CSC 311: Introduction to Machine Learning

Lecture 2 - Decision Trees & Bias-Variance Decomposition

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University of Toronto, Fall 2024

Outline

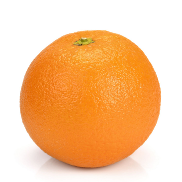
Introduction

Today

- Announcement: HW1 (will be) released this week
- Decision Trees
  - Simple but powerful learning algorithm
  - Used widely in Kaggle competitions
  - Lets us motivate concepts from information theory (entropy, mutual information, etc.)
- Bias-variance decomposition
  - Concept to motivate combining different classifiers.
- Ideas we will need in today's lecture
  - Trees [from algorithms]
  - Expectations, marginalization, chain rule [from probability]

Decision Trees

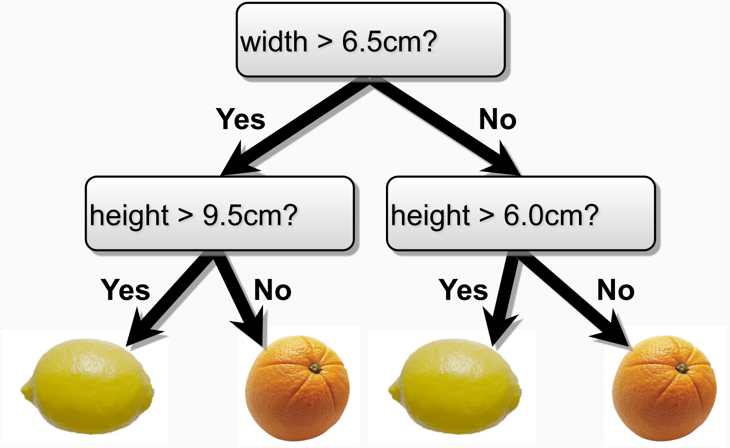
Lemons or Oranges

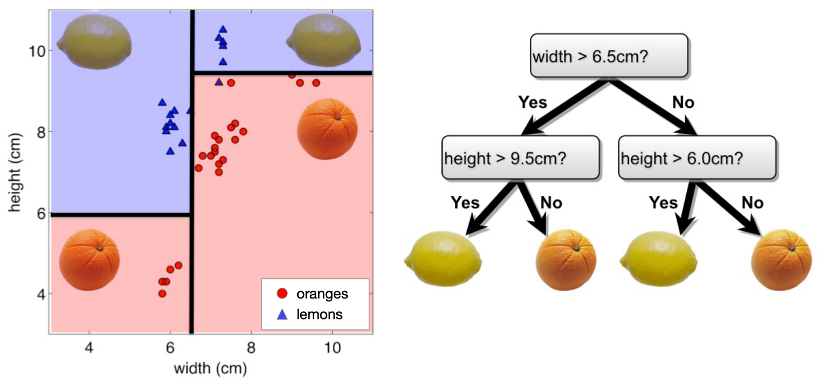
Scenario: You run a sorting facility for citrus fruits

- Binary classification: lemons or oranges
- Features measured by sensor on conveyor belt: height and width

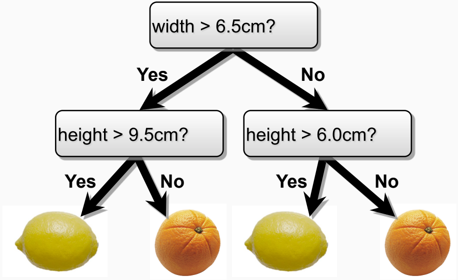
- Make predictions by splitting on features according to a tree structure.



- Split *continuous features* by checking whether that feature is greater than or less than some threshold.
- Decision boundary is made up of axis-aligned planes.



Decision Trees

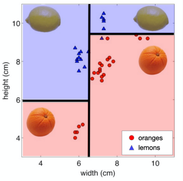


- Internal nodes test a feature
- Branching is determined by the feature value
- Leaf nodes are outputs (predictions)

Question: What are the hyperparameters of this model?

Decision Trees—Classification and Regression

- Each path from root to a leaf defines a region R_m of input space
- Let $\{(x^{(m_1)}, t^{(m_1)}), \ldots, (x^{(m_k)}, t^{(m_k)})\}$ be the training examples that fall into R_m
- $m = 4$ on the right



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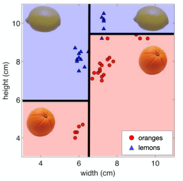
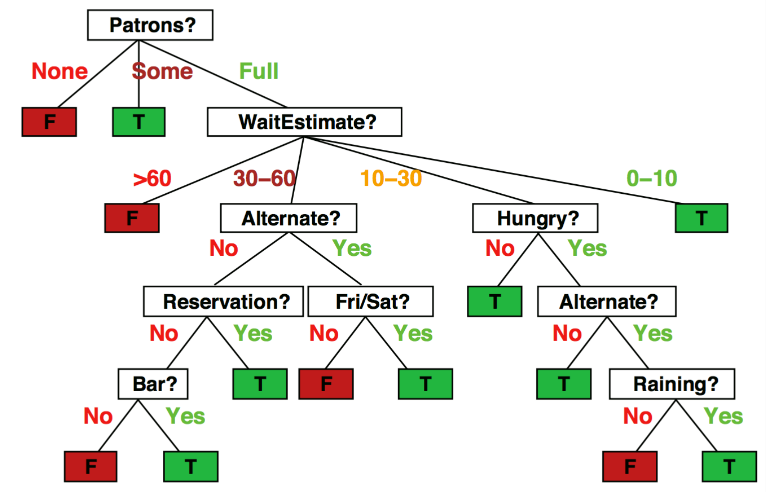


