CSC 311: Introduction to Machine Learning

Lecture 3 - Bagging, Linear Models I

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Outline

Announcements

- HW1 released last week and is due next Wednesday.
- Go to the earliest possible TA OH you can attend.
- **Manage your time well!** If you wait till the last TA session, you may have a long wait to ask your question.

Introduction

Bias-Variance Decomposition

Interpretations

$$\mathbb{E}[(y-t)^2] = \underbrace{(y_\star - \mathbb{E}[y])^2}_{\text{bias}} + \underbrace{\mathrm{Var}(y)}_{\text{variance}} + \underbrace{\mathrm{Var}(t)}_{\text{Bayes error}}$$

Bias/variance decomposes the expected loss into three terms:

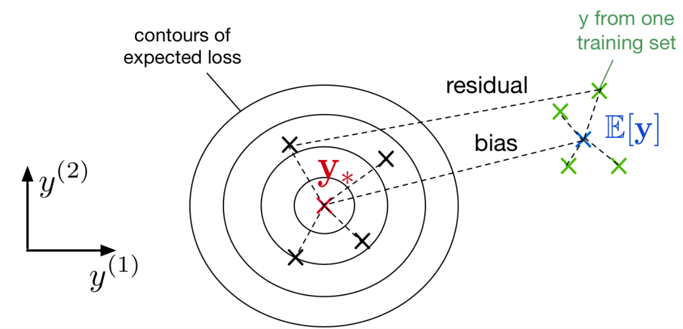
- **bias**: how wrong the expected prediction is (corresponds to under-fitting)
- **variance**: the amount of variability in the predictions (corresponds to over-fitting)
- **Bayes error**: the inherent unpredictability of the targets

Often loosely use “bias” for “under-fitting” and “variance” for “over-fitting”.

Overly Simple Model

An overly **simple** model (e.g. KNN with large k) might have

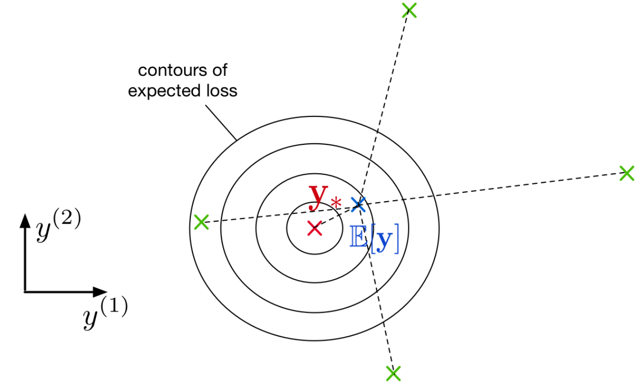
- **high bias**
  (cannot capture the structure in the data)
- **low variance**
  (enough data to get stable estimates)



Overly Complex Model

An overly **complex** model (e.g. KNN with $k = 1$) might have

- **low bias**
  (learns all the relevant structure)
- **high variance**
  (fits the quirks of the data you happened to sample)



Bagging

Bagging Motivation

- Sample m independent training sets from p_{sample}.
- Compute the prediction y_i using each training set.
- Compute the average prediction $y = \frac{1}{m}\sum_{i=1}^{m} y_i$.
- How does this affect the three terms of the expected loss?
  - **Bias:** unchanged,
    since the averaged prediction has the same expectation

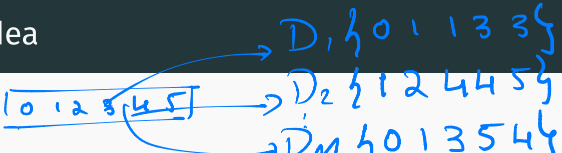
$$\mathbb{E}[y] = \mathbb{E}\left[\frac{1}{m}\sum_{i=1}^{m} y_i\right] = \mathbb{E}[y_i]$$

  - **Variance:** reduced,
    since we are averaging over independent predictions

$$\mathrm{Var}[y] = \mathrm{Var}\left[\frac{1}{m}\sum_{i=1}^{m} y_i\right] = \frac{1}{m^2}\sum_{i=1}^{m}\mathrm{Var}[y_i] = \frac{1}{m}\mathrm{Var}[y_i].$$

  - **Bayes error:** unchanged,
    since we have no control over it

Bagging: The Idea



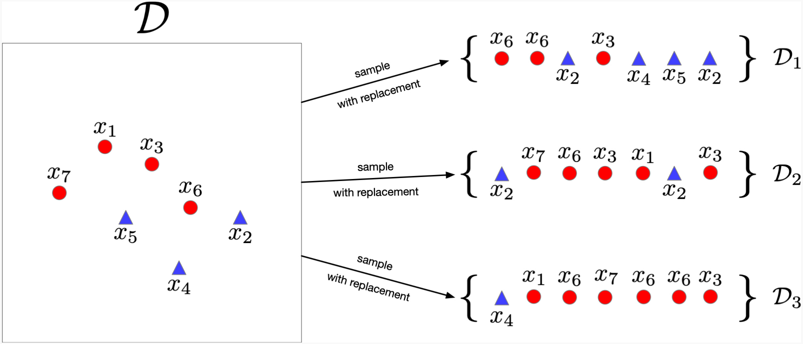
- In practice, p_{sample} is often expensive to sample from. So training separate models on independently sampled datasets is very wasteful of data!

