Flow-Based Methods in Manifold Reconstruction

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- Ambient-isotopic
- Co-dimension 1 submanifold of $\mathbb{R}^n$

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Surface (manifold) reconstruction

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- Any submanifold of $\mathbb{R}^n$
- Co-dimension 1 submanifold of $\mathbb{R}^n$
- Topological space
- Homotopy equivalent
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- Homeomorphic
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- Hausdorff distance relative to lfs
Reconstructing the complement

Unlike homeomorphism, homotopy equivalence does not preserve dimension.
Reconstructing the complement

Unlike homeomorphism, homotopy equivalence does not preserve dimension.

All these knots have the same homotopy type, but not their complements.
Chapter 0 Some Underlying Geometric Notions

Naturally we would like $f(t)$ to depend continuously on both $t$ and $x$, and this will be true if we have each $x \in X - X$ move along its line segment at constant speed so as to reach its image point in $X$ at time $t = 1$, while points $x \in X$ are stationary, as remarked earlier.

Examples of this sort lead to the following general definition. A deformation retraction of a space $X$ onto a subspace $A$ is a family of maps $f_t: X \to X$, $t \in I$, such that $f_0 = 1$ (the identity map), $f_1(X) = A$, and $f_t|_A = 1$ for all $t$. The family $f_t$ should be continuous in the sense that the associated map $X \times I \to X$, $(x, t) \mapsto f_t(x)$, is continuous.

It is easy to produce many more examples similar to the letter examples, with the deformation retraction $f_t$ obtained by sliding along line segments. The figure on the left below shows such a deformation retraction of a Möbius band onto its core circle.

The three figures on the right show deformation retractions in which a disk with two smaller open subdisks removed shrinks to three different subspaces.

In all these examples the structure that gives rise to the deformation retraction can be described by means of the following definition. For a map $f: X \to Y$, the mapping cylinder $Mf$ is the quotient space of the disjoint union $(X \times I) \bigsqcup Y$ obtained by identifying each $(x, 1) \in X \times I$ with $f(x) \in Y$. In the letter examples, the space $X$ is the outer boundary of the thick letter, $Y$ is the thin letter, and $f: X \to Y$ sends the outer endpoint of each line segment to its inner endpoint. A similar description applies to the other examples. Then it is a general fact that a mapping cylinder $Mf$ deformation retracts to the subspace $Y$ by sliding each point $(x, t)$ along the segment $\{x\} \times I \subset Mf$ to the endpoint $f(x) \in Y$.

Not all deformation retractions arise in this way from mapping cylinders, however. For example, the thick $X$ deformation retracts to the thin $X$, which in turn deformation retracts to the point of intersection of its two crossbars. The net result is a deformation retraction of $X$ onto a point, during which certain pairs of points follow paths that merge before reaching their final destination. Later in this section we will describe a considerably more complicated example, the so-called 'house with two rooms,' where a deformation retraction to a point can be constructed abstractly, but seeing the deformation with the naked eye is a real challenge.

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• Reconstruction as an iso-surfaces:
  0-set of signed dist functs [Hoppe et al’92, Curless et al’96]
  Nearest Neighbor Interpolation [Boissonnat-Cazals’02]
  Mean Least Square [Levin’98, Alexa et al’01, Amenta-Kil’04, Kolluri’05, Dey et al’05]
  SVM [Schölkopf et al’04]

• Delaunay Methods:
  [Boissonnat’84, Amenta-Bern’99, Amenta et al’91, Amenta-Choi-Kolluri’01]

• Using distance functions:
  [Edelsbrunner’04, Chaine’03, Giesen-John’03, Dey-Giesen-Ramos-S’05]
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$P$ is a discrete set of points.
Generalized gradient

[Diagram of generalized gradient]
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Generalized gradients can be defined for distance to any *compact* set.
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Generalized gradients can be defined for distance to any **compact** set.

\[ \sum d(x) v(x) \]