- Regression tree:
  - continuous output
  - leaf value y^m typically set to the mean value in $\{t^{(m_1)}, \dots, t^{(m_k)}\}$
- Classification tree (we will focus on this):
  - discrete output
  - leaf value y^m typically set to the most common value in $\{t^{(m_1)}, \dots, t^{(m_k)}\}$

Decision Trees—Discrete Features

- Will I eat at this restaurant?



Decision Trees—Discrete Features

- Split *discrete features* into a partition of possible values.

Example	Input Attributes										Goal
	Alt	*Bar*	*Fri*	*Hun*	*Pat*	*Price*	*Rain*	*Res*	*Type*	*Est*	*WillWait*
$\mathbf{x}_1$	Yes	No	No	Yes	Some	$$$	No	Yes	French	0–10	$y_1 =$ Yes
$\mathbf{x}_2$	Yes	No	No	Yes	Full	$	No	No	Thai	30–60	$y_2 =$ No
$\mathbf{x}_3$	No	Yes	No	No	Some	$	No	No	Burger	0–10	$y_3 =$ Yes
$\mathbf{x}_4$	Yes	No	Yes	Yes	Full	$	Yes	No	Thai	10–30	$y_4 =$ Yes
$\mathbf{x}_5$	Yes	No	Yes	No	Full	$$$	No	Yes	French	>60	$y_5 =$ No
$\mathbf{x}_6$	No	Yes	No	Yes	Some	$$	Yes	Yes	Italian	0–10	$y_6 =$ Yes
$\mathbf{x}_7$	No	Yes	No	No	None	$	Yes	No	Burger	0–10	$y_7 =$ No
$\mathbf{x}_8$	No	No	No	Yes	Some	$$	Yes	Yes	Thai	0–10	$y_8 =$ Yes
$\mathbf{x}_9$	No	Yes	Yes	No	Full	$	Yes	No	Burger	>60	$y_9 =$ No
$\mathbf{x}_{10}$	Yes	Yes	Yes	Yes	Full	$$$	No	Yes	Italian	10–30	$y_{10} =$ No
$\mathbf{x}_{11}$	No	No	No	No	None	$	No	No	Thai	0–10	$y_{11} =$ No
$\mathbf{x}_{12}$	Yes	Yes	Yes	Yes	Full	$	No	No	Burger	30–60	$y_{12} =$ Yes

Features:

1.	Alternate: whether there is a suitable alternative restaurant nearby.
2.	Bar: whether the restaurant has a comfortable bar area to wait in.
3.	Fri/Sat: true on Fridays and Saturdays.
4.	Hungry: whether we are hungry.
5.	Patrons: how many people are in the restaurant (values are None, Some, and Full).
6.	Price: the restaurant's price range ($, $$, $$$).
7.	Raining: whether it is raining outside.
8.	Reservation: whether we made a reservation.
9.	Type: the kind of restaurant (French, Italian, Thai or Burger).
10.	WaitEstimate: the wait estimated by the host (0-10 minutes, 10-30, 30-60, >60).

Implementing Decision Trees

- Step 1: Understand the problem (is it prediction, learning a good representation). Regression or classification
- Step 2: Formulate the problem mathematically (create notation for your inputs and outcomes and model). similar to KNN - vectorize inputs and labels
- Step 3: Formulate an objective function that represents success for your model.
- Let $\mathcal{D} = \{(\mathbf{x}^1, t^1), \ldots, (\mathbf{x}^N, t^N)\}$ be the training set, $\mathcal{T}$ be the space of valid decision trees and $y(\mathbf{x})$ be the label predicted by running the decision tree on an input.
- Objective: $\mathcal{L} = \min_{\mathcal{T}} \sum_{i=1}^{N} \mathbb{I}[y^i \neq t^i]$ is to minimize the number of misclassifications.
- Why is this difficult?

