Connecting Complexity Classes, Weak Formal Theories, and Propositional Proof Systems

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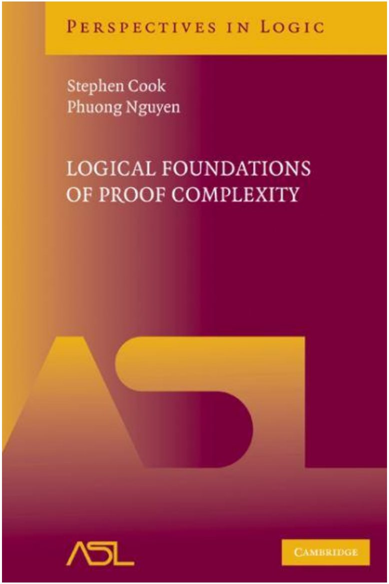
Reference


PERSPECTIVES IN LOGIC
Stephen Cook
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LOGICAL FOUNDATIONS
OF PROOF COMPLEXITY
CAMBRIDGE

Propositional Proof Systems

- A proof system is a polytime map

$$f : \{0,1\}^* \xrightarrow{\text{onto}} \{\text{tautologies}\}$$

  If $f(x) = A$, then x is a proof of A.

- The system is polybounded iff for some polynomial $p(n)$, every tautology of length n has a proof of length at most $p(n)$.

Simple Fact

NP = **co-NP** iff there exists a polybounded proof system.

Conjecture

NP $\neq$ **co-NP** (i.e. there is no polybounded proof system).

- **Activity:** Try to prove specific proof systems are not super.

Frege Systems for Propositional Calculus (Hilbert Style systems)

- Finitely many axiom schemes and rule schemes.
- Must be implicationally complete.

Example for connectives $\vee, \neg$

- **Axiom scheme:** $\neg A \vee A$
- **Rules:** $\dfrac{A}{B \vee A}$ $\quad \dfrac{A \vee A}{A}$ $\quad \dfrac{(A \vee B) \vee C}{A \vee (B \vee C)}$ $\quad \dfrac{A \vee B \qquad \neg A \vee C}{B \vee C}$

- All Frege systems p-simulate each other.

Definition

Proof system f p-simulates proof system g if $\exists$ polytime T such that

$$f(T(x)) = g(x)$$

- Gentzen's propositional LK is p-equivalent to every Frege system.

Are Frege systems polybounded?

To disprove this, we need a family of hard tautologies.

Possible example:

Pigeonhole Principle:

If $n+1$ pigeons are placed in n holes, some hole has at least 2 pigeons.

Atoms $p_{ij}, i \in [n+1], j \in [n]$ (pigeon i placed in hole j)

$\neg\mathrm{PHP}_n^{n+1}$ is the conjunction of clauses:

1. $(p_{i1} \vee \ldots \vee p_{in}), i \in [n+1]$ (pigeon i placed in some hole)
2. $(\neg p_{ik} \vee \neg p_{jk}), i < j \in [n+1], k \in [n]$ (pigeons i, j not both in hole k)

- $\neg\mathrm{PHP}_n^{n+1}$ is unsatisfiable
- $O(n^3)$ clauses

Conjecture (C. 1979)

The tautologies $\{\mathrm{PHP}_n^{n+1}\}$ do not have polysize Frege proofs.

Milestone Result:

Theorem (Haken 1985)

$\{\neg\mathrm{PHP}_n^{n+1}\}$ *do not have polysize resolution refutations.*

Theorem (Buss 1987)

$\{\mathrm{PHP}_n^{n+1}\}$ *have polysize Frege proofs*

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Proof.

- Counting is in $\mathbf{NC}^1$ (i.e. polynomial formula size).
- Define $Count_{n,k}(p_1, ..., p_n) \leftrightarrow$ Exactly k of $p_1, ..., p_n$ are true.
- Family $\langle Count_{n,k} \rangle$ has poly formula size $(n^{O(1)})$
- Hence there are polysize formulas

  $$A_k(\vec{p_{ij}}) \equiv \text{“Pigeons } 1, ..., k \text{ occupy at least } k \text{ holes”}$$

- Prove if no two pigeons occupy same hole, then

  $$A_1 \rightarrow A_2 \rightarrow ... \rightarrow A_{n+1}$$

  to get a contradiction.

$\square$

- So the tautologies $\{\mathrm{PHP}_n^{n+1}\}$ are not hard for Frege systems.
- The question of whether Frege systems are polybounded remains wide open.
- Later we will give tautology families that might be hard for Frege systems.

