1. Recall that a set $A \subseteq \Sigma^*$ is semidecidable iff $A = L(M)$ for some Turing machine $M$.

Recall that an enumerator $E$ is a Turing machine which has one ‘work tape’ and one write-only output tape. $E$ enumerates a set $A \subseteq \Sigma^*$ by starting with a blank work tape, and writing the members of $A$ (in any order, and possibly with repetitions) on its output tape. If $A$ is infinite, then $E$ never halts.

Show that a set $A$ is semidecidable if and only if $A$ is enumerated by some enumerator.

Solution:

$(\Leftarrow)$ Let $A$ be the set enumerated by an enumerator $E$. Define a Turing machine $M_A$ to do the following:

$M_A$ on input $w$ runs the enumerator $E$. If $E$ writes $w$ on its output tape, then $M_A$ accepts $w$. Otherwise $M_A$ never halts.

Clearly $L(M_A) = A$. Hence $A$ is semidecidable.

$(\Rightarrow)$ Let $A = L(M)$ for some TM $M$. Let $\Sigma^* = \{w_1, w_2, \ldots\}$ where we have written $w_1, w_2, \ldots$ in lexicographic order.

Let an enumerator $E$ follow the algorithm below:

for $t = 1, \ldots, \infty$
    for $i = 1 \ldots t$
        Run $M$ on input $w_i$ for $t$ steps.
        If $M$ accepts $w_i$ then write $w_i$ on output tape.
    end for
end for

Then $E$ enumerates $A$. Clearly every $w_i$ written is in $A$, and if $w_i \in A$, then $M$ accepts $w_i$ in at most $t$ steps for some $t$, and hence $w_i$ is written on the output tape.

2. For this problem assume $\Sigma = \{0, 1\}$. Recall that $\text{PAL} = \{ww^R \mid w \in \{0,1\}^*\}$.

Let

$$A = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \subseteq \text{PAL}\}$$

Is $A$ semidecidable? Is $\overline{A}$ semidecidable. Justify your answers.

DO NOT USE RICE's THEOREM

(You may continue your solution on the next page.)

Solution:

$A \not\in SD$ but $\overline{A} \in SD$.

To prove $\overline{A} \in SD$ we use the certificate characterization of $SD$. Define $R(x, y)$ to be true iff either $x$ is not of the form $\langle M \rangle$, where $M$ is a TM, or $x = \langle M \rangle$ for some TM $M$ and $y$ codes an accepting computation of $M$ on some input $w$ such that $w \notin \text{PAL}$. Clearly $R(x, y)$ is computable. Also $x \in \overline{A}$ iff $\exists y R(x, y)$.
To prove $A \not\in SD$ we show $\overline{HB} \leq_m A$. This suffices, because $\overline{HB} \not\in SD$. Given a TM $M$ we construct a TM $M'$ such that $M'$ on input $w$ runs $M$ on a blank tape. If $M$ halts, then $M'$ accepts $w$.

Thus if $\langle M \rangle \in \overline{HB}$, then $L(M') = \emptyset$, so $\langle M' \rangle \in A$.

If $\langle M \rangle \in HB$, then $L(M') = \Sigma^*$, so so $\langle M' \rangle \not\in A$.

3. Let $B$ be an infinite set of Turing machine descriptions such that for each $\langle M \rangle \in B$

   (a) $M$ is a Turing machine that halts on all inputs, and

   (b) $M$ computes a function $f_M : \Sigma^* \rightarrow \Sigma^*$.

Suppose that $B$ is semi-decidable. Prove that there is a computable function $g : \Sigma^* \rightarrow \Sigma^*$ such that $g \neq f_M$, for all $\langle M \rangle \in B$. (Make sure that $g(w)$ is defined for all strings $w \in \Sigma^*$.)

**Solution:**

We use a diagonal argument.

Since $B \in SD$, there is an enumerator $E$ which enumerates $L(B)$. Let the output of the enumerator be $\langle M_1 \rangle, \langle M_2 \rangle, \langle M_3 \rangle, \ldots$.

Let $\{w_1,w_2,w_3,\ldots\}$ be $\Sigma^*$ in lexicographic order.

Define $g(w_i) = f_{M_i}(w_i)a$, where $a \in \Sigma$.

Then $g$ is computable, because, given $w_i$, we can watch the output of $E$ to find $\langle M_i \rangle$, and then compute $f_{M_i}(w_i)$, and then add $a$.

If $g = f_M$ for some $M \in B$, then $g = f_{M_i}$ for some $i$. But $g(w_i) = f_{M_i}(w_i)a \neq f_{M_i}(w_i)$, so $g \neq f_{M_i}$. Therefore $g \neq f_M$, for all $\langle M \rangle \in B$. 