Hardness of learning Decision Trees

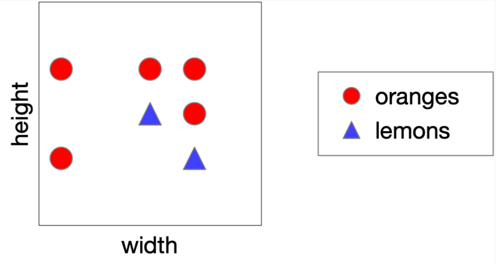
- Decision trees are universal function approximators.
  - For any training set we can construct a decision tree that has exactly the one leaf for every training point, but it probably won't generalize.
  - Example - If all D features were binary, and we had $N = 2^D$ unique training examples, a **Full Binary Tree** would have one leaf per example.
- Finding the smallest decision tree that correctly classifies a training set is NP complete.
  - If you are interested, check: Hyafil & Rivest'76.
- So, how do we construct a useful decision tree?

Learning Decision Trees

- Resort to a greedy heuristic:
  - Intuition: Do the sensible thing locally and then repeat!
  - Start with the whole training set and an empty decision tree.
  - Pick a feature and candidate split that would most reduce a loss
  - Split on that feature and recurse on subpartitions.
- What is a loss?
  - When learning a model, we use a scalar number to assess whether we're on track
  - Scalar value: low is good, high is bad
- Which loss should we use?

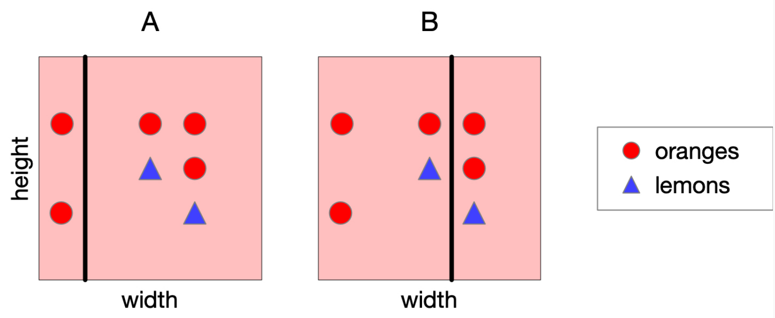
Choosing a Good Split

- Consider the following data. Let's split on width.
- Classify by majority.



Choosing a Good Split

- Which is the best split? Vote!



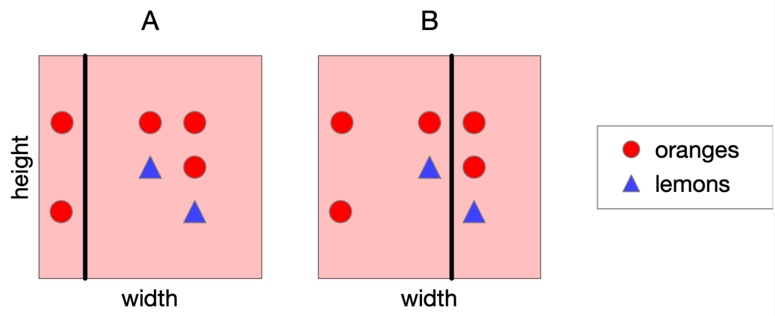
Probability in review

Three concepts you should page into memory for the next fifteen minutes:

- Expectation: $\mathbb{E}_x[f(x)] = \sum_{x \in X} p(x) f(x)$
- Chain rule of probabilities: $p(y|x)p(x) = p(x, y)$
- Marginalization of joint probabilities: $p(x) = \sum_y p(x, y)$

Choosing a Good Split

- A feels like a better split, because the left-hand region is very certain about whether the fruit is an orange.
- Can we quantify this?



Choosing a Good Split

- How can we quantify uncertainty in prediction for a given leaf node?
  - If all examples in leaf have same class: good, low uncertainty
  - If each class has same amount of examples in leaf: bad, high uncertainty
- **Idea:** Use counts at leaves to define probability distributions; use a probabilistic notion of uncertainty to decide splits.
- A brief detour through information theory...

Entropy - Quantifying uncertainty

- You may have encountered the term entropy quantifying the state of chaos in chemical and physical systems,
- In statistics, it is a property of a random variable,
- The entropy of a discrete random variable is a number that quantifies the uncertainty inherent in its possible outcomes.
- The mathematical definition of entropy that we give in a few slides may seem arbitrary, but it can be motivated axiomatically.
  - If you're interested, check: *Information Theory* by Robert Ash or Elements of Information Theory by Cover and Thomas.
- To explain entropy, consider flipping two different coins...

We Flip Two Different Coins

Each coin is a binary random variable with outcomes 1 or 0:

Sequence 1:

```
0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 ... ?
```

Sequence 2:

```
0 1 0 1 0 1 1 1 0 1 0 0 1 1 0 1 0 1 ... ?
```

We Flip Two Different Coins

Each coin is a binary random variable with outcomes 1 or 0:

Sequence 1:
0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 ... ?

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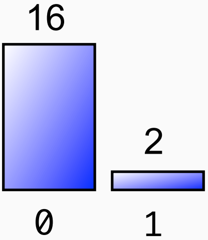
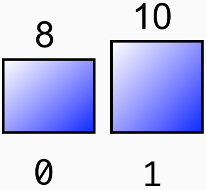
0 1 0 1 0 1 1 1 0 1 0 0 1 1 0 1 0 1 ... ?