- Given training set $\mathcal{D}$, use the empirical distribution $p_{\mathcal{D}}$ as a proxy for p_{sample}. This is called bootstrap aggregation or bagging .
    - Take a dataset $\mathcal{D}$ with n examples.
    - Generate m new datasets ("resamples" or "bootstrap samples")
    - Each dataset has n examples sampled from $\mathcal{D}$ with replacement.
    - Average the predictions of models trained on the m datasets.

- One of the most important ideas in statistics!
    - Intuition: As $|\mathcal{D}| \to \infty$, we have $p_{\mathcal{D}} \to p_{\text{sample}}$.

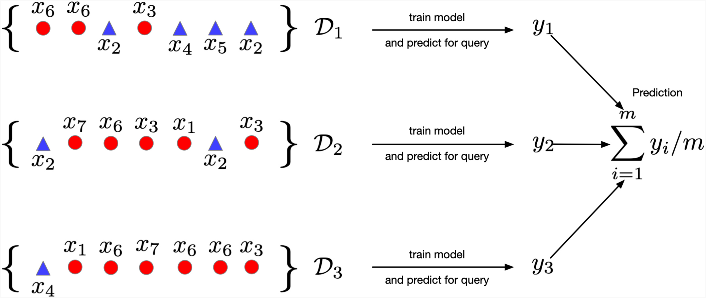
Bagging Example 1/2

Create $m = 3$ datasets by sampling from $\mathcal{D}$ with replacement.
Each dataset contains $n = 7$ examples.



Bagging Example 2/2

Generate prediction y_i using dataset $\mathcal{D}_i$.
Average the predictions.



Aggregating Predictions for Binary Classification

- Classifier i outputs a prediction y_i
- y_i can be real-valued $y_i \in [0, 1]$ or a binary value $y_i \in \{0, 1\}$
- Average the predictions and apply a threshold.

$$y_{\text{bagged}} = \mathbb{I}\left(\frac{1}{m}\sum_{i=1}^{m} y_i > 0.5\right)$$

- Same as majority vote.

Bagging Properties

- A bagged classifier can be stronger than the average model.
  - E.g. on "Who Wants to be a Millionaire", "Ask the Audience" is much more effective than "Phone a Friend".
- But, if m datasets are NOT independent, don't get the $\frac{1}{m}$ variance reduction.
- Reduce correlation between datasets by introducing *additional* variability
  - Invest in a diversified portfolio, not just one stock.
  - Average over multiple algorithms, or multiple configurations of the same algorithm.

Random Forests

- A trick to reduce correlation between bagged decision trees:
  For each node, choose a random subset of features
  and consider splits on these features only.
- Probably the best black-box machine learning algorithm.
  - works well with no tuning.
  - widely used in Kaggle competitions.

Bagging Summary

Reduces over-fitting by averaging predictions.

In most competition winners.
A small ensemble often better than a single great model.

Limitations:

- Does not reduce bias in case of squared error.
- Correlation between classifiers means less variance reduction.
  Add more randomness in Random Forests.
- Weighting members equally may not be the best.
  Weighted ensembling often leads to better results if members are very different.

Linear Regression

Step 1: Task - supervised learning.

Step 2: Model

$y = w^T x + b$

Assume the model predictions are linear functions of input

Step 3: Goodness of fit

e.g Regression

$\mathcal{L}(y^i, t^i) = \frac{1}{2}(y^i - t^i)^2$

$\approx J(w, b) = \frac{1}{2N} \sum_{i=1}^{N} \mathcal{L}(y^{(i)}, t^{(i)})$

Step 4: Optimize

Direct solution.

min $J(w, b)$ by setting $\nabla J(w, b) = 0$

Iterative

take small gradient steps.

Step 5: Regularize (Iterate to find better solutions)

Step 1 & 2: Linear Regression

- Define the task and a strategy on solving it
- Task: predict scalar-valued targets (e.g. stock prices)
- Architecture: linear function of the inputs

Step 3: A Modular Approach to ML

- choose a model describing relationships between variables
- define a loss function quantifying how well the model fits the data
- choose a regularizer expressing preference over different models
- fit a model that minimizes the loss function and satisfies the regularizer's constraint/penalty, possibly using an optimization algorithm

Mixing and matching these modular components give us a lot of different ML methods.