Thesis

If a hard tautology family (for Frege systems) comes from a combinatorial principle, then that principle should not be provable using $\mathbf{NC}^1$ concepts.

- This motivates associating a first-order theory **VC** with a complexity class **C**. The theorems of **VC** are those that can be proved using concepts from **C**.
- Associated with **VC** is a propositional proof system **CFrege**.
- Each universal theorem of **VC** can be translated to a tautology family with polysize proofs in **CFrege**.

The three-way connection

1. **C** is a complexity class.
2. **VC** is a theory whose proofs use concepts from **C**.
3. **CFrege** is a propositional proof system such that the lines in a **CFrege**-proof express concepts from **C**.

Note that $\mathbf{NC}^1$**Frege** is the same as **Frege**.

Example triple: $\{\mathbf{NC}^1, \mathbf{VNC}^1, \mathbf{Frege}\}$

Theories for Polytime reasoning:

- **PV** [C. 75] Equational theory with function symbols for all polytime functions $f : \mathbb{N}^k \to \mathbb{N}$. Inspired by Skolem's Primitive Recursive Arithmetic (1923).

- **PV** functions introduced via Cobham's 1963 characterization of polytime functions:
  - The least class containing initial functions and closed under composition and limited recursion on notation.
  - The axioms and rules include recursive defining equations for eachfunction symbol and
  - **Rule:** Equational Induction on (binary) notation

$$\frac{f(0) = g(0), \qquad \{f(xi) = h_i(x, f(x)), g(xi) = h_i(x, g(x)) : i = 0, 1\}}{f(x) = g(x)}$$

The first-order version of PV

- **PV Nowadays**
  - a first-order theory with polytime function symbols as before, and
  - universal axioms based on Cobham's theorem, but
  - the rule induction on notation is replaced by the axiom scheme induction on notation:
    $$\Big[\varphi(0) \wedge \forall x \Big(\varphi(x) \supset (\varphi(x0) \wedge \varphi(x1))\Big)\Big] \supset \forall y \varphi(y)$$
    where $\varphi(x)$ is a quantifier-free formula.
  - Note that an induction proof of $\varphi(x)$ can be unwound in just $|x|$ steps, where $|x|$ is the binary length of x.
- First-order **PV** is a conservative extension of equational **PV**.

Theorem (Dowd)

PV *proves the induction scheme for open formulas* φ*:*

$$\Big[\varphi(0) \wedge \forall x \Big(\varphi(x) \supset \varphi(x+1)\Big)\Big] \supset \forall y \varphi(y)$$

- But this induction proof of $\varphi(x)$ requires $2^{|x|}$ steps to unwind.
- Dowd's theorem is proved using binary search.

PV Witnessing Theorem

If **PV** $\vdash \forall\vec{x}\exists y\varphi(\vec{x}, y)$, where φ is open (i.e. expresses a polytime predicate) then there is a polytime f such that

$$\mathbf{PV} \vdash \forall\vec{x}\varphi(\vec{x}, f(\vec{x}))$$

Proof.

Since **PV** is a universal theory, this is an easy consequence of the Herbrand Theorem. □

- $\mathbf{S}^1_2$ [Buss 86]: Finitely axiomatizable first-order theory, including induction on notation for **NP** formulas, associated with class **P**.
- **Theorem [Buss 86].** **PV** and $\mathbf{S}^1_2(\mathbf{PV})$ prove the same $\forall\exists\varphi$ theorems, where φ expresses a polytime predicate.
- A function $f(x)$ is *provably total* in a theory T if

$$\mathrm{T} \vdash \forall x\exists y\varphi(x, y)$$

  where $\varphi(x, y)$ is a Σ^b_1 formula expressing $y = f(x)$.
- The provably total functions of **PV** (and of $\mathbf{S}^1_2$) are the polytime functions.

PV is a ROBUST MINIMAL THEORY for **P**.

Observations: ('Polytime proof' means PV proof.)

1. 'Natural' polytime algorithms usually have polytime correctness proofs.
2. Combinatorial theorems of interest in computer science often have polytime proofs.
   - Kuratowski's Theorem
   - Hall's Theorem
   - Menger's Theorem
   - Linear Algebra (Cayley-Hamilton, properties of determinants,...)
   - Extended Euclidean Algorithm

Possible counter-example to 1: Primes in **P**. [AKS 04]

- The correctness statement implies

$$\neg Prime(n) \wedge n \geq 2 \supset \exists d(1 < d < n \wedge d | n)$$

- If **PV** proves this, then by the Witnessing Theorem, the divisor d can be computed from n in polytime, so this implies a polytime integer factoring algorithm.
- (The same reasoning applies to any polytime algorithm for Primes.)