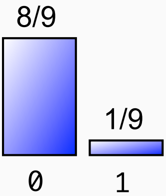
versus

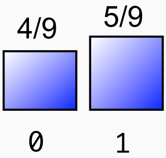


Quantifying Uncertainty

- The entropy of a loaded coin with probability p of heads is given by

$$-p\log_2(p) - (1-p)\log_2(1-p)$$



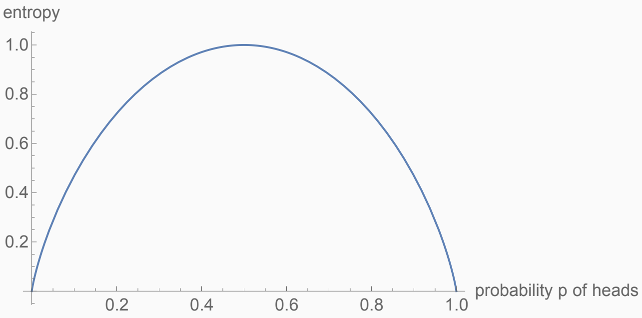


$$-\frac{8}{9}\log_2\frac{8}{9} - \frac{1}{9}\log_2\frac{1}{9} \approx \frac{1}{2} \qquad -\frac{4}{9}\log_2\frac{4}{9} - \frac{5}{9}\log_2\frac{5}{9} \approx 0.99$$

- Notice: the coin whose outcomes are more certain has a lower entropy.
- In the extreme case $p = 0$ or $p = 1$, we were certain of the outcome before observing. So, we gained no certainty by observing it, i.e., entropy is 0.

Quantifying Uncertainty

- Can also think of entropy as the expected information content of a random draw from a probability distribution.



- Claude Shannon showed: you cannot store the outcome of a random draw using fewer expected bits than the entropy without losing information.
- So units of entropy are bits; a fair coin flip has 1 bit of entropy.

Entropy

- More generally, the entropy of a discrete random variable Y is given by

$$H(Y) = -\sum_{y \in Y} p(y) \log_2 p(y)$$

- **"High Entropy"**:
  - Variable has a uniform like distribution over many outcomes
  - Flat histogram
  - Values sampled from it are less predictable

[Slide credit: Vibhav Gogate]

Entropy

- More generally, the **entropy** of a discrete random variable Y is given by

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- "High Entropy":
  - Variable has a uniform like distribution over many outcomes
  - Flat histogram
  - Values sampled from it are less predictable
- "Low Entropy"
  - Distribution is concentrated on only a few outcomes
  - Histogram is concentrated in a few areas
  - Values sampled from it are more predictable

[Slide credit: Vibhav Gogate]

Entropy

- Suppose we observe partial information X about a random variable Y
  - For example, $X = \mathrm{sign}(Y)$.
- We want to work towards a definition of the expected amount of information that will be conveyed about Y by observing X.
  - Or equivalently, the expected reduction in our uncertainty about Y after observing X.

Entropy of a Joint Distribution

- Example: $X = \{\text{Raining, Not raining}\}, Y = \{\text{Cloudy, Not cloudy}\}$

	Cloudy	Not Cloudy
Raining	24/100	1/100
Not Raining	25/100	50/100

$$
\begin{aligned}
H(X,Y) &= -\sum_{x\in X}\sum_{y\in Y} p(x,y)\log_2 p(x,y) \\
&= -\frac{24}{100}\log_2\frac{24}{100} - \frac{1}{100}\log_2\frac{1}{100} - \frac{25}{100}\log_2\frac{25}{100} - \frac{50}{100}\log_2\frac{50}{100} \\
&\approx 1.56\text{bits}
\end{aligned}
$$

Conditional Entropy

- Example: $X = \{\text{Raining, Not raining}\}, Y = \{\text{Cloudy, Not cloudy}\}$

	Cloudy	Not Cloudy
Raining	24/100	1/100
Not Raining	25/100	50/100

- What is the entropy of cloudiness Y, **given that it is raining**?

$$
\begin{aligned}
H(Y|X=x) &= -\sum_{y \in Y} p(y|x) \log_2 p(y|x) \\
&= -\frac{24}{25} \log_2 \frac{24}{25} - \frac{1}{25} \log_2 \frac{1}{25} \\
&\approx 0.24\text{bits}
\end{aligned}
$$

- We used: $p(y|x) = \frac{p(x,y)}{p(x)}$, and $p(x) = \sum_y p(x,y)$ (sum in a row)

Conditional Entropy

	Cloudy	Not Cloudy
Raining	24/100	1/100
Not Raining	25/100	50/100

- The expected conditional entropy:

$$\begin{aligned} H(Y|X) &= \mathbb{E}_x[H[Y|x]] \\ &= \sum_{x \in X} p(x) H(Y|X=x) \\ &= -\sum_{x \in X} \sum_{y \in Y} p(x,y) \log_2 p(y|x) \end{aligned}$$

Conditional Entropy

- Example: $X = \{\text{Raining, Not raining}\}$, $Y = \{\text{Cloudy, Not cloudy}\}$