Supervised Learning Setup

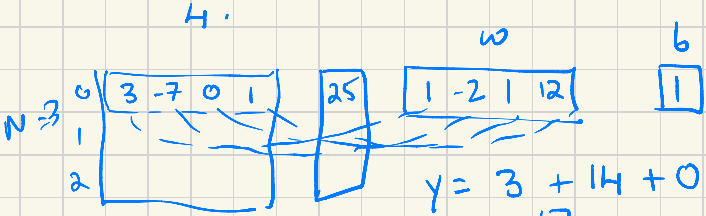
- Input $\mathbf{x} \in \mathcal{X}$ (a vector of features)
- Target $t \in \mathcal{T}$
- Data $\mathcal{D} = \{(\mathbf{x}^{(i)}, t^{(i)})$ for $i = 1, 2, ..., N\}$
- Objective: learn a function $f : \mathcal{X} \to \mathcal{T}$ based on the data such that $t \approx y = f(\mathbf{x})$

Model

Model: a *linear* function of the features $\mathbf{x} = (x_1, \ldots, x_D) \in \mathbb{R}^D$ to make prediction $y \in \mathbb{R}$ of the target $t \in \mathbb{R}$:

$$y = f(\mathbf{x}) = \sum_j w_j x_j + b = \mathbf{w}^\top \mathbf{x} + b$$

- Parameters are weights $\mathbf{w}$ and the bias/intercept b
- Want the prediction to be close to the target: $y \approx t$.
- Highly interpretable model, useful for debugging.



$$y = 3 + 14 + 0 + 12 + 1$$
$$= 17$$
$$ 12$$
$$= 29 + 1 = 30$$

$$t = 25$$

$$L(y, t) = \frac{1}{2}(30 - 25)^2$$
$$= 25/2 = 12.5$$

Loss Function

Loss function $\mathcal{L}(y, t)$ defines how badly the algorithm's prediction y fits the target t for some example $\mathbf{x}$.

Squared error loss function: $\mathcal{L}(y, t) = \frac{1}{2}(y - t)^2$

- $y - t$ is the residual, and we want to minimize this magnitude
- $\frac{1}{2}$ makes calculations convenient.

Cost function: loss function averaged over all training examples also called *empirical* or *average loss*.

$$\mathcal{J}(\mathbf{w}, b) = \frac{1}{2N}\sum_{i=1}^{N}\left(y^{(i)} - t^{(i)}\right)^2 = \frac{1}{2N}\sum_{i=1}^{N}\left(\mathbf{w}^\top \mathbf{x}^{(i)} + b - t^{(i)}\right)^2$$

Vectorization

Loops v.s. Vectorized Code

- We can compute prediction for one data point using a for loop:

```
y = b
for j in range(M):
    y += w[j] * x[j]
```

- But, excessive super/sub scripts are hard to work with, and Python loops are slow.
- Instead, we express algorithms using vectors and matrices.

$$\mathbf{w} = (w_1, \ldots, w_D)^\top \qquad \mathbf{x} = (x_1, \ldots, x_D)^\top$$

$$y = \mathbf{w}^\top \mathbf{x} + b$$

- This is simpler and executes much faster:

```
y = np.dot(w, x) + b
```

Benefits of Vectorization

Why vectorize?

- The code is simpler and more readable. No more dummy variables/indices!
- Vectorized code is much faster
  - Cut down on Python interpreter overhead
  - Use highly optimized linear algebra libraries (hardware support)
  - Matrix multiplication very fast on GPU

You will practice switching in and out of vectorized form.

- Some derivations are easier to do element-wise
- Some algorithms are easier to write/understand using for-loops and vectorize later for performance

Predictions for the Dataset

- Put training examples into a design matrix $\mathbf{X}$.
- Put targets into the target vector $\mathbf{t}$.
- We can compute the predictions for the whole dataset.

$$\mathbf{X}\mathbf{w} + b\mathbf{1} = y$$

$$\begin{pmatrix} x_1^{(1)} & x_2^{(1)} & \dots & x_D^{(1)} \\ x_1^{(2)} & x_2^{(2)} & \dots & x_D^{(2)} \\ \vdots & \vdots & & \vdots \\ x_1^{(N)} & x_2^{(N)} & \dots & x_D^{(N)} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_D \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} y^{(1)} \\ \vdots \\ y^{(N)} \end{pmatrix}$$

Computing Squared Error Cost

We can compute the squared error cost across the whole dataset.

$$\mathbf{y} = \mathbf{X}\mathbf{w} + b\mathbf{1}$$

$$\mathcal{J} = \frac{1}{2N}\|\mathbf{y} - \mathbf{t}\|^2$$

Sometimes we may use $\mathcal{J} = \frac{1}{2}\|\mathbf{y} - \mathbf{t}\|^2$, without a normalizer. This would correspond to the sum of losses, and not the averaged loss.

