1 Computability and Kolmogorov complexity

1.1 Computable and uncomputable functions

The following question is a basic exercise: if you understood the definitions, you should be able to do it easily.

Question 1. *Prove or disprove: For any language $L \subseteq \{0,1\}^*$ we have that*

$$L \text{ is decidable } \iff \bar{L} \text{ is decidable },$$

where $\bar{L} = \{0,1\}^* \setminus L$ is the set of inputs that are not in L (i.e., the complement of L).

Many of the following questions ask about undecidable languages, and you can prove that using reductions from undecidable languages we saw in class (e.g., the halting problem). There are also some easily decidable languages thrown into the mix below, to make sure you develop intuition as to what separates easy computational problems from impossible computational problems.

Question 2. *For each of the following language, say whether they're decidable or not, and prove your answer:*[1]

1. $L = \{\langle M \rangle : M(\langle M \rangle) = 1\}$.
2. $L = \{\langle M \rangle, x : M(x) \text{ prints your first name}\}$.
3. $L = \{\langle M \rangle : M(\langle M \rangle) \text{ doesn't halt}\}$.
4. $L = \{\langle M \rangle, x : M(x) \text{ halts}\}$.
5. $L = \left\{\langle M \rangle, x : M(x) \text{ halts after at most } |x|^3 \text{ steps}\right\}$.
6. $L = \{\langle M \rangle : \text{the first bit in the description of } M \text{ is } 0\}$.
7. $L = \{\langle M \rangle : \exists M' \text{ such that } L(M) \cap L(M') = \emptyset\}$.
8. $L = \{\langle M \rangle : \exists x \text{ such that } M(x) = x\}$.

1.2 Kolmogorov complexity

We learned that $K(\cdot)$ as a function is uncomputable, but how about an undecidable language? The next question asks you to find one, mostly in order to see that there is no big gap between the two notions in this context.

Question 3. *In class we showed that Kolmogorov complexity, as a function, is uncomputable. Show a language (i.e., a decision problem) based on Kolmogorov complexity that is undecidable (as always, prove your answer).*

[1] Do not use Rice's theorem.

The answer to the following question is somewhat counter-intuitive (at least for me),[2] and finding it requires a trick.

Question 4. *Prove or disprove: For infinitely many $n \in \mathbb{N}$ there exists $x \in \{0,1\}^n$ such that $K(x) \leq \text{loglog}(n)$. Would your answer change if we replace "loglog" by "loglogloglogloglog"?*

The following question stress-tests your precise low-level understanding of the definition of Kolmogorov complexity, in terms of the exact length of a small description M, w for a string x.

Question 5. *Show that for any $x, y \in \{0,1\}^*$ we have $K(xy) \leq K(x) + K(y) + O(\log(K(x))$, where "$xy$" is the concatenation of x and y. Would your answer change if we replace "$\log(K(x))$" by "$\log(K(y))$"?*

The last question is very hard if you haven't internalized the definitions and basic results around Kolmogorov complexity, and becomes much easier to solve once you understand them. The crux of the question refers to the existence of n-bit strings with Kolmogorov complexity essentially k, for every $k \leq n$; the foregoing sentence says "essentially" because the low-level implementation requires a slack of $k \pm O(\log n)$ (rather than exactly k).

Question 6. *Prove or disprove: For every large enough $n \in \mathbb{N}$ and every $k \leq n - 1$ there is x such that $K(x) \in [k - O(\log k), k + O(\log n)]$.*

1.3 Verifiable functions

The following two questions verify that you understand the relevant definitions (and they do not require special creativity otherwise). Mainly, they check that you understood what $L(M)$ means for a TM M.

Question 7. *Show that for any $L \subseteq \{0,1\}^*$, if there's a TM M such that $L(M) = L$ then L is verifiable.*[3]

Question 8. *Show that for any $L \subseteq \{0,1\}^*$ we have that L is decidable if and only if there's a TM M that halts on every input and $L(M) = L$.*

Recall that in class we defined R and RE, which are classes of languages. We will now define a notion of "complement classes", and develop an alternative formulation of the class RE.

Definition 1. *For a class $\mathcal{C}$ of languages, we define the complement class as $co\mathcal{C} = \{\bar{L} : L \in \mathcal{C}\}$, i.e. $co\mathcal{C}$ includes all complements of languages in $\mathcal{C}$ (recall that $\bar{L} = \{0,1\}^* \setminus L$).*

Question 9. *Phrase what you proved in Question 1 in terms of R and coR.*

Question 10. *Prove that $\mathsf{R} = \mathsf{RE} \cap \mathsf{coRE}$.*

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[2] Hint: My initial intuition would be that any n-bit string x would have $K(x) \geq \log(n)$, since we want to at least specify the length n of x (and the binary representation of n has length $\lceil \log(n) \rceil$). Alas, initial intuitions may not always be correct!

[3] Recall that we skipped this part in class, and mentioned that the proof idea imitates a proof we've seen for the fact that the halting problem is verifiable.