Due: Friday, October 20, beginning of tutorial

1. Do Exercise 11, page 50 of the Notes. (Application of compactness.)

Solution:
We are given a set $\Phi = \{A_1, A_2, \ldots\}$ of $L$-sentences such that for all sufficiently large $i$

$$\{A_1, \ldots, A_{i-1}\} \not\models A_i \quad (1)$$

We are to prove that $\Phi$ is not finitely axiomatizable.

Suppose to the contrary that $\Phi$ is finitely axiomatizable. Then there is a finite set $\Gamma$ of $L$-sentences such that $\Phi$ and $\Gamma$ have the same set of models. Since $\Gamma$ is finite, there is a formula $B$ which is the conjunction of all the formulas in $\Gamma$. Then $\Phi$ and $B$ have the same set of models, and hence by definition of $\models$,

$$\Phi \models B$$

Thus by compactness, $B$ is a logical consequence of some finite subset $\{A_1, \ldots, A_{i-1}\}$ of $\Phi$, where we may assume that $i$ is 'sufficiently large' in the sense of (1). Then every model of $\{A_1, \ldots, A_{i-1}\}$ is a model of $B$, and hence a model of $\Phi$, and hence a model of $A_i$. But then $\{A_1, \ldots, A_{i-1}\} \models A_i$, which contradicts (1).

2. Do Exercise 13 on page 51 of the Notes (Show that the theory of successor is not finitely axiomatizable.)

Solution:
We are to show that $Th(s)$ (as defined in the Notes on page 51) is not finitely axiomatizable. By Problem 3 above, it suffices to show that for all $i \geq 4$, axiom $S_{i+1}$ is not a logical consequence of axioms $S_1 \ldots S_i$. For this it suffices to define a structure $M$ which satisfies axioms $S_1 \ldots S_i$ but fails to satisfy axiom $S_{i+1}$.

Here is the structure $M$. The universe $M$ is the disjoint union of $\mathbb{N}$ and the set of $i - 2$ new elements $\{a_0, \ldots, a_{i-3}\}$. We define $0^M = 0$, and define $s^M$ to be the true successor on elements of $\mathbb{N}$, and define $s^M(a_j) = a_k$, where $k = j + 1 \mod (i - 2)$. Thus the new elements $a_j$ form a loop of size $i - 2$ under the successor function. Then it is easy to see that axioms $S_1$ to $S_i$ are satisfied, but axiom $S_{i+1}$ is not satisfied, since there is a loop of size $i - 2$.

3. Let $A$ be the sentence $\forall x \neg Pxx$.
Let $B$ be the sentence $\forall x \forall y \forall z ((\neg Pxy \lor \neg Pyz) \lor Pxz)$.
Let $C$ be the sentence $\forall x \forall y (\neg Pxy \lor \neg Pyx)$.

Give an LK proof of the sequent $A, B \rightarrow C$. You may use $A$, $B$, $C$ as abbreviations in your proof, and you can leave out exchanges and weakenings.

Solution:
In the LK proof below, we have omitted weakenings and exchanges.
\[
\begin{align*}
\frac{Pbc \to Pbc}{\neg Pbc, Pbc \to (\neg left)} & \quad \frac{Pcb \to Pcb}{\neg Pcb, Pcb \to (\neg left)} & \quad \frac{Pbb \to Pbb}{\neg Pbb, Pbb \to (\neg left)} \\
\frac{(\neg Pbc \lor \neg Pcb), Pbc, Pcb \to}{A, ((\neg Pbc \lor \neg Pcb) \lor Pbb), Pbc, Pcb \to (\forall left)} & \quad \frac{A, Pbb \to}{(\forall left)} \\
\frac{A, ((\neg Pbc \lor \neg Pcb) \lor Pbb), Pbc, Pcb \to}{A, \forall z((\neg Pbc \lor \neg Pcb) \lor Pbz), Pbc, Pcb \to (\forall left)} & \quad \frac{A, \forall z((\neg Pby \lor \neg Pyz) \lor Pbz), Pbc, Pcb \to (\forall left)}{A, \forall yz((\neg Pby \lor \neg Pyz) \lor Pbz), Pbc, Pcb \to (\forall left)} \\
\frac{A, B, Pbc, Pcb \to (\neg right)}{A, B, Pcb \to \neg Pbc (\neg right)} & \quad \frac{A, B \to \neg Pbc, \neg Pcb (\forall left)}{A, B \to (\neg Pbc \lor \neg Pcb) (\forall right (twice))} \quad \frac{A, B \to C}{(\forall right (twice))}
\end{align*}
\]

4. Give an LK proof of the sequent

\[\forall x(x + 0 = x) \to \forall x \forall y(x + (y + 0) = x + y)\]

You do not need to put in weakenings or exchanges. Indicate which LK equality axioms you use.

**Solution:**
The LK equality axioms are

L1: \[ \to a = a \]
L4: \[ a = a, b + 0 = b \to a + (b + 0) = a + b \]

Here is the LK proof:

\[
\begin{align*}
A, B, Pbc, Pcb & \to (\neg right) \\
A, B, Pcb & \to \neg Pbc (\neg right) \\
A, B & \to \neg Pbc, \neg Pcb (\forall right) \\
A, B & \to (\neg Pbc \land \neg Pcb) (\forall right (twice)) \\
A, B & \to C
\end{align*}
\]

Practice Exercises (Do not hand in):
Do Exercise 10, page 25 of the Notes. (Give a sentence whose finite models are finite unions of disjoint directed cycles, and give an infinite model.)