The minimizer does not depend on N (but optimization might!).

Combining Bias and Weights

We can combine the bias and the weights and add a column of 1's to design matrix.

Our predictions become

$$\mathbf{y} = \mathbf{X}\mathbf{w}.$$

$$\mathbf{X} = \begin{bmatrix} 1 & [\mathbf{x}^{(1)}]^\top \\ 1 & [\mathbf{x}^{(2)}]^\top \\ 1 & \vdots \end{bmatrix} \in \mathbb{R}^{N \times (D+1)} \text{ and } \mathbf{w} = \begin{bmatrix} b \\ w_1 \\ w_2 \\ \vdots \end{bmatrix} \in \mathbb{R}^{D+1}$$

Optimization

Step 4: Solving the Minimization Problem

Goal is to minimize the cost function $\mathcal{J}(\mathbf{w})$.

Recall: the minimum of a smooth function (if it exists) occurs at a **critical point**, i.e. point where the derivative is zero.

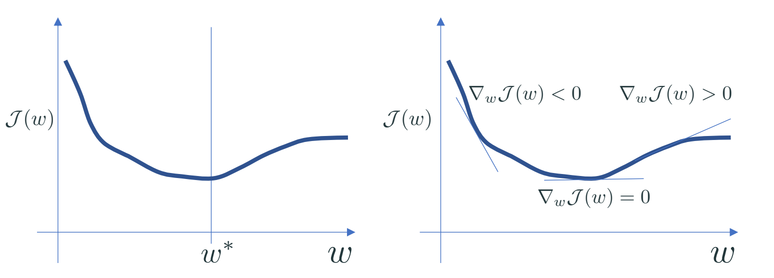
$$\nabla_{\mathbf{w}} \mathcal{J} = \frac{\partial \mathcal{J}}{\partial \mathbf{w}} = \begin{pmatrix} \frac{\partial \mathcal{J}}{\partial w_1} \\ \vdots \\ \frac{\partial \mathcal{J}}{\partial w_D} \end{pmatrix}$$

Solutions may be direct or iterative.

- **Direct solution**: set the gradient to zero and solve in closed form — directly find provably optimal parameters.
- **Iterative solution**: repeatedly apply an update rule that gradually takes us closer to the solution.

Minimizing 1D Function

- Consider $\mathcal{J}(w)$ where w is 1D.
- Seek $w = w^*$ to minimize $\mathcal{J}(w)$.
- The gradients can tell us where the maxima and minima of functions lie
- **Strategy:** Write down an algebraic expression for $\nabla_w \mathcal{J}(w)$. Set $\nabla_w \mathcal{J}(w) = 0$. Solve for w.



Direct Solution for Linear Regression

$\frac{1}{2}\|Xw - t\|^2 = \frac{1}{2}(Xw - t)^T(Xw - t)$

- Seek $\mathbf{w}$ to minimize $\mathcal{J}(\mathbf{w}) = \frac{1}{2}\|\mathbf{X}\mathbf{w} - \mathbf{t}\|^2$
- Taking the gradient with respect to $\mathbf{w}$ and setting it to $\mathbf{0}$, we get:

$$\nabla_{\mathbf{w}}\mathcal{J}(\mathbf{w}) = \mathbf{X}^{\top}\mathbf{X}\mathbf{w} - \mathbf{X}^{\top}\mathbf{t} = \mathbf{0}$$

  See course notes for derivation.

- Optimal weights:

$$\mathbf{w}^{*} = (\mathbf{X}^{\top}\mathbf{X})^{-1}\mathbf{X}^{\top}\mathbf{t}$$

- Few models (like linear regression) permit direct solution.

$X^T X w = X^T t$

$w = (X^T X)^{-1} X^T t$

Iterative Solution: Gradient Descent

- Many optimization problems don't have a direct solution.
- A more broadly applicable strategy is gradient descent.
- Gradient descent is an iterative algorithm, which means we apply an update repeatedly until some criterion is met.
- We initialize the weights to something reasonable (e.g. all zeros) and repeatedly adjust them in the direction of steepest descent.

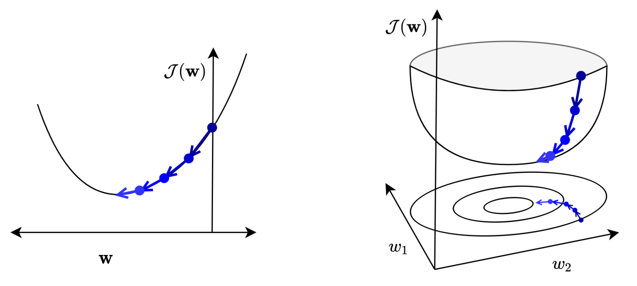
Deriving Update Rule

Observe:

- if $\partial\mathcal{J}/\partial w_j > 0$, then decreasing $\mathcal{J}$ requires decreasing w_j.
- if $\partial\mathcal{J}/\partial w_j < 0$, then decreasing $\mathcal{J}$ requires increasing w_j.