	Cloudy	Not Cloudy
Raining	24/100	1/100
Not Raining	25/100	50/100

- What is the entropy of cloudiness, given the knowledge of whether or not it is raining?

$$\begin{aligned} H(Y|X) &= \sum_{x \in X} p(x) H(Y|X = x) \\ &= \frac{1}{4} H(\text{cloudy|is raining}) + \frac{3}{4} H(\text{cloudy|not raining}) \\ &\approx 0.75 \text{ bits} \end{aligned}$$

Conditional Entropy

- Some useful properties:
    - H is always non-negative
    - Chain rule: $H(X,Y) = H(X|Y) + H(Y) = H(Y|X) + H(X)$
    - If X and Y independent, then X does not affect our uncertainty about Y: $H(Y|X) = H(Y)$
    - But knowing Y makes our knowledge of Y certain: $H(Y|Y) = 0$
    - By knowing X, we can only decrease uncertainty about Y: $H(Y|X) \leq H(Y)$

Information Gain

	Cloudy	Not Cloudy
Raining	24/100	1/100
Not Raining	25/100	50/100

- How much more certain am I about whether it's cloudy if I'm told whether it is raining? My uncertainty in Y minus my expected uncertainty that would remain in Y after seeing X.
- This is called the information gain $IG(Y|X)$ in Y due to X, or the mutual information of Y and X

$$IG(Y|X) = H(Y) - H(Y|X) \tag{1}$$

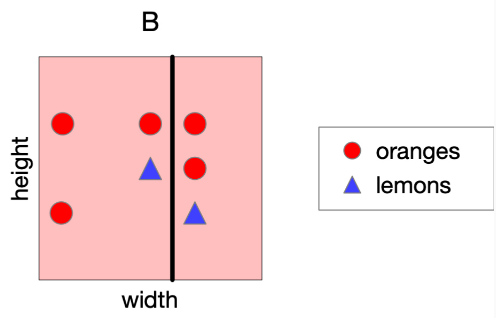
- If X is completely uninformative about Y: $IG(Y|X) = 0$
- If X is completely informative about Y: $IG(Y|X) = H(Y)$

Revisiting Our Original Example

- Information gain measures the informativeness of a variable, which is exactly what we desire in a decision tree split!
- The information gain of a split: how much information (over the training set) about the class label Y is gained by knowing which side of a split you're on.

Information Gain of Split B

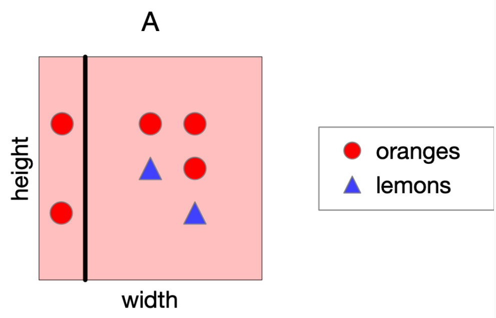
- What is the information gain of split B? Not terribly informative...



- Entropy of class outcome before split: $H(Y) = -\frac{2}{7}\log_2(\frac{2}{7}) - \frac{5}{7}\log_2(\frac{5}{7}) \approx 0.86$
- Conditional entropy of class outcome after split: $H(Y|left) \approx 0.81$, $H(Y|right) \approx 0.92$
- $IG(split) \approx 0.86 - (\frac{4}{7} \cdot 0.81 + \frac{3}{7} \cdot 0.92) \approx 0.006$

Information Gain of Split A

- What is the information gain of split A? Very informative!



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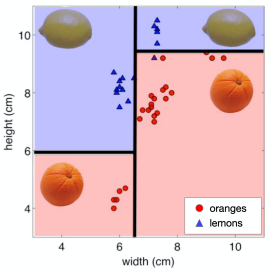
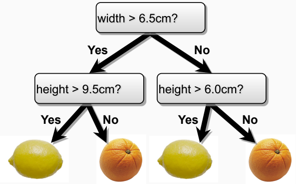
  $H(Y) = -\frac{2}{7}\log_2(\frac{2}{7}) - \frac{5}{7}\log_2(\frac{5}{7}) \approx 0.86$
- Conditional entropy of class outcome after split:
  $H(Y|left) = 0$, $H(Y|right) \approx 0.97$
- $IG(split) \approx 0.86 - (\frac{2}{7} \cdot 0 + \frac{5}{7} \cdot 0.97) \approx 0.17$!!

Constructing Decision Trees





- At each level, one must choose:
    1. Which feature to split.
    2. Possibly where to split it.
- Choose them based on how much information we would gain from the decision! (choose feature that gives the highest gain)

Decision Tree Construction Algorithm

- Simple, greedy, recursive approach, builds up tree node-by-node
    1. pick a feature to split at a non-terminal node
    2. split examples into groups based on feature value
    3. for each group:
        - if no examples – return majority from parent
        - else if all examples in same class – return class
        - else loop to step 1
- Terminates when all leaves contain only examples in the same class or are empty.
- Questions for discussion:
    - How do you choose the feature to split on?
    - How do you choose the threshold for each feature?

