CSC 411: Lecture 06: Decision Trees

Raquel Urtasun & Rich Zemel

University of Toronto

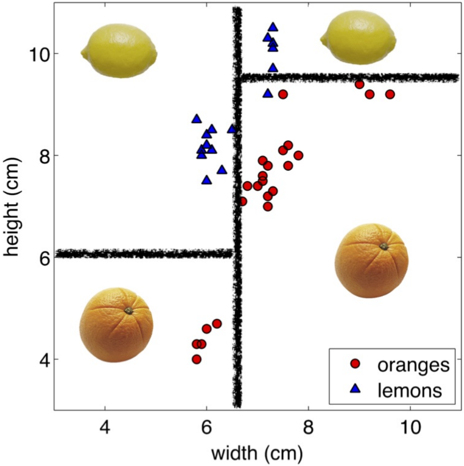
Sep 30, 2015

Today

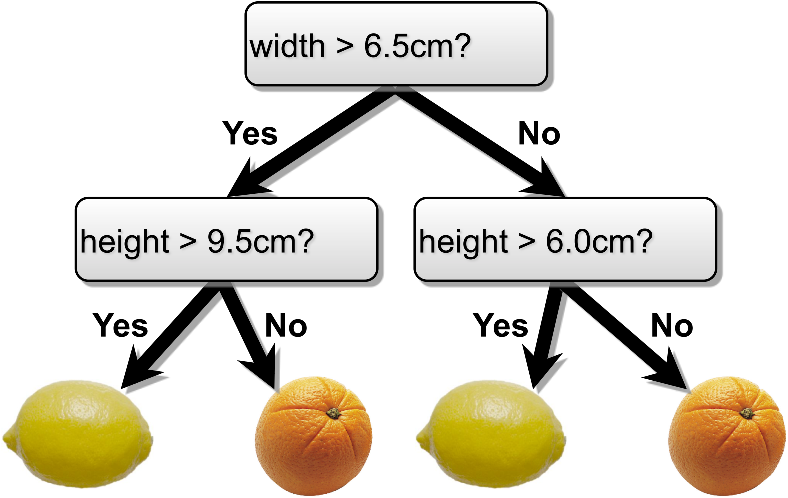
- Decision Trees
  - entropy
  - mutual information

Another Classification Idea

- We could view the decision boundary as being the composition of several simple boundaries.



Decision Tree: Example



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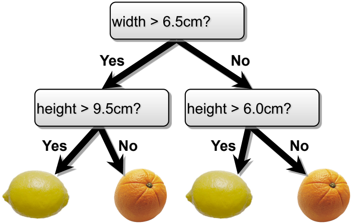
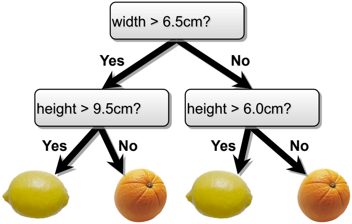


- Internal nodes test attributes

Decision Trees



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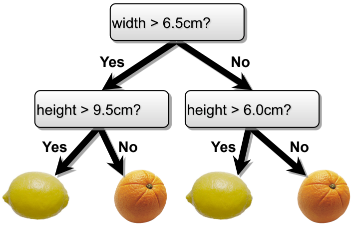
Decision Trees



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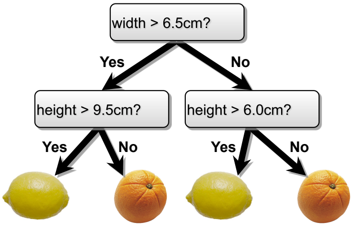

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- Branching is determined by attribute value
- Leaf nodes are outputs (class assignments)
- In general, a decision tree can represent any binary function

Decision Tree: Algorithm

- Choose an attribute on which to descend at each level.

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Two Binary Sequences

Sequence 1:
0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 ... ?

Sequence 2:
0 1 0 1 0 1 1 1 0 1 0 0 1 1 0 1 0 1 ... ?

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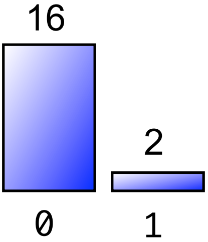
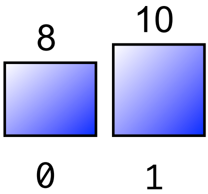
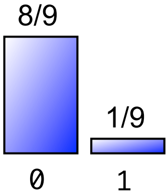


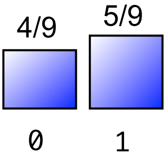
versus



Quantifying Uncertainty

$$H(X) = -\sum_{x \in X} p(x) \log_2 p(x)$$





$$-\frac{8}{9}\log_2\frac{8}{9} - \frac{1}{9}\log_2\frac{1}{9} \approx \frac{1}{2}$$

$$-\frac{4}{9}\log_2\frac{4}{9} - \frac{5}{9}\log_2\frac{5}{9} \approx 0.99$$

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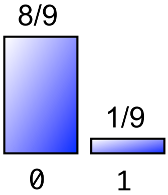

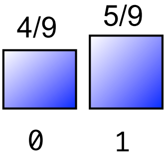

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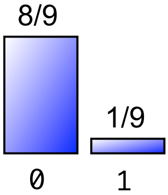

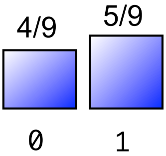

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- How surprised are we by a new value in the sequence?
- How much information does it convey?