The following update always decreases the cost function for small enough α (unless $\partial\mathcal{J}/\partial w_j = 0$):

$$w_j \leftarrow w_j - \alpha \frac{\partial \mathcal{J}}{\partial w_j}$$



Setting Learning Rate

Gradient descent update rule:

$$w_j \leftarrow w_j - \alpha \frac{\partial \mathcal{J}}{\partial w_j}$$

$\alpha > 0$ is a learning rate (or step size).

- The larger α is, the faster $\mathbf{w}$ changes.
- Values are typically small, e.g. 0.01 or 0.0001.
- We'll see later how to tune the learning rate.
- If minimizing total loss rather than average loss, needs a smaller learning rate ($\alpha' = \alpha/N$).

Gradient Descent Intuition

- Gradient descent gets its name from the gradient, the direction of fastest *increase*.

$$\nabla_{\mathbf{w}} \mathcal{J} = \frac{\partial \mathcal{J}}{\partial \mathbf{w}} = \begin{pmatrix} \frac{\partial \mathcal{J}}{\partial w_1} \\ \vdots \\ \frac{\partial \mathcal{J}}{\partial w_D} \end{pmatrix}$$

- Update rule in vector form:

$$\mathbf{w} \leftarrow \mathbf{w} - \alpha \frac{\partial \mathcal{J}}{\partial \mathbf{w}}$$

  Update rule for linear regression:

$$\mathbf{w} \leftarrow \mathbf{w} - \frac{\alpha}{N} \sum_{i=1}^{N} (y^{(i)} - t^{(i)})\, \mathbf{x}^{(i)}$$

- Gradient descent updates $\mathbf{w}$ in the direction of fastest *decrease*.
- Once it converges, we get a critical point, i.e. $\frac{\partial \mathcal{J}}{\partial \mathbf{w}} = \mathbf{0}$.

Why Use Gradient Descent?

- Applicable to a much broader set of models.
- Easier to implement than direct solutions.
- More efficient than direct solution for regression in high-dimensional space.
  - The linear regression direction solution $(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{t}$ requires matrix inversion, which is $\mathcal{O}(D^3)$.
  - Gradient descent update costs $\mathcal{O}(ND)$ or less with stochastic gradient descent.
  - Huge difference if D is large.

Feature Mappings

Steps for linear regression

Below is a categorization of ML problems that you will see time, and time-again throughout this semester.

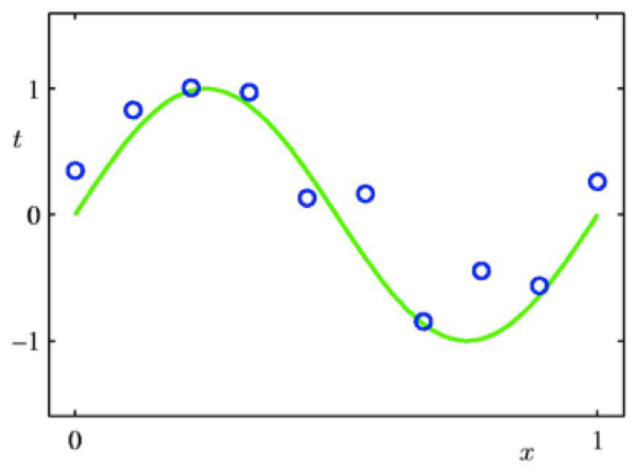
- Step 1: Understand the problem (is it prediction, learning a good representation). **Regression**
- Step 2: Formulate the problem mathematically (create notation for your inputs and outcomes and model).**Linear function of inputs**
- Step 3: Formulate an objective function that represents success for your model. **Mean squared error**
- Step 4: Find a strategy to solve the optimization problem on pencil and paper. **Direct or gradient based optimization**
- Step 5: Translate the algorithm into code. **Part of future homework excercises**
- Step 6: Analyze, iterate, improve design choices in your model and algorithm

Feature Mapping

Can we use linear regression to model a non-linear relationship?

- Map the input features to another space $\boldsymbol{\psi}(\mathbf{x}) : \mathbb{R}^D \to \mathbb{R}^d$.
- Treat the mapped feature (in $\mathbb{R}^d$) as the input of a linear regression procedure.