Back to Our Example

Example	Input Attributes										Goal
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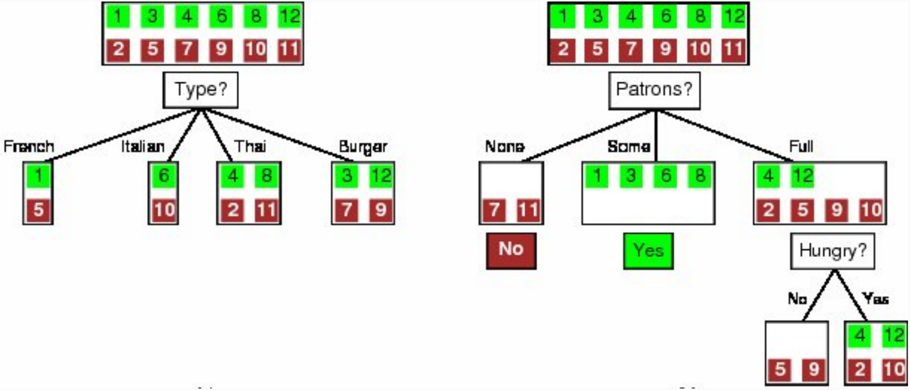
Features:

1.	Alternate: whether there is a suitable alternative restaurant nearby.
2.	Bar: whether the restaurant has a comfortable bar area to wait in.
3.	Fri/Sat: true on Fridays and Saturdays.
4.	Hungry: whether we are hungry.
5.	Patrons: how many people are in the restaurant (values are None, Some, and Full).
6.	Price: the restaurant's price range ($, $$, $$$).
7.	Raining: whether it is raining outside.
8.	Reservation: whether we made a reservation.
9.	Type: the kind of restaurant (French, Italian, Thai or Burger).
10.	WaitEstimate: the wait estimated by the host (0-10 minutes, 10-30, 30-60, >60).

[from: Russell & Norvig]

Feature Selection

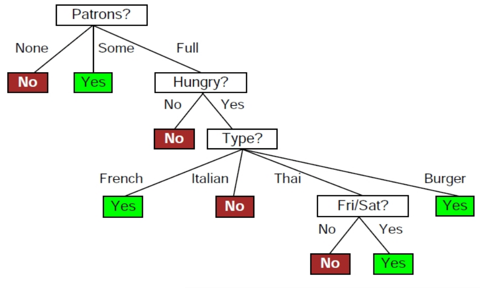


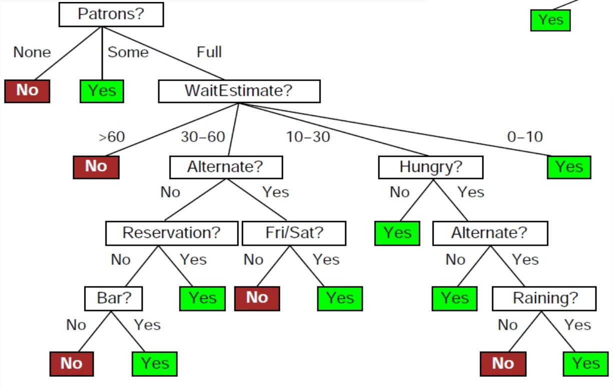
$$IG(Y) = H(Y) - H(Y|X)$$

$$IG(type) = 1 - \left[\frac{2}{12}H(Y|\text{Fr.}) + \frac{2}{12}H(Y|\text{It.}) + \frac{4}{12}H(Y|\text{Thai}) + \frac{4}{12}H(Y|\text{Bur.})\right] = 0$$

$$IG(Patrons) = 1 - \left[\frac{2}{12}H(0,1) + \frac{4}{12}H(1,0) + \frac{6}{12}H(\frac{2}{6},\frac{4}{6})\right] \approx 0.541$$

Which Tree is Better? Vote!





What Makes a Good Tree?

- Not too small: need to handle important but possibly subtle distinctions in data

What Makes a Good Tree?

- Not too small: need to handle important but possibly subtle distinctions in data
- Not too big:
  - Computational efficiency (avoid redundant, spurious attributes)
  - Avoid over-fitting training examples
  - Human interpretability

What Makes a Good Tree?

- Not too small: need to handle important but possibly subtle distinctions in data
- Not too big:
  - Computational efficiency (avoid redundant, spurious attributes)
  - Avoid over-fitting training examples
  - Human interpretability
- "Occam's Razor": find the simplest hypothesis that fits the observations
  - Useful principle, but hard to formalize (how to define simplicity?)
  - See Domingos, 1999, "The role of Occam's razor in knowledge discovery"

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- We desire small trees with informative nodes near the root

Steps to building decision trees

Below is a categorization of ML problems that you will see time, and time-again throughout this semester.