Theses: ('Polytime proof' means PV proof.)

1. 'Natural' polytime algorithms usually have polytime correctness proofs.
2. Combinatorial theorems of interest in computer science often have polytime proofs.

Possible counter-example to 2:

- Fermat's Little Theorem:

$$\mathrm{Prime}(n) \wedge 1 \leq a < n \ \rightarrow \ a^{n-1} \equiv 1 (\bmod n)$$

- Contrapositive:

$$\forall a, n \exists d < n (a^{n-1} \not\equiv 1 (\bmod n) \ \rightarrow \ d \neq 1 \wedge d | n)$$

- Thus if **PV** proves this then by the Witnessing Theorem, d can be found from a, n in time polynomial in $|n|$.
- This leads to a probabilistic polytime algorithm for factoring.

Propositional proof system associated with P?

- Recall: Frege systems are associated with $\mathbf{NC}^1$
- A problem is in $\mathbf{NC}^1$ iff it can be solved by a uniform polysize family of Boolean formulas.
- A Frege proof consists of a sequence of Boolean formulas, where each formula is an axiom or follows from earlier formulas by simple rules.
- NOTE: A problem is in **P** iff it can be solved by a uniform polysize family of Boolean circuits.
- So a proof for a polytime propositional proof system should be a sequence of Boolean circuits, with axioms and rules as for Frege systems.
- A Boolean circuit can be described by a straight line program in which each line defines the value of a gate in terms of previous gate values.
- So we abbreviate circuit outputs by introducing new *extension variables* defined by formulas.

Extended Frege Systems (EFrege systems, or "P-Frege Systems")

- Extend Frege systems by allowing new extension variables and their definitions:

$$p \leftrightarrow A$$

  for any atom p and formula A, provided p does not occur in A, or earlier in the proof, or in the conclusion.

- p may occur in a later formula A'.
  This allows *lines* in a Frege proof to be massively abbreviated.

$$p_1 \leftrightarrow A_1,\ \ p_2 \leftrightarrow A_2(p_1), \ldots, p_n \leftrightarrow A_n(p_1, \ldots, p_{n-1})$$

- Lines in an Extended Frege proof are like Boolean circuits. (The new atoms are like gates in the circuit.)

Historical Notes

- Extended Resolution (**ER**) introduced by G.S. Tseitin in 1966.
  - ER extends the resolution proof system by allowing clauses defining new variables.
  - For example, to introduce p so that $p \leftrightarrow (q \vee r)$, add clauses

$$\overline{p} \vee q \vee r, \qquad p \vee \overline{q}, \qquad p \vee \overline{r}$$

- (C. 75) Introduced **PV** and indicated that theorems of **PV** can be translated into polysize families of **ER** proofs.
- (C. 75) also outlined a proof that **PV** proves the soundness of **ER** (reflection principle).
- (C.-Reckhow 74 and 79) Introduced 'Frege Systems' and **EFrege** systems and pointed out the latter are *p*-equivalent to **ER**.

Recall the three-way connection

- **C** is a complexity class.
- **VC** is a theory whose proofs use concepts from **C**.
- **CFrege** is a propositional proof system such that the lines in a **CFrege**-proof express concepts from **C**.

$\mathbf{NC}^1$**Frege** is the same as **Frege**. **PFrege** is the same as **EFrege**.

Example triples

- $\{\mathbf{NC}^1, \mathbf{VNC}^1, \mathbf{Frege}\}$
- $\{\mathbf{P}, \mathbf{PV}, \mathbf{EFrege}\}$

Theorem

- **VC** *proves soundness of* **CFrege**
- *If* **VC** *proves the soundness of proof system* **S**, *then* **CFrege** *p-simulates* **S**.

