1 Asymptotics: A refresher

Throughout the course we will use the following mathematical slang: Instead of denoting $f \in O(g)$ and $f \in \omega(g)$, etc., we will denote $f = O(g)$ and $f = \omega(g)$. Sometimes we will even denote $f \leq O(g)$ and $f \geq \omega(g)$, etc. This is syntactic sugar, and the precise mathematical meaning of the expression is obtained by replacing the relevant symbol (i.e., "$=$" or "$\leq$" or "$\geq$") by the correct formal symbol "$\in$".

Question 1. *When writing $f \in O(g)$, what is the mathematical object represented by "f"? Is it a number, a function, a graph, a matrix? Which type of number/function/graph/matrix, i.e. what is the domain or range of the elements in it? Does the same apply to "g"?*

Sometimes in the course we will use another informal notation, by writing functions in asymptotic expressions as receiving a generic unspecified variable n (e.g., writing "$f(n) = \omega(1)$"). This is again mathematical slang, intended to remind that f is a function that receives an integer as input; the true meaning of the expression is obtained by removing "(n)" (e.g., "$f(n) \geq \omega(1)$" just means "$f \in \omega(1)$").

Question 2. *Spell out the full definition of "$f \in O(g)$". Now answer the following question: If $f \in O(g)$, is it true that there is a constant C such that for* each and every integer $n \in \mathbb{N}$ *we have $f(n) \leq C \cdot g(n)$?*

Question 3. *Let $f(n)$ be the smallest prime larger than n. Your friend claims that for every n the number $f(n)$ is some constant number, and hence $f(n) = O(1)$. Is your friend correct?*

Question 4. *Let $f(n) = n^{c_n}$ where for every $n \in \mathbb{N}$ we have that c_n is some number that depends on n. Does this necessarily mean that f is a polynomial?*

The following question uses operators on functions to obtain other functions, in the natural way. For example, we denote by $f(n) + g(n)$ the function h such that $h(n) = f(n) + g(n)$ (and similarly for other operations, such as multiplication).

Question 5. *Prove or disprove each of the following statements.*

- *For any f, if $g(n) = f(n) + O(1)$ then $g(n) = O(f(n))$.*
- *If $f = \omega(g)$ and $h = \Theta(1)$ then $g + h = o(f)$.*
- *If $f = \omega(g)$ and $f = \omega(h)$ then $g + h = o(f)$.*
- *If $f = \omega(g)$ and $f = \omega(h)$ then $g(n) \cdot h(n) = o(f(n))$.*
- *If $f(n) = o(n)$ and $g(n) = o(n)$ then $f(n) + g(n) = o(n)$.*[1]

If the questions above were not easy for you, this means that you need a thorough refresher on asymptotic notation.

[1] When parsing an expression such as "$f(n) = o(n)$", pause and ask yourself what is the meaning of "n" in the expression "$o(n)$". If "n" represents a function, what is this function?