Modeling a Non-Linear Relationship

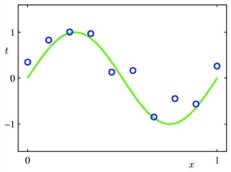


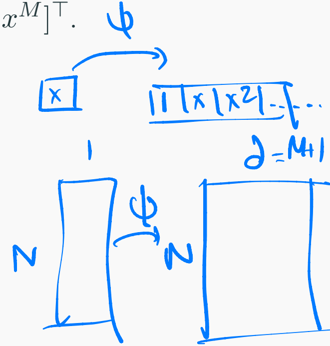
Polynomial Feature Mapping

Fit the data using a degree-M polynomial function of the form:

$$y = w_0 + w_1 x + w_2 x^2 + ... + w_M x^M = \sum_{i=0}^{M} w_i x^i$$

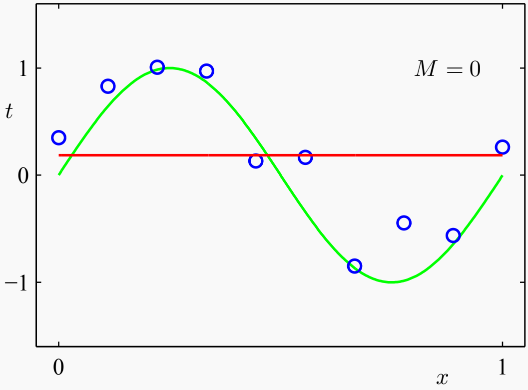
- The feature mapping is $\boldsymbol{\psi}(x) = [1, x, x^2, ..., x^M]^\top$.
- $y = \boldsymbol{\psi}(x)^\top \mathbf{w}$ is linear in $w_0, w_1, ...$.
- Use linear regression to find $\mathbf{w}$.





Polynomial Feature Mapping with $M = 0$

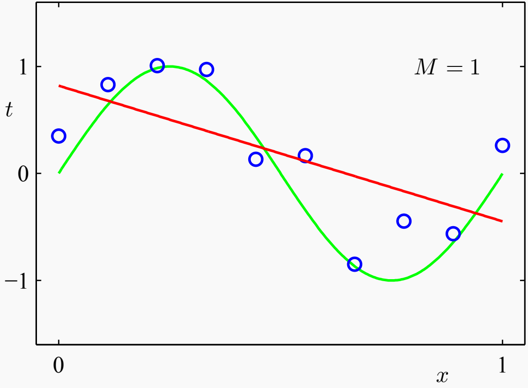
$$y = w_0$$



[Pattern Recognition and Machine Learning, Christopher Bishop.]

Polynomial Feature Mapping with $M = 1$

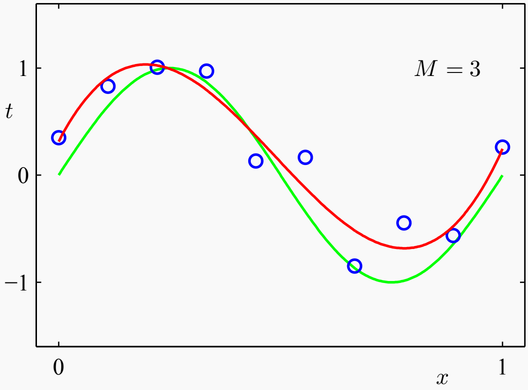
$$y = w_0 + w_1 x$$



[Pattern Recognition and Machine Learning, Christopher Bishop.]

Polynomial Feature Mapping with $M = 3$

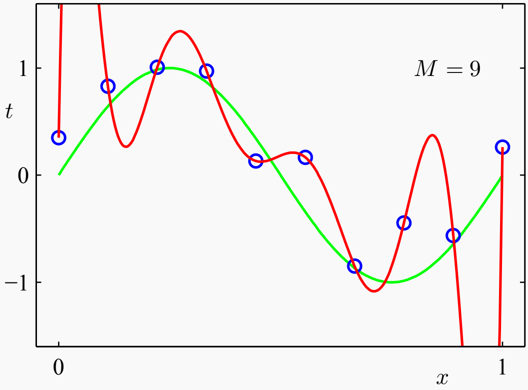
$$y = w_0 + w_1 x + w_2 x^2 + w_3 x^3$$



[Pattern Recognition and Machine Learning, Christopher Bishop.]

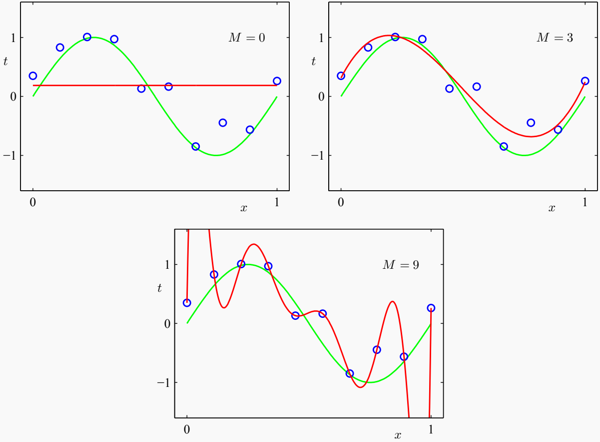
Polynomial Feature Mapping with $M = 9$

$$y = w_0 + w_1 x + w_2 x^2 + w_3 x^3 + \ldots + w_9 x^9$$



[Pattern Recognition and Machine Learning, Christopher Bishop.]