Recall Historical Notes

- Extended Resolution (**ER**) introduced by G.S. Tseitin in 1966.
- (C. 75) Introduced **PV** and indicated that theorems of **PV** can be translated into polysize families of **ER** proofs.
- (C. 75) also outlined a proof that **PV** proves the soundness of **ER** (reflection principle).
- (C.-Reckhow 74 and 79) Introduced 'Frege Systems' and **EFrege** systems and pointed out the latter are *p*-equivalent to **ER**.
- (Clote 90) 'ALOGTIME and a conjecture of S. A. Cook' introduced first theory **ALV** for $\mathbf{NC}^1$ with translations to Frege systems.
- (Arai 91, 00) 'A bounded arithmetic **AID** for Frege systems' Showed his system **AID** is equivalent to Clote's **ALV**, and proves soundness of Frege using a result of Buss.
- (Krajíček 95) 'Bounded Arithmetic, Propositional Logic, and Complexity Theory' expounded the three way connection.

Complexity Classes

(Google: Complexity Zoo)

$$\mathbf{AC^0} \subset \mathbf{AC^0}(m) \subseteq \mathbf{TC^0} \subseteq \mathbf{NC}^1 \subseteq \mathbf{NC}^2 \subseteq \mathbf{P}$$

Defined by uniform polysize Boolean circuit families

- $\mathbf{AC^0}$ – bounded-depth circuits with unbounded fanin $\wedge, \vee$ (Immerman's **FO**)
- $\mathbf{AC^0}(m)$ – Allow mod m gates $(p_1 + p_2 + \ldots + p_k)$ mod m in above circuits.
- $\mathbf{TC^0}$ – Allow threshold gates (counting class)
- $\mathbf{NC}^1$ – polynomial formula size
- $\mathbf{NC}^2$ – polysize $\log^2$ depth families of Boolean circuits (contains matrix inverse, determinant, etc)
- $\mathbf{P}$ – polysize families of Boolean circuits

Complexity Classes

$$\mathbf{AC^0} \subset \mathbf{AC^0}(m) \subseteq \mathbf{TC^0} \subseteq \mathbf{NC}^1 \subseteq \mathbf{NC}^2 \subseteq \mathbf{P}$$

- **Open question:** $\mathbf{P} = \mathbf{NP}$?
- **Also open:** $\mathbf{AC^0}(6) = \mathbf{NP}$?

Theorem (Razborov-Smolensky 87)

$\mathbf{AC^0}(p^k) \subsetneq \mathbf{TC^0}$, *for every* $k \geq 1$ *and prime* p.

$\mathbf{AC^0} \subset \mathbf{AC^0}(m) \subseteq \mathbf{TC^0} \subseteq \mathbf{NC}^1 \subseteq \mathbf{NC}^2 \subseteq \mathbf{P}$

[C.-Nguyen 2010] presents a unified way to define a first-order theory **VC** (over a two-sorted language) corresponding to a complexity class **C**, including all of the above classes.

In particular:

- $\mathbf{VNC}^1$ is a simplified version of Clote's **ALV** and Arai's **AID**.
- **VP** is a finitely axiomatized theory for polynomial time.
- **VPV** is the two-sorted version **PV**, with function symbols for all polytime functions.
- **VPV** is a conservative extension of **VP**.

Also the book describes propositional translations of the theories to the corresponding proof systems.

Two-sorted theories cont'd

- The base theory $\mathbf{V^0}$ ($= \mathbf{VAC}^0$) corresponds to $\mathbf{AC^0}$.
- The pigeonhole principle $\mathrm{PHP}(n, X)$ is expressed by the following two-sorted formula where X is a bit-array, and $X(i,j)$ means that pigeon i is mapped to hole j:

$$\forall i \leq n \exists j < n X(i,j) \rightarrow \exists i, k \leq n \exists j < n (i < k \wedge X(i,j) \wedge X(k,j))$$

- For each constant n, This translates into a propositional formula equivalent to PHP_n^{n+1}
- Each bit $X(i,j)$ translates to a Boolean variable p_{ij}^X.
- Bounded quantifiers $\exists i \leq n$ and $\forall i \leq n$ translate to

$$\bigvee_{i=1}^{n} \qquad \bigwedge_{i=1}^{n}$$

respectively.
- Does $\mathbf{V^0}$ prove $\mathrm{PHP}(n, X)$?

What is the proof system AC^0-Frege?

- Answer: Restrict formulas in a Frege proof to have depth $\leq d$, for some constant d.

Theorem (Ajtai 88)

There are no polysize $\mathbf{AC^0}$*-Frege proofs of* $\{\mathrm{PHP}_n^{n+1}\}$

- Since Σ_0^B theorems of $\mathbf{V^0}$ translate to polysize families of $\mathbf{AC^0}$-Frege proofs, we answer our earlier question:

Corollary

$\mathbf{V^0}$ *does not prove* $\mathrm{PHP}(n, X)$.