Quantifying Uncertainty: Shannon Entropy

$$H(X) = -\sum_{x \in X} p(x) \log_2 p(x)$$

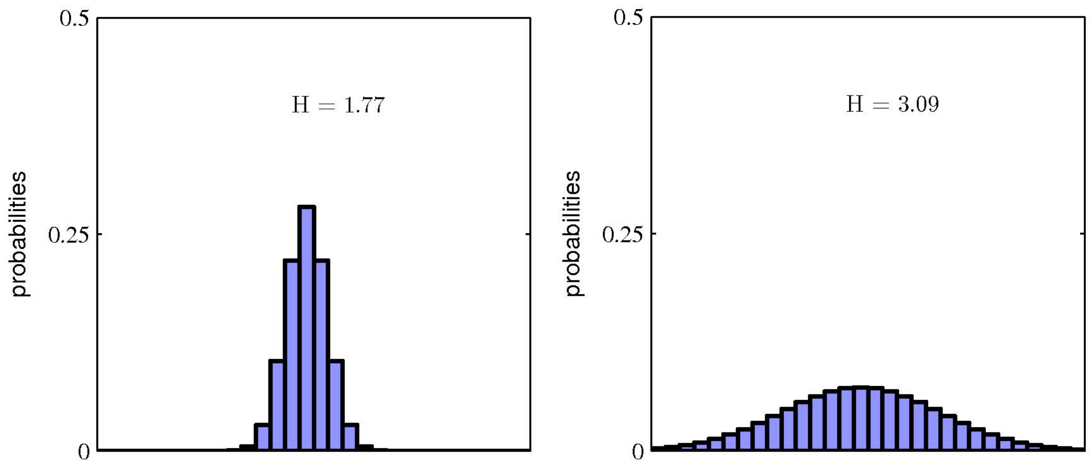
- Shannon Entropy is an extremely powerful concept.
- It tells you how much you can compress your data!

Entropy

- For a discrete random variable X, where $P(X=x_i) = p(x_i)$, the entropy of a random variable is:

$$\mathcal{H}(p) = -\sum_i p(x_i) \log p(x_i).$$

- Distributions that are sharply picked around a few values will have a relatively low entropy, whereas those that are spread more evenly across many values will have higher entropy



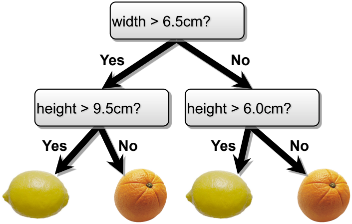
- Histograms of two probability distributions over 30 bins.

- The largest entropy will arise from a uniform distribution $H = -\ln(1/30) = 3.40$.

- For a density defined over continuous random variable, the differential entropy is given by:

$$\mathcal{H}(p) = -\int p(x) \log p(x) \mathrm{d}x.$$

Decision Tree: Algorithm



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Entropy of a Joint Distribution

	Cloudy	Not Cloudy
Raining	24/100	1/100
Not Raining	25/100	50/100

$$
\begin{aligned}
H(X,Y) &= -\sum_{x\in X}\sum_{y\in Y} p(x,y)\log_2 p(x,y) \\
&= -\frac{24}{100}\log_2\frac{24}{100} - \frac{1}{100}\log_2\frac{1}{100} - \frac{25}{100}\log_2\frac{25}{100} - \frac{50}{100}\log_2\frac{50}{100} \\
&\approx 1.56\text{bits}
\end{aligned}
$$

Specific Conditional Entropy

	Cloudy	Not Cloudy
Raining	24/100	1/100
Not Raining	25/100	50/100

- What is the entropy of cloudiness, given that it is raining?

$$
\begin{aligned}
H(X|Y=y) &= \sum_{x \in X} p(x|y) \log_2 p(x|y) \\
&= -\frac{24}{25} \log_2 \frac{24}{25} - \frac{1}{25} \log_2 \frac{1}{25} \\
&\approx 0.24 \text{bits}
\end{aligned}
$$

