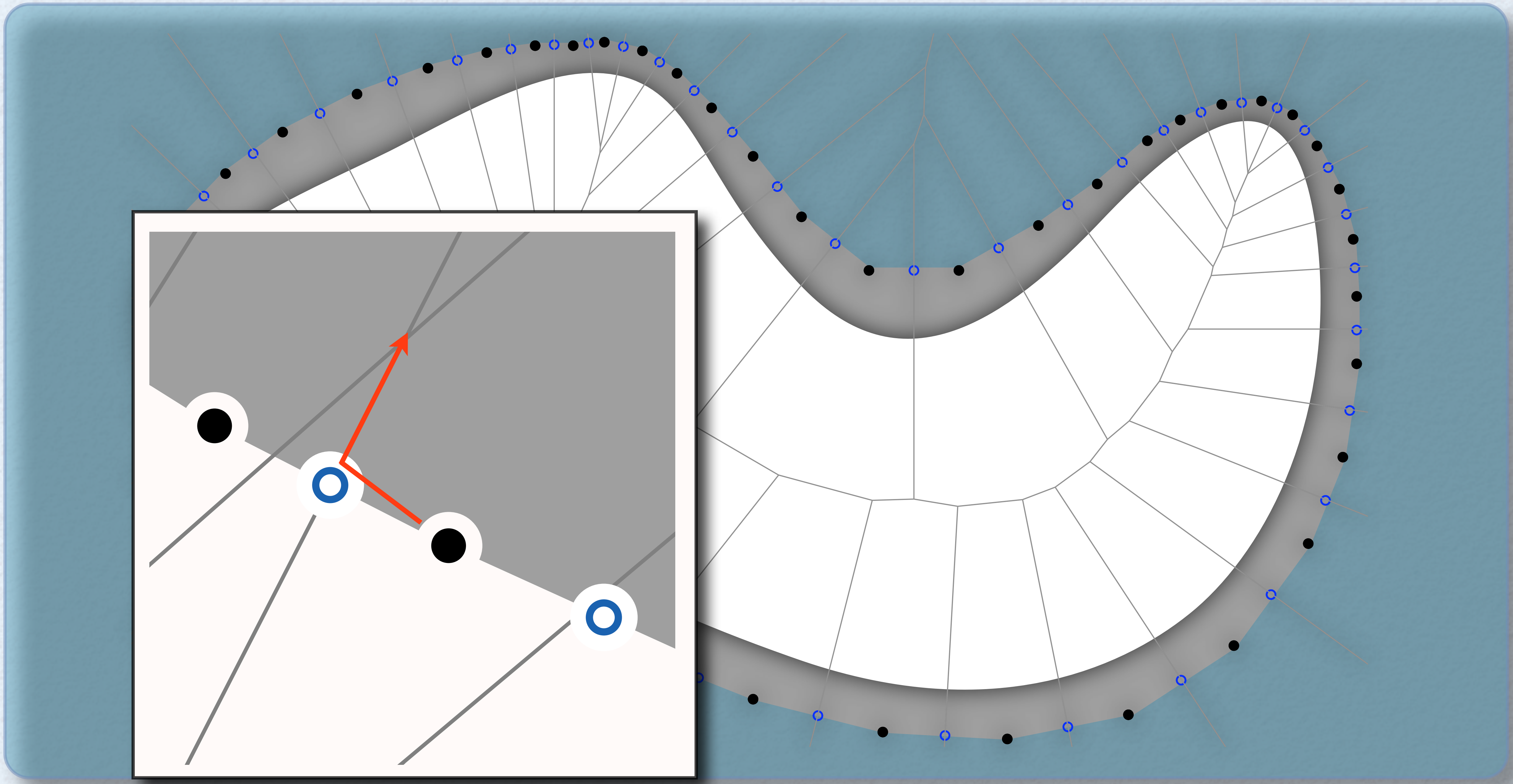

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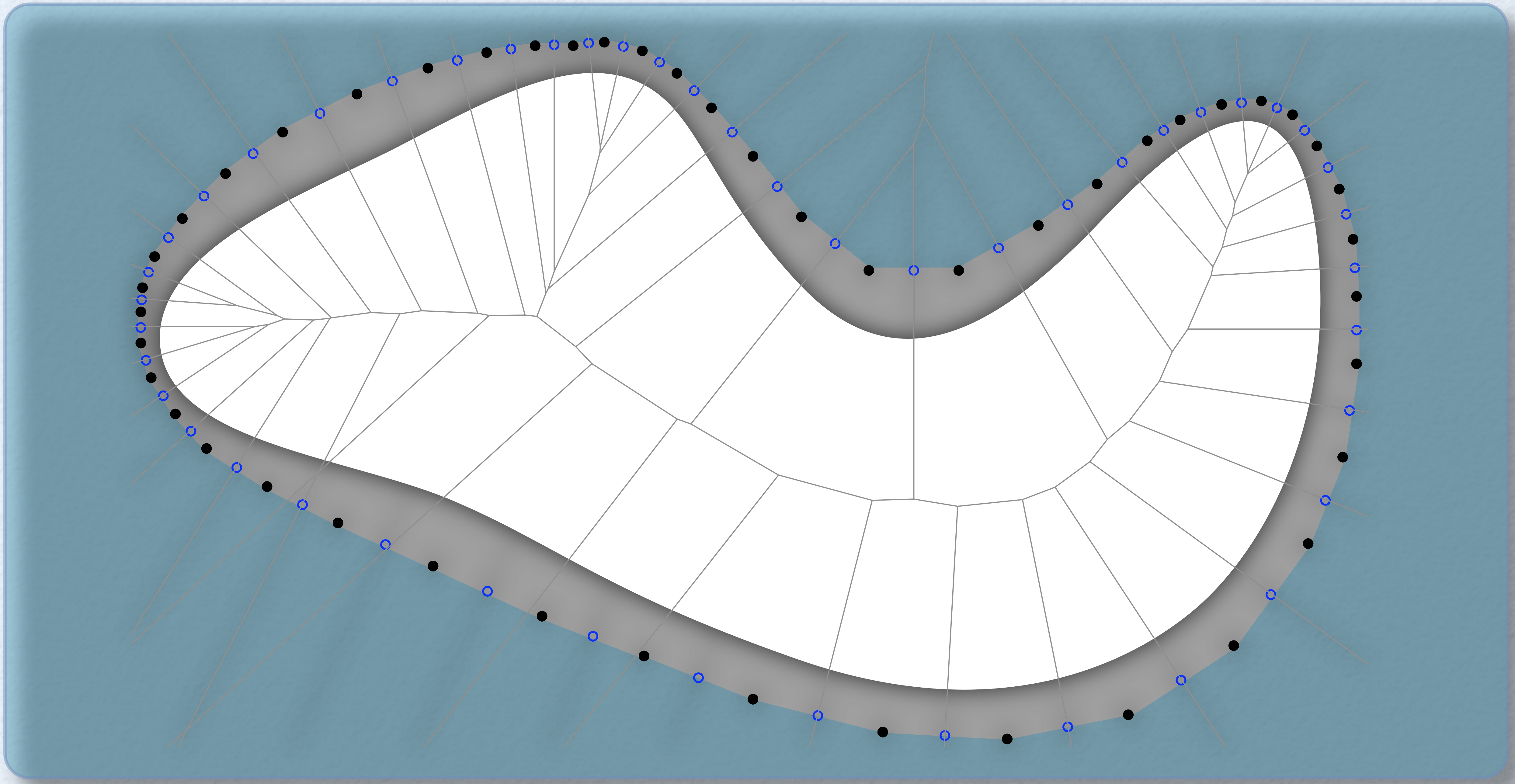