- $\mathbf{VTC}^0$ corresponds to $\mathbf{TC^0}$ (the counting class), so it is easy to see that
$$\mathbf{VTC}^0 \vdash \mathrm{PHP}(n, X)$$
- Since $\mathbf{VTC}^0 \subseteq \mathbf{VNC}^1$, if follows that $\mathbf{VNC}^1 \vdash \mathrm{PHP}(n, X)$, so we obtain Buss's Theorem that $\{\mathrm{PHP}_n^{n+1}\}$ has polysize Frege proofs as a corollary.
- (Recall that $\mathbf{NC}^1$-Frege = Frege.)

We can associate propositional systems with other classes

- $\mathbf{AC^0}(m)$-Frege
- $\mathbf{TC^0}$-Frege Has polysize proofs of $\{\mathrm{PHP}_n^{n+1}\}$
- $\mathbf{NC}^1$-Frege = Frege
- $\mathbf{NC}^2$-Frege
- **PFrege** = **EFrege** (Extended Frege):
  - Allows introduction of new variables by definition, corresponding to gates in a circuit

Surprising open question

Is $\mathbf{AC^0}(2)$**Frege** polybounded?

This is open, despite the [Razborov-Smolensky 87] proof that $\mathbf{AC^0}(p) \not\subseteq \mathbf{TC^0}$ for any prime p.

Conjecture

PHP_n^{n+1} do not have polysize $\mathbf{AC^0}(2)$**Frege** proofs.

A weaker conjecture:

$$\mathbf{VAC}^0(2) \nvdash \mathrm{PHP}(n, X)$$

but this is also open.

Hard tautology families for Frege systems?

Consider the 'hard matrix identity'

$$AB = I \to BA = I$$

where A, B are $n \times n$ matrices.

- If the entries are in $GF(2)$ (or even in $\mathbb{Z}$ or $\mathbb{Q}$) this translates into a polysize family $\{\varphi_n\}$ of tautologies.
- Proofs of these identities seem to require tools from linear algebra, such as Gaussian Elimination, or the Cayley-Hamilton Theorem.
- Note that computing matrix inverses (over finite fields or $\mathbb{Z}$ or $\mathbb{Q}$) can be done in $\mathbf{NC}^2$, but apparently not in $\mathbf{NC}^1$.

Conjecture (e.g. [Soltys-C. 04])

$\{\varphi_n\}$ do not have polysize Frege proofs.

- This conjecture remains open.

Hard matrix tautologies cont'd

$$AB = I \rightarrow BA = I$$

where A, B are $n \times n$ matrices.

- In [Solys-C. 04] we develop formal theories for linear algebra. Although the standard linear algebra operators are in $\mathbf{NC}^2$, proving their properties seems to require **VP** rather than $\mathbf{VNC}^2$.
- **Question:** Do these matrix identies have polysize $\mathbf{NC}^2$-Frege proofs? **Answer:** [Hrubeš -Tzameret 2011]: Yes, and they have quasi-polysize Frege proofs.
- But [Hrubeš -Tzameret] leave open the question of whether the theory $\mathbf{VNC}^2$ proves the identities.

What about hard tautologies for EFrege systems?

- It's difficult to think of interesting universal combinatorial theorems involving polytime functions, which cannot be proved in **VPV**.
- However mathematical logic suggests consistency statements.
- We know [Gödel 31] **con**(**VPV**) is universal sentence not provable in **VPV**.
- It seems plausible to conjecture that the corresponding tautology family does not have polysize **EFrege** proofs.
- For that matter what about **con**(**PA**), or **con**(**ZF**)?
- Let $[\mathbf{con}(\mathbf{ZF})]_n$ be a propositional tautology asserting **ZF** has no proof of $0 \neq 0$ of length n or less.
- It's hard to imagine how the family $\left\{[\mathbf{con}(\mathbf{ZF})]_n\right\}$ could have polysize **EFrege** proofs, unless **EFrege** is polybounded.

Concluding thought

- Given the extreme difficulty of proving lower bounds even for simple proof systems (such as $\mathbf{AC^0}(2)\mathbf{Frege}$), perhaps we should contemplate the possibility

$$\mathbf{NP} = \mathbf{coNP}$$

- This might surprise complexity-theorists, but would not otherwise have the potentially earth-shaking consequences of

$$\mathbf{P} = \mathbf{NP}$$