Model Complexity and Generalization

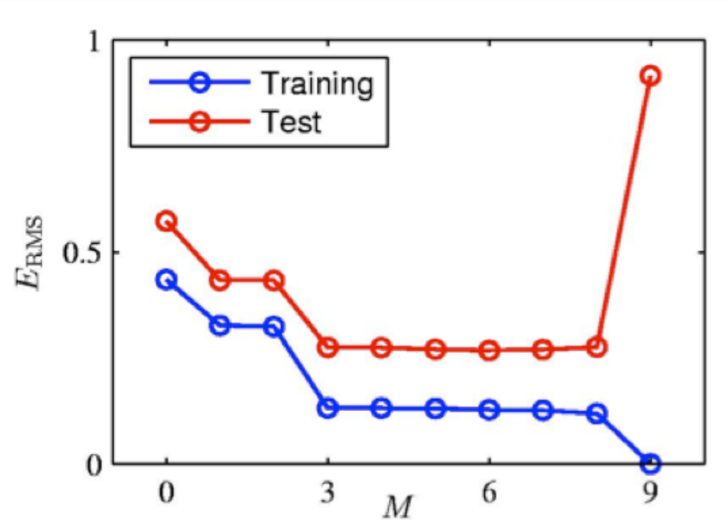


Under-fitting ($M = 0$): Model is too simple, doesn't fit data well.

Good model ($M = 3$): Small test error, generalizes well.

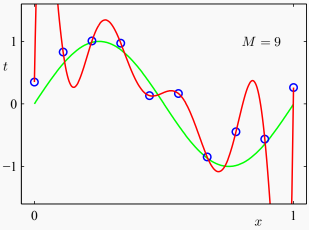
Over-fitting ($M = 9$): Model is too complex, fits data perfectly.

Model Complexity and Generalization



Model Complexity and Generalization

	$M=0$	$M=1$	$M=3$	$M=9$
$w_0^\star$	0.19	0.82	0.31	0.35
$w_1^\star$		-1.27	7.99	232.37
$w_2^\star$			-25.43	-5321.83
$w_3^\star$			17.37	48568.31
$w_4^\star$				-231639.30
$w_5^\star$				640042.26
$w_6^\star$				-1061800.52
$w_7^\star$				1042400.18
$w_8^\star$				-557682.99
$w_9^\star$				125201.43



- As M increases, the magnitude of coefficients gets larger.
- For $M=9$, the coefficients have become finely tuned to the data.
- Between data points, the function exhibits large oscillations.

Regularization

Controlling Model Complexity

How can we control the model complexity?

- A crude approach: restrict # of parameters / basis functions.
  For polynomial expansion, tune M using a validation set.

- Another approach: regularize the model.
  Regularizer is a function that quantifies how much we prefer one hypothesis vs. another.

L^2 (or ℓ_2) Regularization

- Encourage the weights to be small
  by choosing the ℓ_2 **penalty** as our regularizer.

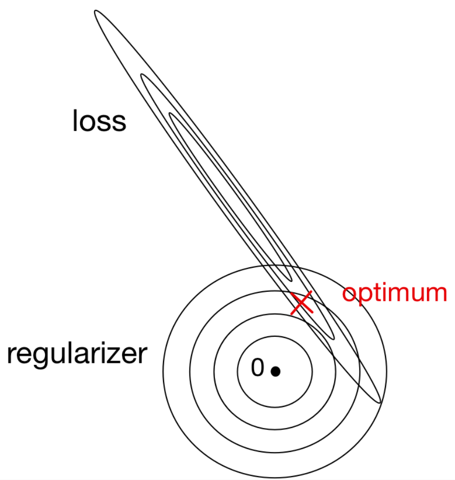
$$\mathcal{R}(\mathbf{w}) = \tfrac{1}{2}\|\mathbf{w}\|_2^2 = \frac{1}{2}\sum_j w_j^2.$$

- The regularized cost function makes a trade-off
  between the fit to the data and the norm of the weights.

$$\mathcal{J}_{\text{reg}}(\mathbf{w}) = \mathcal{J}(\mathbf{w}) + \lambda\mathcal{R}(\mathbf{w}) = \mathcal{J}(\mathbf{w}) + \frac{\lambda}{2}\sum_j w_j^2.$$

- If the model fits training data poorly, $\mathcal{J}$ is large.
  If the weights are large in magnitude, $\mathcal{R}$ is large.
- Large λ penalizes weight values more.
- Tune hyperparameter λ with a validation set.