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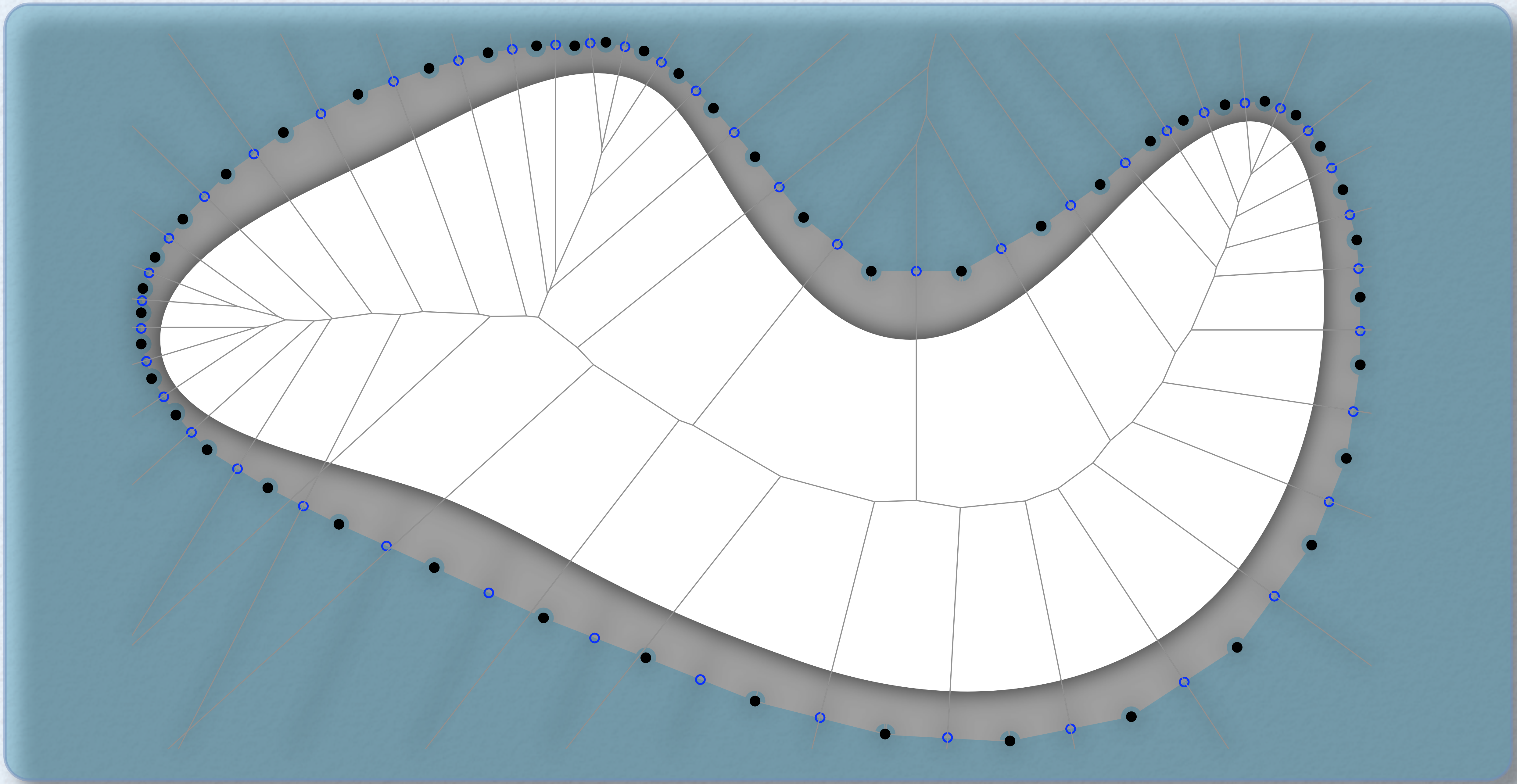

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