[10] 1. Prove that if $A$ is a language and both $A$ and $\overline{A}$ are semidecidable, then $A$ is decidable.

Solution:
Suppose $A = \mathcal{L}(M_1)$ and $\overline{A} = \mathcal{L}(M_2)$. We define a Turing Machine $M_3$ to execute the following algorithm:

On input $w \in \Sigma^*$
for $t = 1 \ldots \infty$
    run $M_1$ and $M_2$ on input $w$ for $t$ steps.
    if $M_1$ accepts $w$ then ACCEPT
    else if $M_2$ accepts $w$ then REJECT
end for

Clearly $M_3$ halts on all inputs, and accepts $w$ iff $w \in A$. 
Let $E$ be an enumerator, and let $\mathcal{L}(E)$ be the set of strings enumerated by $E$. If every string in $\mathcal{L}(E)$ has the form $\langle M \rangle$, where $M$ is a decider (i.e. $M$ is a Turing machine which halts on all inputs), is it possible that for every decidable set $A$ there is a string $\langle M \rangle$ in $\mathcal{L}(E)$ such that $\mathcal{L}(M) = A$? Justify your answer.

**Solution:**
The answer is NO.
We may assume that $\mathcal{L}(E)$ is infinite, because there are infinitely many decidable languages. Let

$$\Sigma^* = w_1, w_2, w_3, \ldots$$

in lexicographic order.

For $i = 1, 2, \ldots$ let $\langle M_i \rangle$ be the $i$th string output by the enumerator $E$.

Define $A = \{w_i \mid w_i \notin \mathcal{L}(M_i)\}$.

Then $A$ is decidable, since $M_i$ is a decider and halts on all inputs, and given $w_i$, the number $i$ can be determined, and hence $\langle M_i \rangle$ can be determined by counting the outputs of the enumerator until the $i$th output has been written.

Now suppose $A = \mathcal{L}(M_n)$ for some $n \in \mathbb{N}$.

Then $w_n \in A \iff w_n \in \mathcal{L}(M_n)$ because $A = \mathcal{L}(M_n)$,

but $w_n \in A \iff w_n \notin \mathcal{L}(M_n)$ by definition of $A$.

This is a contradiction. Hence the decidable set $A \neq \mathcal{L}(M)$ for all $\langle M \rangle \in \mathcal{L}(E)$. 

3. Let $B = \{ (G, G') | G$ and $G'$ are context free grammars and $\mathcal{L}(G) \subseteq \mathcal{L}(G') \}$. Is $B$ semidecidable? Is $\overline{B}$ semidecidable? Justify your answers using results presented in lectures. (You may complete your answer on the next page.)

**Answer:**

(1) $B$ is not semidecidable, but (2) $\overline{B}$ is semidecidable.

To prove (1), we show $\text{All}_{CFG} \leq_m B$.

This proves (1) because $\text{All}_{CFG}$ is not semidecidable.

Recall that $\text{All}_{CFG} = \{ (G) | G$ is a CFG and $\mathcal{L}(G) = \Sigma^* \}$.

Given $(G)$ (where $G$ is a CFG) we must construct a pair $(G_1, G_2)$ of CFG’s such that

$$\mathcal{L}(G) = \Sigma^* \Leftrightarrow \mathcal{L}(G_1) \subseteq \mathcal{L}(G_2)$$

Let $G_1$ be any CFG such that $\mathcal{L}(G_1) = \Sigma^*$, and let $G_2 = G$.

Clearly this suffices.

To prove (2), it suffices to use the certificate characterization of SD. Given a pair $(G, G')$ of CFG’s, a certificate showing $\mathcal{L}(G) \not\subseteq \mathcal{L}(G')$ is a string $w$ such that $w \in \mathcal{L}(G)$ and $w \notin \mathcal{L}(G')$. Here we use the fact that $\text{A}_{CFG}$ is decidable, where

$$\text{A}_{CFG} = \{ (G, w) | G$ is a CFG and $w \in \mathcal{L}(G) \}$$