Due: Friday, November 17, beginning of tutorial

1. Prove or disprove:

There is a primitive recursive function \( f(x, y) \) such that for every unary primitive recursive function \( g(x) \) there is a number \( a \in \mathbb{N} \) such that

\[
g(x) = f(a, x)
\]

for all \( x \in \mathbb{N} \).

Solution:
We will disprove this by a simple diagonal argument.

Suppose such a primitive recursive function \( f(x, y) \) exists.

Let \( D(x) = f(x, x) + 1 \) Then \( D \) is a primitive recursive function, so there is a number \( d \) such that

\[
D(x) = f(d, x)
\]

for all \( x \in \mathbb{N} \). In particular \( D(d) = f(d, d) \). But by definition \( D(d) = f(d, d) + 1 \), which is a contradiction. So there can be no such \( f \).

2. Do Exercise 9, page 64 in the Notes. (Show that the functions \( \text{Bit}(x, i) \) and \( \text{NumOnes}(x) \) are primitive recursive.) The functions shown to be primitive recursive on page 65 should be helpful.

Solution:

\( \text{Bit}(x, i) = \text{rm}(q(x, 2^i), 2) \), so Bit is obtained by composition from functions which have been shown to be primitive recursive in the Notes (see page 65), and hence Bit is itself primitive recursive.

\( \text{NumOnes}(x) = \sum_{0 \leq i \leq x} \text{Bit}(x, i) \), so that NumOnes is obtained by a bounded sum (see page 63) from a primitive recursive function, so that it is itself primitive recursive.

3. Let Primes be the set of prime numbers.

Let \( A = \{x \mid \text{dom} \{x\}_1 = \text{Primes}\} \).

Is \( A \) r.e.? Is \( A^c \) r.e.? Justify your answers. (Do not use Rice’s Theorem.)

Solution:

Neither \( A \) nor \( A^c \) is r.e.

To show \( A \) is not r.e. it suffices to show \( H^c \leq_m A \), since \( H^c \) is not r.e (see page 73 of the Notes for the definition of \( H \)).

Thus we want a total computable function \( f(x) \) such that program \( \{x\} \) fails to halt on input 0 iff \( \text{dom} \{f(x)\}_1 = \text{Primes} \).

By the special case of the S-m-n theorem (page 73) it suffices to give a computable function \( g(x, y) \) such that \( g(x, y) = \infty \) iff \( y \in \text{Primes} \), so \( \{f(x)\}_1(y) = \infty \) iff \( y \in \text{Primes} \). Now simply note that this definition of \( g(x, y) \) is computable, by definition by cases.
To show $A^c$ is not r.e. it suffices to show $H^c \leq_m A^c$, which is the same as showing $H \leq_m A$. The proof is almost the same as above, except we replace Primes with the complement of Primes in defining $g(x, y)$.

Thus we want a total computable function $f(x)$ such that program $\{x\}$ halts on input $0$ iff $\text{dom}(\{f(x)\}_1) = \text{Primes}$.

By the special case of the S-m-n theorem it suffices to give a computable function $g(x, y)$ such that $g(x, y) = \infty$ iff $y \notin \text{Primes}$, so $\{f(x)\}_1(y) = \infty$ iff $y \notin \text{Primes}$. Again simply note that this definition of $g(x, y)$ is computable, by definition by cases.

4. Let $B = \{x \mid \text{ran}(\{x\}) \text{ has at most one number}\}$.

Is $B$ r.e.? Is $B^c$ r.e.? Justify your answers. (Do not use Rice’s Theorem.)

**Solution:**

$B$ is not r.e., but $B^c$ is r.e.

To show $B^c$ is r.e., we use the ”certificate” characterization of r.e. sets – this is item iii) in the bottom of page 77 in the Notes. Given an input $x$ to $B$, we must give a certificate $c$ which we can use to verify that $x \notin B$; i.e. we can use $c$ to show that the range of the function $\{x\}$ has at least two distinct numbers. This certificate is a number coding the tuple $(r_1, y_1, r_2, y_2)$, where $r_1$ and $r_2$ are inputs to the program $\{x\}$ and $y_1$ and $y_2$ code the (halting) computations of program $\{x\}$ on inputs $r_1$ and $r_2$ respectively, and $U(y_1) \neq U(y_2)$, where $U$ is the output function (see bottom of page 68 in the Notes).

We can formalize this by using the Kleene $T$-predicate. Thus given an input $x$ we can verify that $(r_1, y_1, r_2, y_2)$ is a suitable certificate for $x$ if

$$T_1(x, r_1, y_1) \land T_1(x, r_2, y_2) \land (U(y_1) \neq U(y_2))$$

To show that $B$ is not r.e., we show $H^c \leq_m B$, which is the same as showing $H \leq_m B^c$. Thus we want a total computable function $f(x)$ such that

$$\{x\}_1(0) \neq \infty \iff \text{ran}(\{f(x)\}_1) \text{ has at least 2 distinct numbers}$$

By the special case of the S-m-n theorem, we want a computable function $g(x, y) = \{f(x)\}_1(y)$ as above. For this, we simply define

$$g(x, y) = y \cdot (\{x\}_1(0) + 1) = y \cdot (\Phi_1(x, 0) + 1)$$

where $\Phi_1$ is the universal function (page 69).