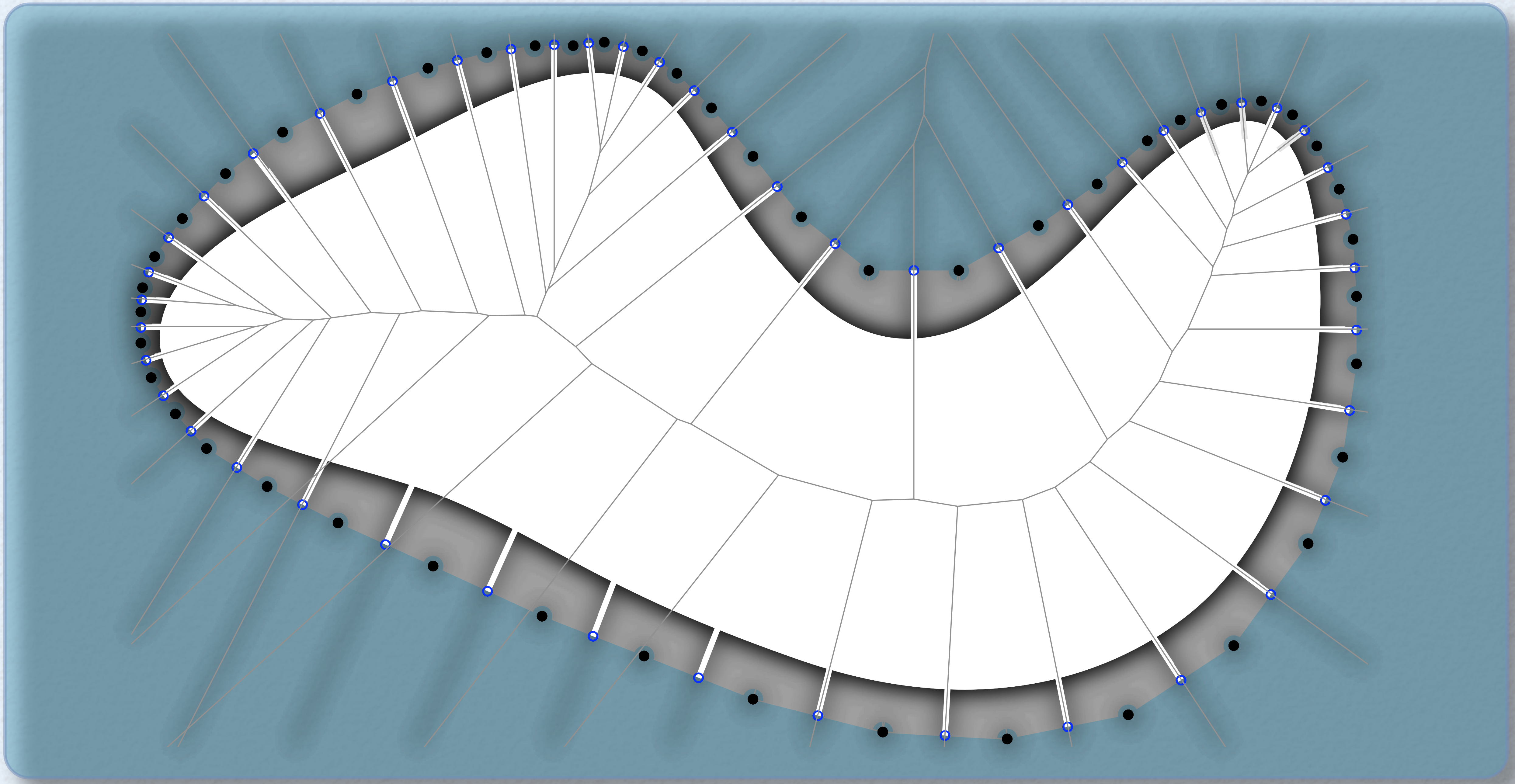

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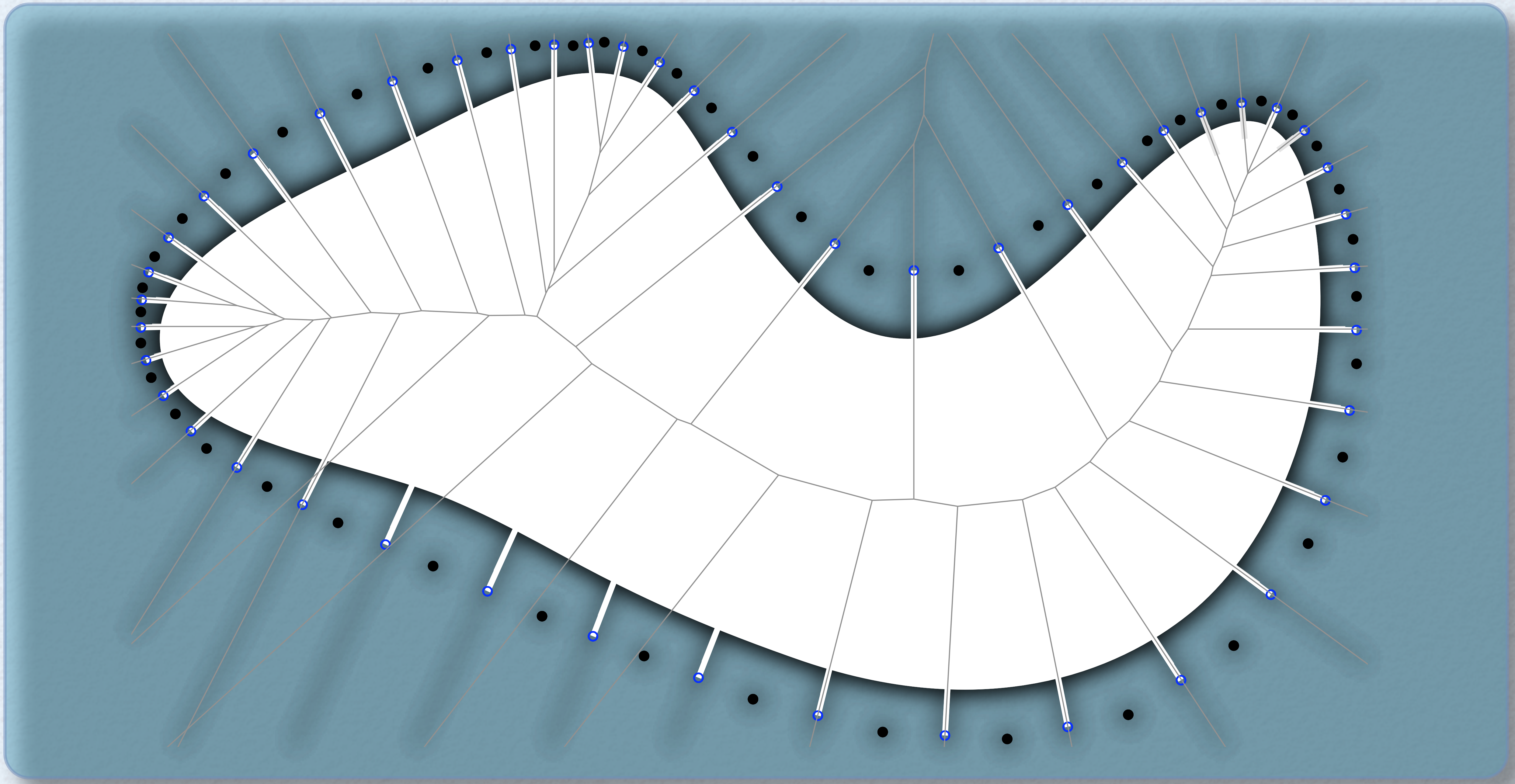