L^2 Regularization Picture



L^2 Regularized Least Squares: Ridge regression

For the least squares problem, we have $\mathcal{J}(\mathbf{w}) = \frac{1}{2N}\|\mathbf{X}\mathbf{w} - \mathbf{t}\|^2$.

- When $\lambda > 0$ (with regularization), regularized cost gives

$$\begin{aligned}\mathbf{w}_\lambda^{\text{Ridge}} = \underset{\mathbf{w}}{\operatorname{argmin}}\, \mathcal{J}_{\text{reg}}(\mathbf{w}) &= \underset{\mathbf{w}}{\operatorname{argmin}}\, \frac{1}{2N}\|\mathbf{X}\mathbf{w} - \mathbf{t}\|_2^2 + \frac{\lambda}{2}\|\mathbf{w}\|_2^2 \\ &= (\mathbf{X}^\top\mathbf{X} + \lambda N\mathbf{I})^{-1}\mathbf{X}^\top\mathbf{t}\end{aligned}$$

- $\lambda = 0$ (no regularization) reduces to least squares solution!
- Can also formulate the problem as

$$\underset{\mathbf{w}}{\operatorname{argmin}}\, \frac{1}{2}\|\mathbf{X}\mathbf{w} - \mathbf{t}\|_2^2 + \frac{\lambda}{2}\|\mathbf{w}\|_2^2$$

  with solution

$$\mathbf{w}_\lambda^{\text{Ridge}} = (\mathbf{X}^\top\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^\top\mathbf{t}.$$

Gradient Descent under the L^2 Regularization

- Gradient descent update to minimize $\mathcal{J}$:

$$\mathbf{w} \leftarrow \mathbf{w} - \alpha \frac{\partial}{\partial \mathbf{w}} \mathcal{J}$$

- The gradient descent update to minimize the L^2 regularized cost $\mathcal{J} + \lambda \mathcal{R}$ results in weight decay:

$$\begin{aligned}
\mathbf{w} &\leftarrow \mathbf{w} - \alpha \frac{\partial}{\partial \mathbf{w}} (\mathcal{J} + \lambda \mathcal{R}) \\
&= \mathbf{w} - \alpha \left(\frac{\partial \mathcal{J}}{\partial \mathbf{w}} + \lambda \frac{\partial \mathcal{R}}{\partial \mathbf{w}} \right) \\
&= \mathbf{w} - \alpha \left(\frac{\partial \mathcal{J}}{\partial \mathbf{w}} + \lambda \mathbf{w} \right) \\
&= (1 - \alpha \lambda) \mathbf{w} - \alpha \frac{\partial \mathcal{J}}{\partial \mathbf{w}}
\end{aligned}$$

Conclusions

Linear regression exemplifies recurring themes of this course:

- choose a model and a loss function
- formulate an optimization problem
- solve the minimization problem
  using direction solution or gradient descent.
- vectorize the algorithm, i.e. represent in terms of linear algebra
- make a linear model more powerful using feature mappings
- improve the generalization by adding a regularizer

Deriving gradients for direct / iterative optimization for Ridge Regression

Cost $J(\omega) = \frac{1}{2N} \|X\omega - t\|^2 + \frac{\lambda}{2} \|\omega\|^2$ **Optimization problem** $\underset{\omega}{\operatorname{argmin}}\ J(\omega)$

$X \in \mathbb{R}^{N \times D}$

$\omega \in \mathbb{R}^{N \times 1}$

$t \in \mathbb{R}^{N \times 1}$

Gradient of Cost with respect to weights.

Rewrite $J(\omega) = \frac{1}{2N}(X\omega - t)^T(X\omega - t) + \frac{\lambda}{2} \omega^T \omega$

Eq (129) in matrix cookbook

$$\nabla_\omega J(\omega) = \frac{1}{2N} \cdot 2 \cdot X^T(X\omega - t) + \frac{\lambda}{2} \cdot 2 \cdot \omega$$

Eq (87) in matrix cookbook

$$= \frac{1}{N} X^T \underbrace{(X\omega - t)} + \frac{\lambda N}{N} \omega$$

Residual = how different is the prediction from the target

Direct Approach

set $\nabla_w J(w) = 0$ & solve for w

$$\nabla_w J(w) = 0 \Rightarrow \frac{1}{N} X^T X w + \frac{\lambda N}{N} w - \frac{X^T t}{N} = 0$$

$$\Rightarrow (X^T X + \lambda N I) w = X^T t$$

$$\Rightarrow \boxed{w = (X^T X + \lambda N I)^{-1} X^T t}$$

Solution to Ridge Regression