- Step 1: Understand the problem (is it prediction, learning a good representation).
- Step 2: Formulate the problem mathematically (create notation for your inputs and outcomes and model).
- Step 3: Formulate an objective function that represents success for your model.
- Step 4: Find a strategy to solve the optimization problem on pencil and paper. **Greedy algorithm to construct trees node by node**
- Step 5: Translate the algorithm into code. **Part of the homework excercise to translate this idea into code**
- Step 6: Analyze, iterate, improve design choices in your model and algorithm

Decision Tree Miscellany

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- Decision trees can also be used for regression on real-valued outputs. Choose splits to minimize squared error, rather than maximize information gain.

KNN versus Decision Trees

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Advantages of KNNs over decision trees

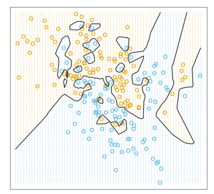
- Few hyperparameters
- Can incorporate interesting distance measures (e.g. shape contexts)

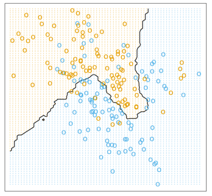
- We've seen many classification algorithms.
- We can combine multiple classifiers into an **ensemble**, which is a set of predictors whose individual decisions are combined in some way to classify new examples
  - E.g., (possibly weighted) majority vote
- For this to be nontrivial, the classifiers must differ somehow, e.g.
  - Different algorithm
  - Different choice of hyperparameters
  - Trained on different data
  - Trained with different weighting of the training examples
- Next lecture, we will study some specific ensembling techniques.

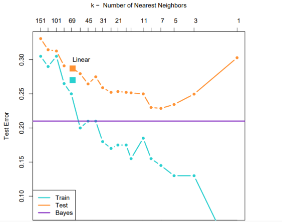
Bias-Variance Decomposition

- Today, we deepen our understanding of generalization through a bias-variance decomposition.
    - This will help us understand ensembling methods.
- What is generalization?
    - Ability of a model to correctly classify/predict from unseen examples (from the same distribution that the training data was drawn from).
    - **Why does this matter?** Gives us confidence that the model has correctly captured the right patterns in the training data and will work when deployed.

- Overly simple models underfit the data, and overly complex models overfit.
- We can quantify underfitting and overfitting in terms of the bias/variance decomposition.







Aside: Quick review of sampling

- Sampling is the process of drawing random variables from a distribution that describes its behavior.
- $x \sim \mathcal{N}(0, 1)$ (univariate sampling from a standard normal distribution). Empirical samples: $\{x^1, x^2, \ldots, x^N\}$, $x^i \in \mathbb{R}$
- $\mathbf{x} \sim \mathcal{N}(0, \Sigma)$ (multivariate sampling from a normal distribution with covariance Σ). Empirical samples: $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$, $\mathbf{x}^i \in \mathbb{R}^d$
- $y \sim \mathcal{N}(5x + 12, 1)$ (univariate sampling from a conditional distribution whose mean is conditional on input). Empirical (conditional) samples: $\{y^1, y^2, \ldots, y^N\}$ given $\{x^1, x^2, \ldots, x^N\}$, $x^i, y^i \in \mathbb{R}$

Aside: Quick review

- Previously, we knew what the distribution was and how they were parameterized.
- The samples are independent and identically distributed.
- For many phenomena, we may not know how data is distributed.
- Make assumptions on how data are distributed, we'll use ideas from statistics to better understand our model's generalization error.

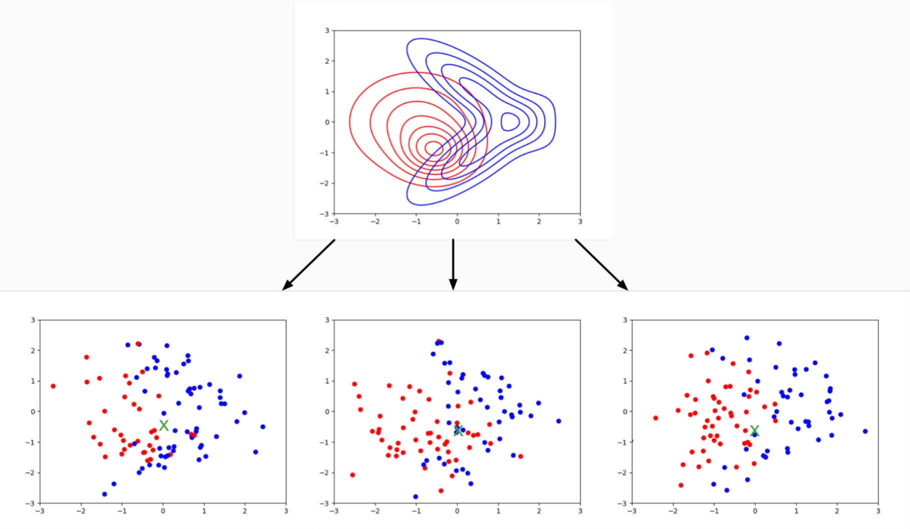
Basic Setup for Classification

- p_{sample} is a data generating distribution.
  For lemons and oranges, $p_{\text{sample}}(x, t)$ characterizes the true heights, widths, and labels.
- Think of this as the (true, but unknown) distribution of heights and widths of oranges and lemons in nature.
- Similarly we have the (true, but unknown) distribution of the target (orange or lemon) conditional on the heights and widths of the fruit nature: $p_{\text{target}}(t|x)$.
- We assume that the training set $\mathcal{D}$ consists of pairs $(\mathbf{x}_i, t_i)$ sampled
  independent and identically distributed (i.i.d.) from p_{sample}.
- We can sample lots of training sets independently from p_{sample}.