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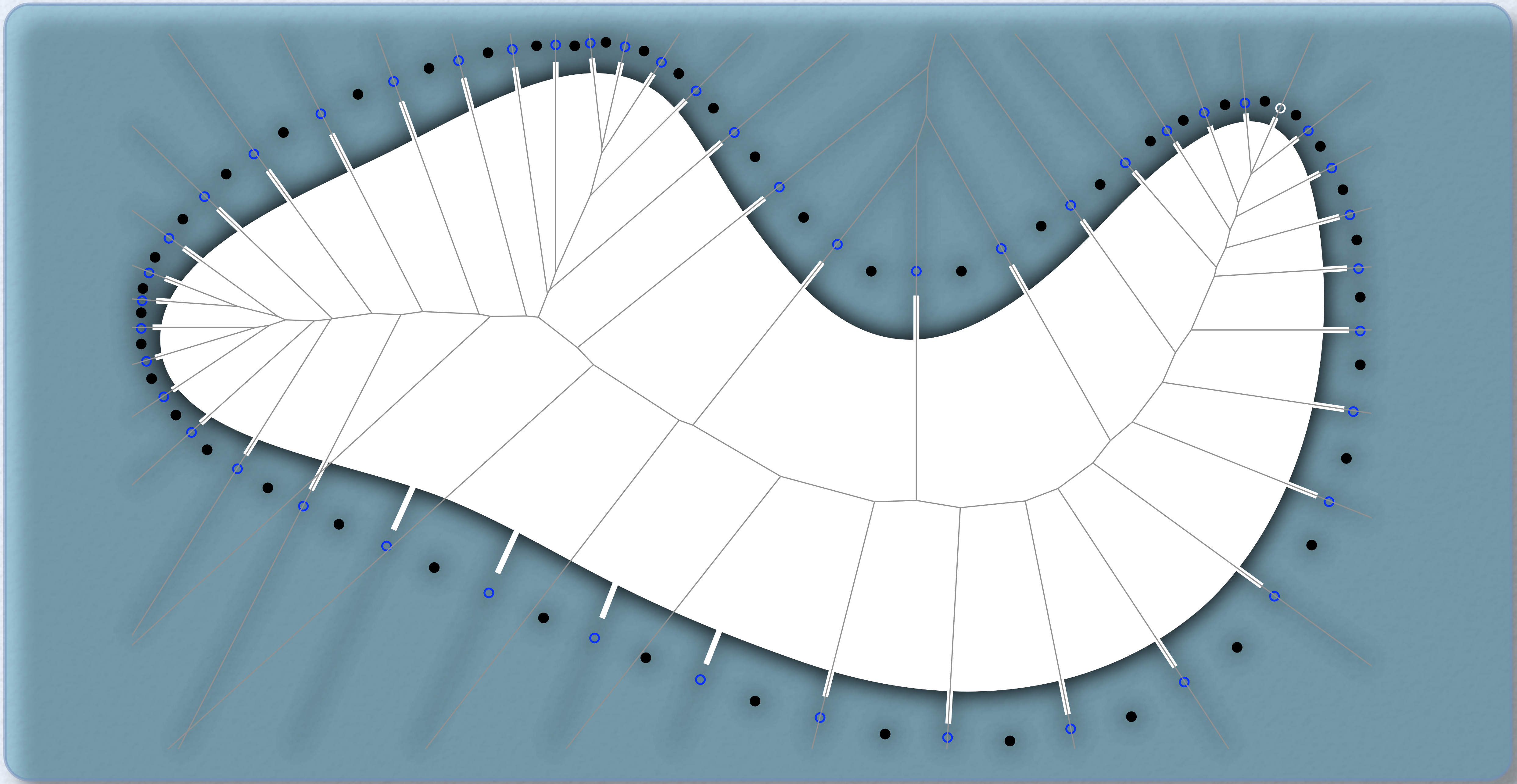

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