1 Lower bounds

1.1 An exponential lower bound for DNFs

Recall that a DNF (Disjunctive Normal Form) is a disjunction of conjunctive terms; that is, a syntactic expression of the form

$$\Phi(x_1, ..., x_n) = (\ell_1 \wedge \ell_2 \wedge ... \wedge \ell_{w_1}) \vee (\ell_{w_1+1} \wedge ... \wedge \ell_{w_2}) \vee ... \vee (\ell_{w_{s-1}} \wedge ... \wedge \ell_{w_s})$$

where each ℓ_i is a literal, i.e. either a variable x_j or a negation of a variable $\neg x_j$. We refer to the expression "$(\ell_{w_{i-1}+1} \wedge ... \wedge \ell_{w_i})$" as the i^{th} term, we call w_i the width of the i^{th} terms, and the number s of terms is the size of the DNF.

A DNF computes a function on $x_1, ..., x_n$ in the natural way, i.e. evaluate each term and take the OR. For example, this is a DNF that computes the parity (i.e., sum modulo 2) of two variables: $\Phi(x_1, x_2) = (x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_2)$.

Question 1. *Show that any DNF computing the function* $f(x_1, ..., x_n) = Parity(x_1, ..., x_n) = \sum_{i \in [n]} x_i \bmod 2$ *must have size at least* 2^{n-1}.

Hint: Could a DNF computing parity have a term with less than n literals?

1.2 The implication of the lower bound for DNF

What does the statement you proved in Question 1 imply?

Question 2. *Your friend claims that they wrote down a DNF that computes the parity of* 1024 *input bits in their notebook. Is this possible? What if your friend is very very clever and can come up with sophisticated DNFs for parity that you haven't thought of?*

Question 3. *A generative AI model claims that it has a sophisticated method for computing the parity of n input bits by a DNF of size $2^{\sqrt{n}}$. Is this possible? What if the generative AI is the best model in the world, and was trained on all existing data for a million years?*

Question 4. *Is it possible for* anyone, anything, any human or machine whatsoever, *to come up with a DNF of size 2^{n-2} that computes the parity of n input inputs?*

1.3 An upper bound

For good measure, show that the lower bound you proved is tight, i.e. we can compute parity with a DNF of size 2^{n-1} (which is better than the trivial bound of 2^n).

Question 5. *Show a DNF of size 2^{n-1} that computes the parity of n variables.*