1. Let \( B \subseteq \Sigma^* \) be a decidable set. For each \( y \in \Sigma^* \) let
\[
B_y = \{ x \in \Sigma^* \mid \langle x, y \rangle \in B \}
\]
Show that there is a decidable language \( C \subseteq \Sigma^* \) such that \( C \neq B_y \) for all \( y \in \Sigma^* \).

**Solution:**
A simple diagonal argument suffices. Let
\[
C = \{ x \mid \langle x, x \rangle \notin B \}
\]
\( C \) is decidable because \( B \) is decidable.
For each \( y \in \Sigma^* \), \( C \) differs from \( B_y \) on input \( y \), because \( y \in C \iff \langle y, y \rangle \notin B \), by definition of \( C \).
But \( y \in B_y \iff \langle y, y \rangle \in B \), by definition of \( B_y \).

2. Let
\[
A = \{ \langle M \rangle : M \text{ is a TM and } L(M) \text{ has at least two strings} \}
\]
Is \( A \) semidecidable? Is \( \overline{A} \) semidecidable?

**Solution:**
\( A \) is semidecidable. To show this we use the certificate characterization of semidecidable.
Define a relation \( R(x, y) \) which defines the following condition:
\[ [x = \langle M \rangle] \text{ for some Turing machine } M, \text{ and } y \text{ codes a tuple } \]
\[
y = \langle x_1, C_1, x_2, C_2 \rangle
\]
where \( x_1 \neq x_2 \) and \( C_i \) codes an accepting computation of \( M \) on input \( x_i \), for \( i \in \{1, 2\} \).
Then \( R \) is decidable, and
\[
\langle M \rangle \in A \iff \exists y R(\langle M \rangle, y)
\]
\( \overline{A} \) is not semidecidable.
It suffices to show \( \overline{HB} \leq_m \overline{A} \), since \( \overline{HB} \) is not semidecidable.
This is equivalent to showing \( HB \leq_m A \). Here is that reduction: Given a Turing machine \( M \), we define a Turing machine \( M' \) which does the following:
\( M' \) on input \( x \) runs \( M \) on a blank tape. If \( M \) halts, then \( M' \) accepts \( x \).
Thus if \( \langle M \rangle \in HB \) then \( M \) halts on a blank tape, so \( L(M') = \Sigma^* \), so \( \langle M' \rangle \in A \).
If \( \langle M \rangle \notin HB \), then \( L(M') = \emptyset \), so \( \langle M' \rangle \notin A \).
3. Prove that every infinite semidecidable set has an infinite decidable subset.

Hint: Consider enumerators.

Solution:
Suppose that $A$ is semidecidable. Then $A$ is the output of some enumerator $E$. Now define $B$ by the condition

$$x \in B \iff x \text{ is output by } E, \text{ and when } x \text{ is output by } E \text{ for the first time,}$$

$$E \text{ has not yet output any string that is lexicographically larger than } x.$$

Then clearly $B \subseteq A$, because every member of $B$ is output by $E$.

Further, $B$ is decidable, because to determine whether a given string $x$ is in $B$, just watch the output of $E$ until either some string lexicographically larger than $x$ is output before $x$ is output (in which case $x$ is not in $B$) or $x$ is output, in which case $x$ is in $B$.

Finally $B$ is infinite, because for every string $x$ in $B$, when $x$ is output by $E$ for the first time, since $A$ is infinite there must be a first time after this that $E$ outputs a string $y$ which is lexicographically larger than $x$, and this $y$ is also in $B$. 