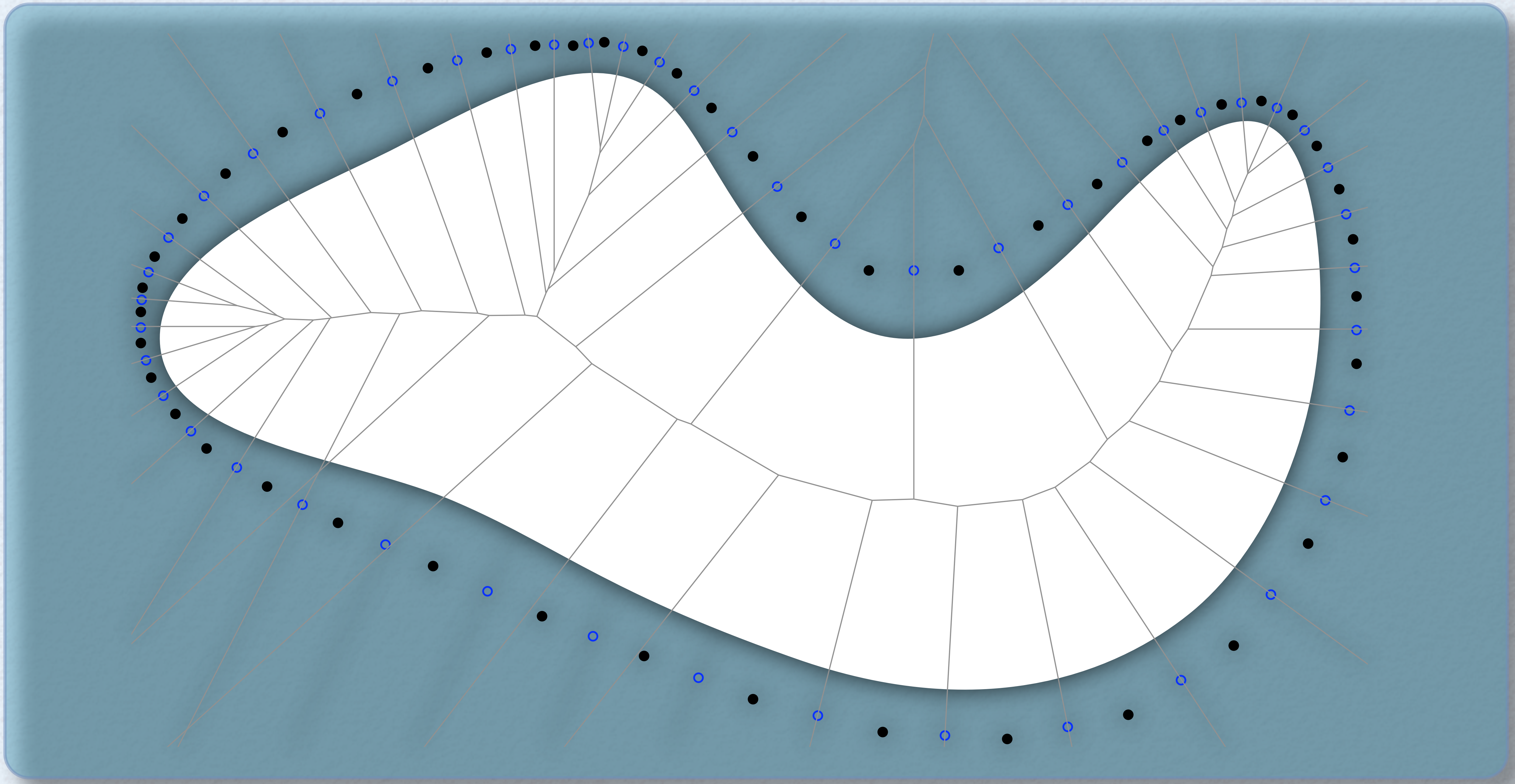

Putting Everything Together

Theorem.
Let $P \subset \mathbb{R}^n$ and h be the induced distance function. If $h(c_1) < \cdots < h(c_k)$ are critical points of h, then for any submanifold Σ of $\mathbb{R}^n$ densely sampled by P, there is a $1 < j < k$, such that:

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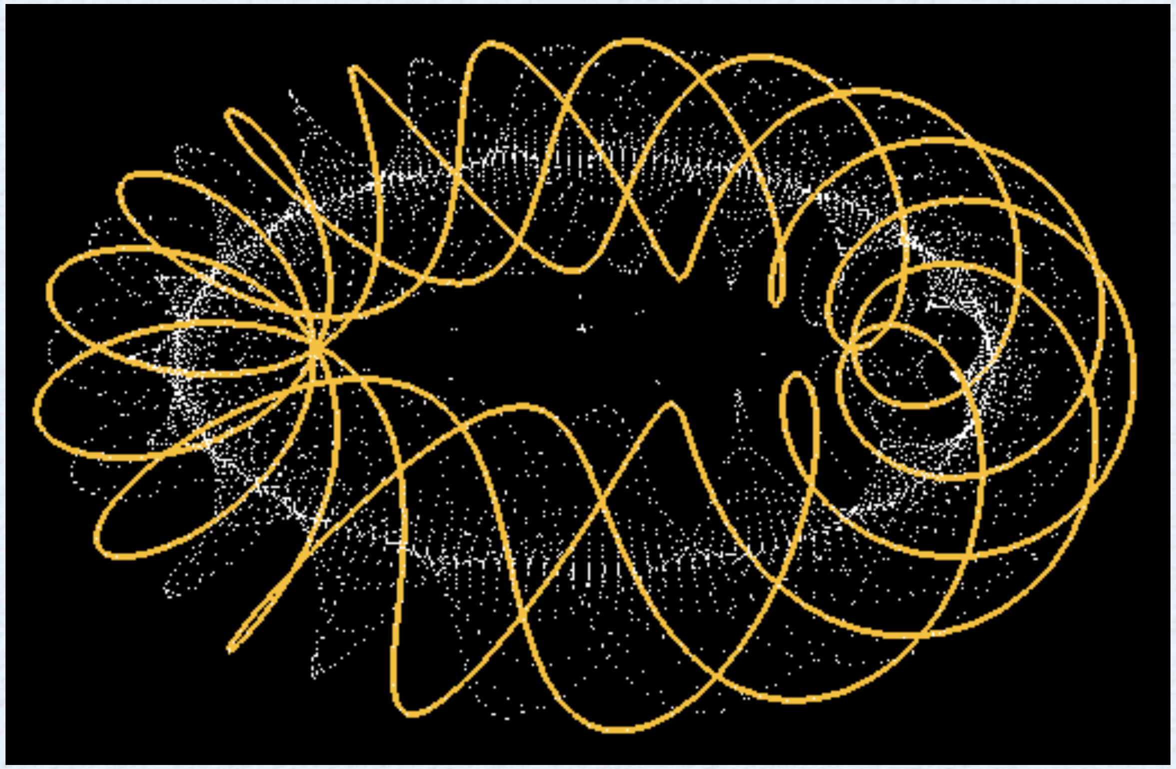

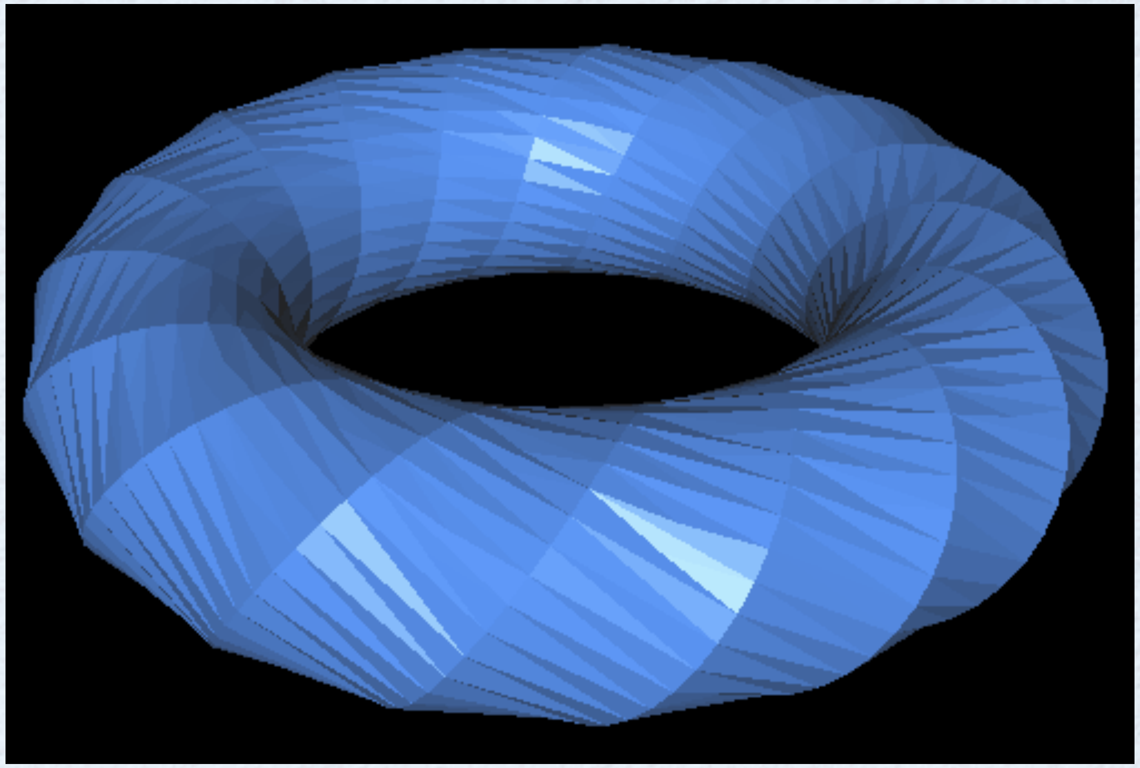