(Non-Specific) Conditional Entropy

	Cloudy	Not Cloudy
Raining	24/100	1/100
Not Raining	25/100	50/100

- The expected conditional entropy:

$$\begin{aligned} H(X|Y) &= \sum_{y \in Y} p(y) H(X|Y=y) \\ &= -\sum_{y \in Y} \sum_{x \in X} p(x,y) \log_2 p(x|y) \end{aligned}$$

(Non-Specific) Conditional Entropy

	Cloudy	Not Cloudy
Raining	24/100	1/100
Not Raining	25/100	50/100

- What is the entropy of cloudiness, given the knowledge of whether or not it is raining?

$$
\begin{aligned}
H(X|Y) &= \sum_{y \in Y} p(y) H(X|Y=y) \\
&= \frac{1}{4} H(\text{clouds}|\text{is raining})) + \frac{3}{4} H(\text{clouds}|\text{not raining}) \\
&\approx 0.75 \text{ bits}
\end{aligned}
$$

Mutual Information

	Cloudy	Not Cloudy
Raining	24/100	1/100
Not Raining	25/100	50/100

- How much information about cloudiness do we get by discovering whether it is raining?

$$
\begin{aligned}
IG(X|Y) &= H(X) - H(X|Y) \\
&\approx 0.25 \text{ bits}
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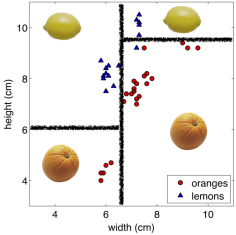
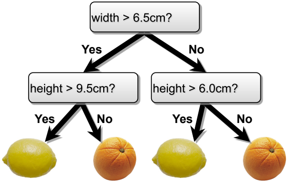
- Also called information gain in X due to Y
- For decision trees, X is the class/label and Y is an attribute

Constructing Decision Trees





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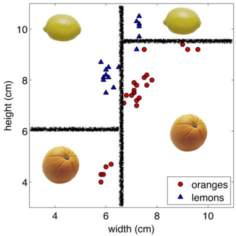

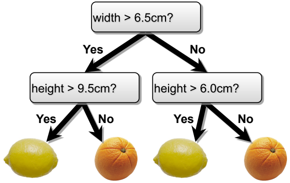

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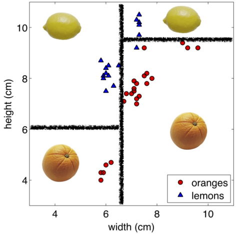

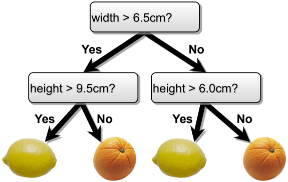

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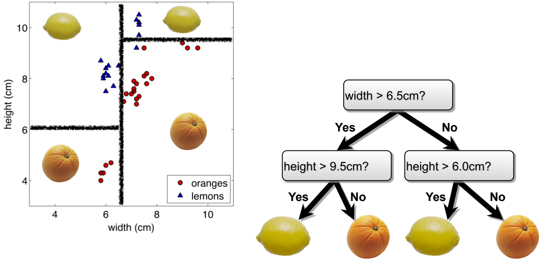

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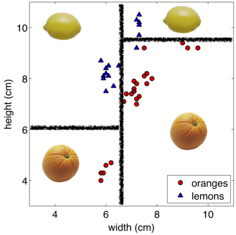

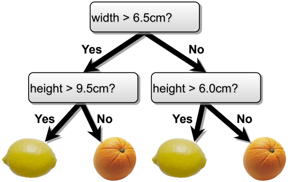

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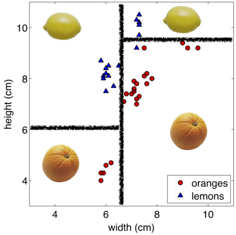

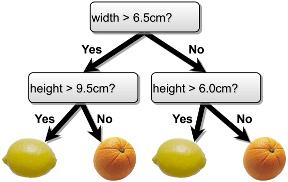

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- We can use the mutual information to automate the process.
- At each level, one must choose:
  1. Which variable to split.
  2. Possibly where to split it.
- Choose them based on how much information we would gain from the decision!

Decision Tree Algorithm

- Simple, greedy, recursive approach, builds up tree node-by-node

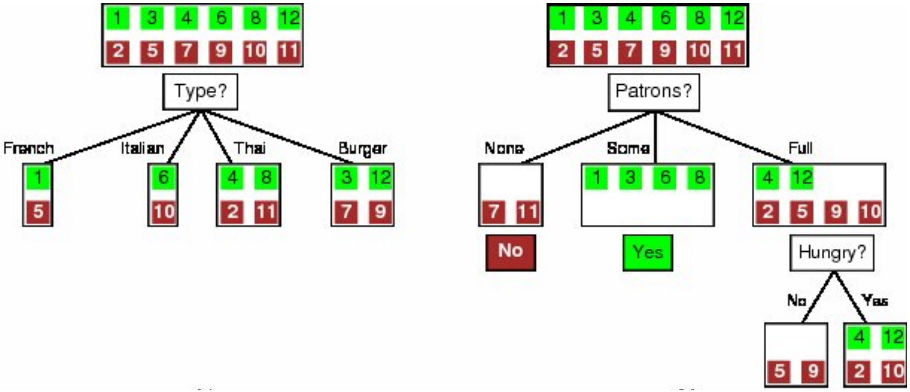
1. pick an attribute to split at a non-terminal node
2. split examples into groups based on attribute value
3. for each group:
    - if no examples – return majority from parent
    - else if all examples in same class – return class
    - else loop to step 1