... or even for (geodesic) distances relative to a compact subset of a Reimannian manifold. [Grove’ 93]
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x(0) &= x_0 \\
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**Theorem** [Lieutier’04]. $\phi$ is continuous.
Proposition. Let $X$ and $Y \subseteq X$ be arbitrary sets and

$$H : [0, 1] \times X \to X$$

be a continuous function (on both variables) satisfying

1. $\forall x \in X : H(0, x) = x$
2. $\forall y \in Y, \forall t \in [0, 1] : H(t, y) \in Y$
3. $\forall x \in X : H(1, x) \in Y$

Then $X$ and $Y$ have the same homotopy type.
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2. $\forall y \in Y, \forall t \in [0, 1] : H(t, y) \in Y$ \hspace{1cm} Nothing leaves $Y$
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Proposition. Let $X$ and $Y \subseteq X$ be arbitrary sets and

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2. $\forall y \in Y, \forall t \in [0,T] : H(t,y) \in Y$ \hspace{1cm} \text{Nothing leaves } Y$

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Lemma [Lieutier’04]. If $Y \subset X$ are bounded and

1. $\phi(X) = X$ and $\phi(Y) = Y$, and
2. $\|v(x)\| \geq c > 0$ for $x \in X \setminus Y$,

then $X$ and $Y$ are homotopy equivalent.
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So, we “push $X$ into $Y$” at speed $> 0$. 
Important flow-tight sets

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Finite unions and intersections of flow-tight sets are flow-tight.
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A point $c$ is critical iff $\{c\} = V(c) \cap D(c)$.

The index of $c$ is the dimension of $D(c)$. 

Stable manifold of a critical point $c$ is the set of all points that flow to $c$.

$$\text{Sm}(c) = \{x : \phi(\infty, x) = c\}.$$
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Computation and properties [Giesen-John’03, Dey-Giesen-John’04, Buchin-Dey-Giesen-John’08, Cazals-Parameswaran-Pion’08]
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Computation and properties [Giesen-John’03, Dey-Giesen-John’04, Dey-Giesen-John’08, Cazals-Parameswaran-Pion’08] [Giesen-John’03] found out that a subcomplex of FC reconstructs manifold.
Separation of critical points
**Theorem** [Dey-Giesen-Ramos-S’05] If $P$ is a uniform $\varepsilon$-sample of $\Sigma$ with $\varepsilon < 1/\sqrt{3}$, then any critical point $c$ of $h$ is either shallow, i.e. $\text{dist}(c, \Sigma) \leq \varepsilon^2 \cdot \tau$ or is deep, i.e. $\text{dist}(c, \Sigma) \geq (1 - 2\varepsilon^2)\tau$, where $\tau = \text{reach}(\Sigma)$. 
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Reconstruction Results

1. [Niyogi, Smale, Weinberger’04] Union of balls of same radius within appr. range centered at points of a **uniform** sample is homotopy equiv to manifold.

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3. [Dey-Giesen-Ramos-S’05] The **boundary** of union of Sm’s of deep crit pts of a **tight adaptive sample** is homeomorphic to surface in $\mathbb{R}^3$.

4. [S’08] Under **uniform sampling**, union of Sm’s of shallow crit pts is homotopy equiv to the sampled manifold and that of **deep** ones is homotopy equiv to the complement of manifold.
Theorem.
Let $P \subset \mathbb{R}^n$ and $h$ be the induced distance function. If $h(c_1) < \cdots < h(c_k)$ are critical points of $h$, then for any submanifold $\Sigma$ of $\mathbb{R}^n$ densely sampled by $P$, there is a $1 < j < k$, such that:

\[
\bigcup_{i=1}^{j} \text{Sm}(c_i) \simeq \Sigma \quad \text{and} \quad \bigcup_{i=j+1}^{k} \text{Sm}(c_i) \simeq \Sigma^c
\]
[Lieutier’04] The medial axis of any bounded open subset of $\mathbb{R}^n$ is homotopy equivalent to it.

[Giesen-Ramos-S’06] Union of unstable manifolds of deep critical points captures the homotopy type of the medial axis and can be used to approximate it.

[Ramos-S’07] Edelsbrunner’s WRAP algorithm can be modified using the separation of critical points to guarantee topology.

[Cohen-Steiner-Lieutier-Chazal’06] Sampling criteria for guaranteed reconstruction of compact sets from a large family of non-smooth objects.
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Thank You!