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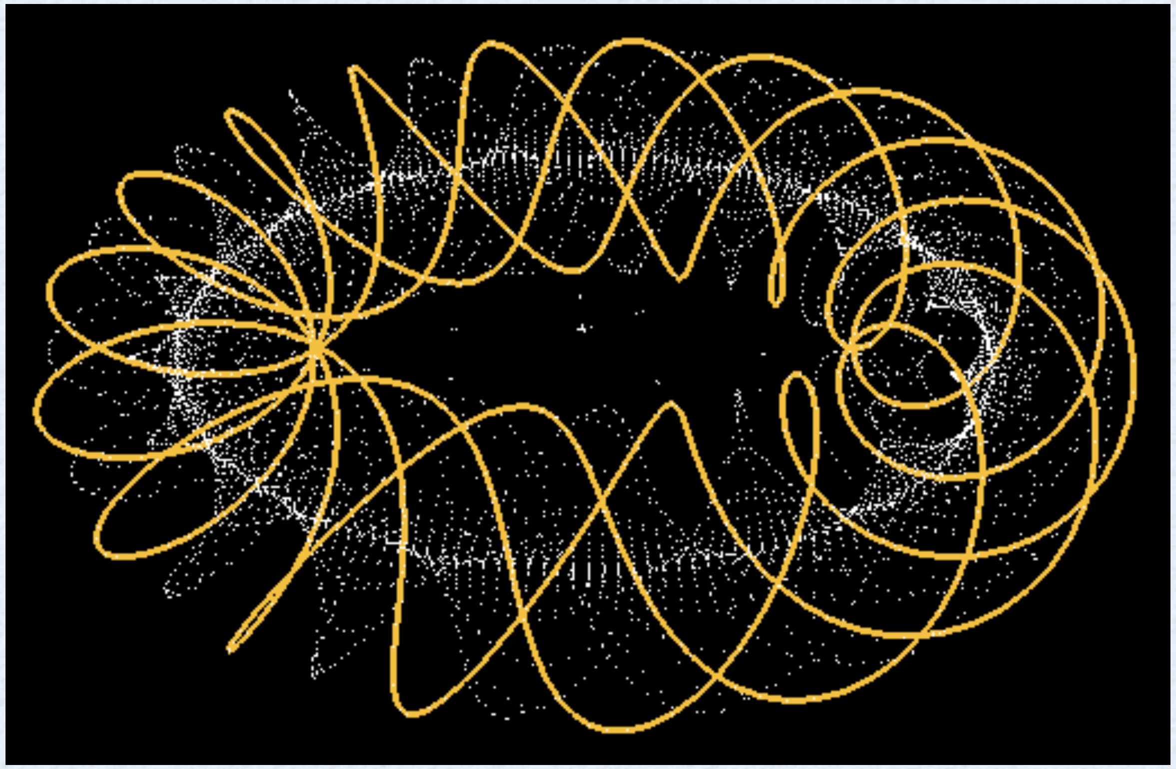

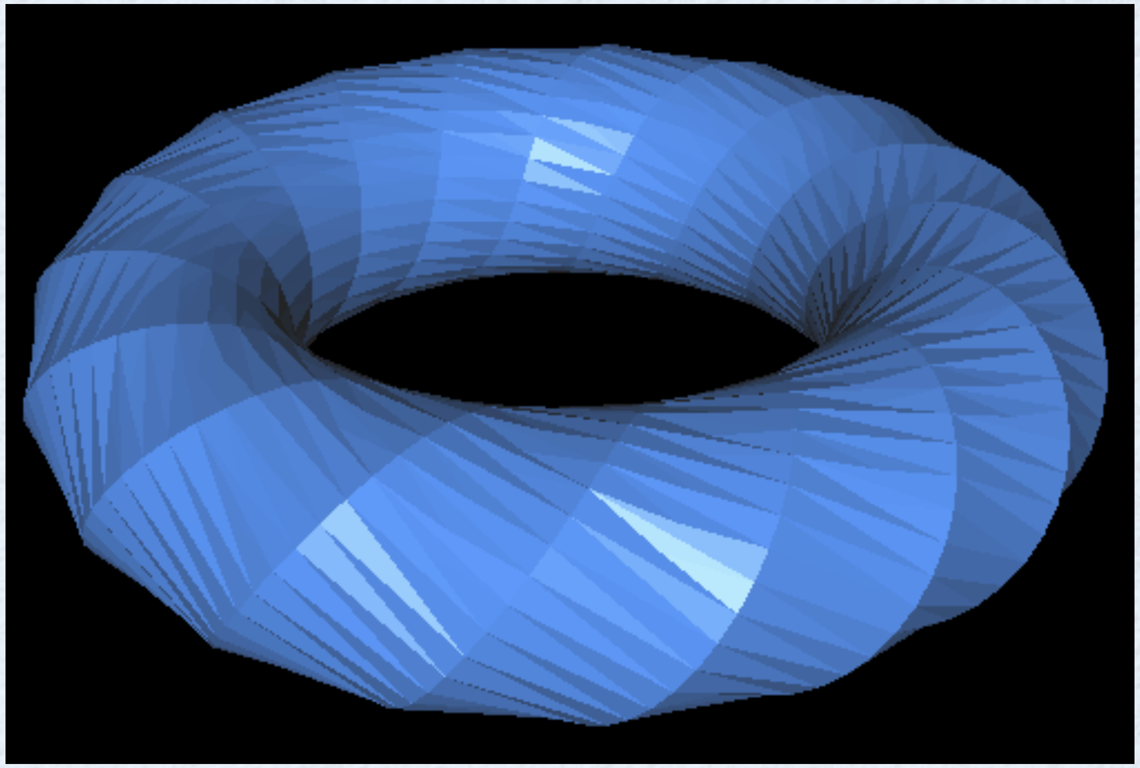

Thank You!