Manifold Homotopy via the Flow Complex

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The Surface Reconstruction Problem

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- Homeomorphic
- Ambient-isotopic
- \( \text{co-dimension 1 submanifold of } \mathbb{R}^n \)
- Hausdorff distance relative to lfs
The Surface Reconstruction Problem

Given a point cloud sampled from a surface $\Sigma$, we want to compute a surface $\hat{\Sigma}$ that has the same topology as $\Sigma$ and closely approximates it geometrically. We consider any submanifold of $\mathbb{R}^n$ co-dimension 1 submanifold of $\mathbb{R}^n$.

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The Surface Reconstruction Problem

Given a point cloud sampled from a surface $\Sigma$, we want to compute a surface $\hat{\Sigma}$ that has the \textit{same topology} as $\Sigma$ and closely approximates it \textit{geometrically}.

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- \textit{homotopy equivalent}

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- homeomorphic
- ambient-isotopic
- homotopy equivalent
Unlike homeomorphism, homotopy equivalence does not preserve dimension.
Chapter 0 Some Underlying Geometric Notions

Naturally we would like \( f_t(x) \) to depend continuously on both \( t \) and \( x \), and this will be true if we have each \( x \in X \setminus X \) move along its line segment at constant speed so as to reach its image point in \( X \) at time \( t = 1 \), while points \( x \in X \) are stationary, as remarked earlier.

Examples of this sort lead to the following general definition. A deformation retraction of a space \( X \) onto a subspace \( A \) is a family of maps \( f_t: X \to X \), \( t \in I \), such that \( f_0 = 1 \) (the identity map), \( f_1(X) = A \), and \( f_t|_A = 1 \) for all \( t \). The family \( f_t \) should be continuous in the sense that the associated map \( X \times I \to X \), \((x, t) \mapsto f_t(x)\), is continuous.

It is easy to produce many more examples similar to the letter examples, with the deformation retraction \( f_t \) obtained by sliding along line segments. The figure on the left below shows such a deformation retraction of a M"obius band onto its core circle. The three figures on the right show deformation retractions in which a disk with two smaller open subdisks removed shrinks to three different subspaces.

In all these examples the structure that gives rise to the deformation retraction can be described by means of the following definition. For a map \( f: X \to Y \), the mapping cylinder \( Mf \) is the quotient space of the disjoint union \((X \times I) \sqcup Y\) obtained by identifying each \((x, 1) \in X \times I\) with \( f(x) \in Y\). In the letter examples, the space \( X \) is the outer boundary of the thick letter, \( Y \) is the thin letter, and \( f: X \to Y \) sends the outer endpoint of each line segment to its inner endpoint. A similar description applies to the other examples. Then it is a general fact that a mapping cylinder \( Mf \) deformation retracts to the subspace \( Y \) by sliding each point \((x, t)\) along the segment \( \{x\} \times I \subset Mf \) to the endpoint \( f(x) \in Y\).

Not all deformation retractions arise in this way from mapping cylinders, however. For example, the thick \( X \) deformation retracts to the thin \( X \), which in turn deformation retracts to the point of intersection of its two crossbars. The net result is a deformation retraction of \( X \) onto a point, during which certain pairs of points follow paths that merge before reaching their final destination. Later in this section we will describe a considerably more complicated example, the so-called 'house with two rooms,' where a deformation retraction to a point can be constructed abstractly, but seeing the deformation with the naked eye is a real challenge.

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An adaptive $\varepsilon$-sample of $\Sigma$ has a point within $\varepsilon \cdot \text{lfs}(x)$ of every $x \in \Sigma$. 
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An adaptive \( \varepsilon \)-sample of Σ has a point within \( \varepsilon \cdot lfs(x) \) of every \( x \in \Sigma \).

