Due: Friday, September 29, beginning of tutorial

NOTE: Each problem set counts 10% of your mark, and it is important to do your own work. You may consult with others concerning the general approach for solving problems on assignments, but you must write up all solutions entirely on your own. Copying assignments is a serious academic offence and will be dealt with accordingly.

1. Do Exercise 9 on page 14 of the course notes. (If $S$ is a propositional sequent, then there is a cut-free PK derivation of $A_S$ from $S$.)

2. Prove that commutativity of addition is not a logical consequence of the first four Peano postulates P1, P2, P3, P4. (See Exercise 4, page 24.)

3. Consider the following axioms for a partial order relation $Pxy$ with biggest element $b$ and smallest element $s$, where $b$ and $s$ are constant symbols:

   $A_1$: $\forall x \neg Pxx$ (irreflexivity)
   $A_2$: $\forall x \forall y \forall z ((Pxy \land Pyz) \supset Pxz)$ (transitivity)
   $A_3$: $\forall x (x = b \lor \exists y (Pxy \land \forall z (\neg Pxz \lor \neg Pzy)))$ (immediate successors exist)
   $A_4$: $\forall x (Px b \lor x = b)$ ($b$ is the biggest)
   $A_5$: $\forall x (x = s \lor \exists y (Pyx \land \forall z (\neg Pyz \lor \neg Pzx)))$ (immediate predecessors exist)
   $A_6$: $\forall x (Pxs \lor x = s)$ ($s$ is the smallest)

(a) Show that the set $\{A_1, ..., A_6\}$ of sentences has models of every positive finite cardinality 1, 2, 3, ... Also show that this set of sentences has an infinite model.

(b) Find a modification $A_3'$ of $A_3$ such that $\{A_1, A_2, A_3'\}$ has an infinite model, but no finite model.

(c) Find two more sentences $B_1, B_2$ such that the finite models of the set

$\{B_1, B_2, A_1, A_2, A_3, A_4\}$

 correspond exactly to the finite binary trees rooted at $b$. The sentence $B_1$ should say that every element has either no (immediate) predecessors or exactly two (immediate) predecessors, and $B_2$ should say that successors are unique.

Practice Exercises (Do not hand in):

- Give a PK proof of the sequent $(P \land (Q \lor R)) \rightarrow ((P \land Q) \lor (P \land R))$.

- Prove the duality theorem, as expressed in Exercise 1 on page 4 of the course notes. Use structural induction on $A$.

- Do Exercise 8 page 12 of the course notes. (Show how to give PK rules for $\implies$.)

- Do Exercise 11 page 16 of the course notes. (Prove the equivalence of the 3 forms of compactness.)