Decision Tree Example: Data

Ex.	Attributes										Target
	Alt	*Bar*	*Fri*	*Hun*	*Pat*	*Price*	*Rain*	*Res*	*Type*	*Est*	*WillWait*
X_1	T	F	F	T	Some	$$$	F	T	French	0–10	T
X_2	T	F	F	T	Full	$	F	F	Thai	30–60	F
X_3	F	T	F	F	Some	$	F	F	Burger	0–10	T
X_4	T	F	T	T	Full	$	F	F	Thai	10–30	T
X_5	T	F	T	F	Full	$$$	F	T	French	>60	F
X_6	F	T	F	T	Some	$$	T	T	Italian	0–10	T
X_7	F	T	F	F	None	$	T	F	Burger	0–10	F
X_8	F	F	F	T	Some	$$	T	T	Thai	0–10	T
X_9	F	T	T	F	Full	$	T	F	Burger	>60	F
X_{10}	T	T	T	T	Full	$$$	F	T	Italian	10–30	F
$X_{1}1$	F	F	F	F	None	$	F	F	Thai	0–10	F
$X_{1}2$	T	T	T	T	Full	$	F	F	Burger	30–60	T

Russell & Norvig example

Attribute Selection



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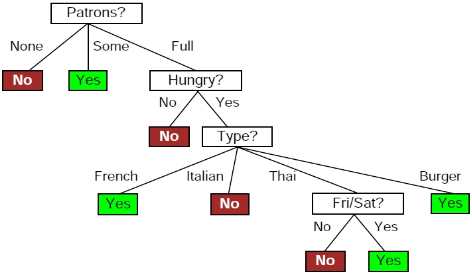
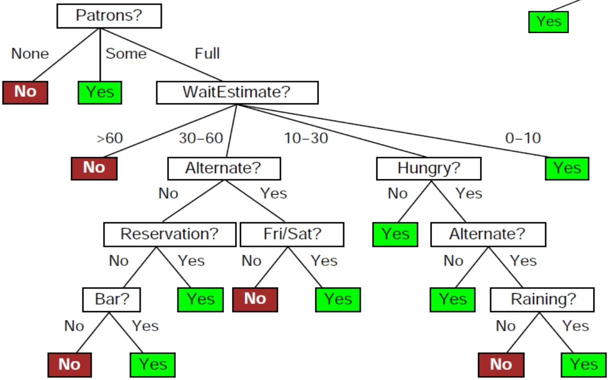
$$IG(type) = 1 - \left[\frac{2}{12}H(\frac{1}{2},\frac{1}{2}) + \frac{2}{12}H(\frac{1}{2},\frac{1}{2}) + \frac{4}{12}H(\frac{2}{4},\frac{2}{4}) + \frac{4}{12}H(\frac{2}{4},\frac{2}{4})\right] = 0$$

$$IG(Patrons) = 1 - \left[\frac{2}{12}H(0,1) + \frac{4}{12}H(1,0) + \frac{6}{12}H(\frac{2}{6},\frac{4}{6})\right] \approx 0.541$$

Which Tree is Better?





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- Occam's Razor: find the simplest hypothesis (smallest tree) that fits the observations
- Inductive bias: small trees with informative nodes near the root

Decision Tree Miscellany

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Decision Tree Miscellany

- Problems:
  - You have exponentially less data at lower levels.
  - Too big of a tree can overfit the data.
  - Greedy algorithms don't necessarily yield the global optimum.
- In practice, one often regularizes the construction process to try to get small but highly-informative trees.
- Decision trees can also be used for regression on real-valued outputs, but it requires a different formalism.

Comparison to k-NN

K-Nearest Neighbors

- Decision boundaries: piece-wise
- Test complexity:
  non-parametric, few parameters besides (all?) training examples

Decision Trees

- Decision boundaries:
  axis-aligned, tree structured
- Test complexity: attributes and splits

Applications of Decision Trees

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- Practical Applications:
  - Flight simulator: 20 state variables; 90K examples based on expert pilot's actions; auto-pilot tree
  - Yahoo Ranking Challenge
  - Random Forests