A uniform \( \varepsilon \)-sample of Σ has a point within \( \varepsilon \cdot \text{reach}(\Sigma) \) of every \( x \in \Sigma \).
The squared distance function induced by $P$ is

\[ h(x) := \min_{p \in P} \| x - p \|^2 \]

$P$ is a discrete set of points.
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The **driver** of \( x \) is the **closest point** to \( x \) in \( D(x) \).

\[ v(x) = 2(x - d(x)) \]
$v(x) = \nabla h(x)$

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Critical Points of Distance Function
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The index of $c$ is the dimension of $D(c)$.
Stable manifold of a critical point $c$ is the set of all points that flow to $c$.

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For a point set $P \subset \mathbb{R}^n$ and $r \in \mathbb{R}$:

union of balls $B^r(P) := \bigcup_{p \in P} B(p, r)$

alpha shape $K^r(P) := \text{Nrv} \bigcup_{p \in P} (B(p, r) \cap V_p)$

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**Theorem** [Edel’95] $B^r(P)$ and $K^r(P)$ are homotopy equivalent.
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**Theorem** [DGJ’03, BG’05] $F^r(P)$ and $K^r(P)$ are homotopy equivalent.
**Theorem** [DGRS’05] If $P$ is a uniform $\varepsilon$-sample of $\Sigma$ with $\varepsilon < 1/\sqrt{3}$, then any critical point $c$ of $h$ is either shallow, i.e. $\text{dist}(c, \Sigma) \leq \varepsilon^2 \cdot \tau$ or is deep, i.e. $\text{dist}(c, \Sigma) \geq (1 - 2\varepsilon^2)\tau$, where $\tau = \text{reach}(\Sigma)$. 
Separation of Critical Points

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Theorem. If $P$ is a uniform $\varepsilon$-sample, then for a shallow $c$, $\text{dist}(c, P) < \sqrt{5/3}\varepsilon\tau$ and for a deep $c$, $\text{dist}(c, P) > (1 - 2\varepsilon^2)\tau$. 
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\[ \bigcup_{c: \text{shallow}} \text{Sm}(c) = F^r(P) \text{ for } r = \sqrt{5/3\varepsilon\tau}. \]
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**Theorem.** [NSW'06] If $P$ is a uniform $\varepsilon$-sample of $\Sigma$ then $B^{(r)}(P)$ is homotopy equivalent to $\Sigma$ (when $r$ and $\varepsilon$ are in the right range).
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Um(c) = \bigcap_{\epsilon > 0} \phi(B(x, \epsilon)) = \phi(V(c)).
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Lemma [Lieutier’04]. If $Y \subset X$ are bounded and

1. $\phi(X) = X$ and $\phi(Y) = Y$, and

2. $\|v(x)\| \geq c > 0$ for $x \in X \setminus Y$,

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So, we “push $X$ into $Y$” at speed $> 0$. 
If $V(x) \cap D(x) = \emptyset$ then

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\|v(x)\| = 2 \cdot \|x - d(x)\| \\
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If $V(x) \cap D(x) = \{c\}$ then $x \in \text{Um}(c)$. 
Idea to Lower Bound the Speed

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If $V(x) \cap D(x) = \{c\}$ then $x \in \text{Um}(c)$.

So, if $\text{Um}(c) \subset Y$ we are fine!
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Theorem.
Let $P \subset \mathbb{R}^n$ and $h$ be the induced distance function. If $h(c_1) < \cdots < h(c_k)$ are critical points of $h$, then for any submanifold $\Sigma$ of $\mathbb{R}^n$ densely sampled by $P$, there is a $1 < j < k$, such that:

$$\bigcup_{i=1}^{j} \text{Sm}(c_i) \simeq \Sigma$$

and

$$\bigcup_{i=j+1}^{k} \text{Sm}(c_i) \simeq \Sigma^c$$
Theorem.
Let $P \subset \mathbb{R}^n$ and $h$ be the induced distance function. If $h(c_1) < \cdots < h(c_k)$ are critical points of $h$, then for any submanifold $\Sigma$ of $\mathbb{R}^n$ densely sampled by $P$, there is a $1 < j < k$, such that:

\[
\bigcup_{i=1}^{j} \text{Sm}(c_i) \simeq \Sigma \\
\text{and} \\
\bigcup_{i=j+1}^{k} \text{Sm}(c_i) \simeq \Sigma^c
\]