Basic Setup for Classification

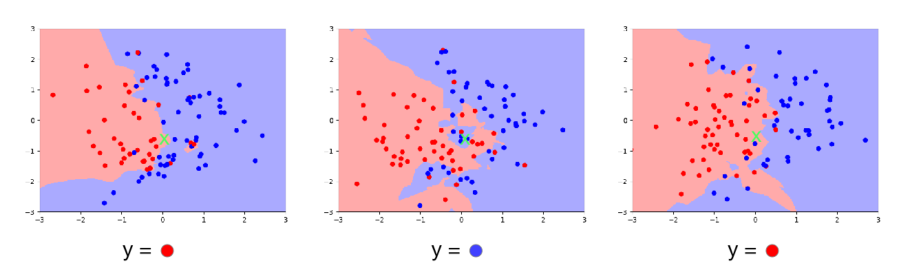
- How do we use the idea of a data generating distribution to understand generalization?
- Generalization is about model performance on a new point – lets pick one!
- Pick a fixed query point $\mathbf{x}$ (denoted with a green x).
  We want to get a prediction y at $\mathbf{x}$.

Basic Setup for Classification

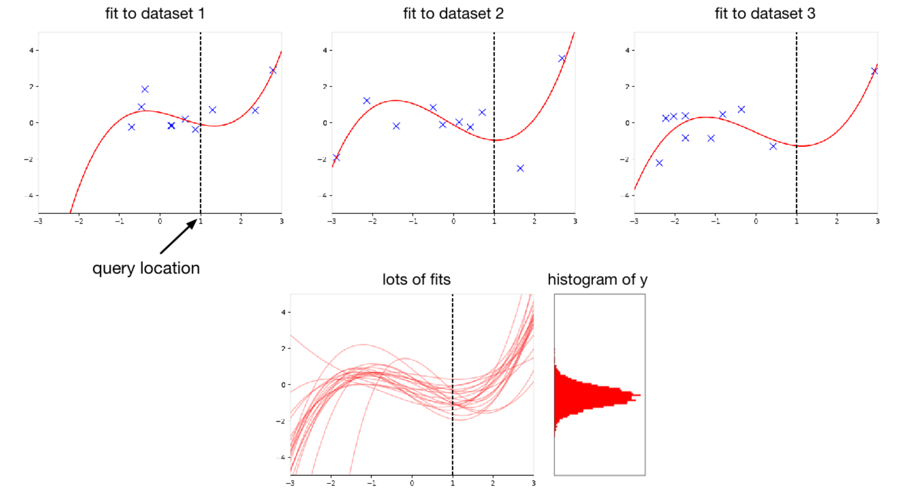


Basic Setup for Classification

- Run our (deterministic) learning algorithm on each training set, and compute its prediction y at the query point $\mathbf{x}$.
- We can view y as a random variable, where the randomness comes from the choice of training set.
- The classification accuracy is determined by the distribution of y.
- Since y is a random variable, we can compute its expectation, variance, etc.



Basic Setup for Regression



Basic Setup

- For a fixed query point $\mathbf{x}$, repeat:
  - Sample a random training set $\mathcal{D}$ i.i.d. from p_{sample}
  - Run the learning algorithm on $\mathcal{D}$ to get a prediction y at $\mathbf{x}$.
  - Sample the (true) target from the conditional distribution $p(t|\mathbf{x})$.
  - Compute the loss $L(y, t)$.

Comments:

- The random variable corresponding to the prediction y is independent of the t – Why?

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- We've made progress! We've precisely written down a mathematical expression corresponding to the generalization error that we incur!
- If our model has generalized, then it means the expected loss is low. When does this happen?

Choosing a prediction y

- For convenience we'll work in regression and assumed the following function to quantify the error in our prediction (square loss), $L(y, t) = \frac{1}{2}(y - t)^2$.
- Imagine that we knew the conditional distribution $p_{\text{target}}(t \mid \mathbf{x})$. What value of y should we predict?
    - Treat t as a random variable and choose y.

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- Imagine that we knew the conditional distribution $p_{\text{target}}(t \mid \mathbf{x})$. What value of y should we predict?
  - Treat t as a random variable and choose y.
- **Claim:** $y_\star = \mathbb{E}_{p_{\text{target}}(t \mid \mathbf{x})}[t \mid \mathbf{x}]$ is the best possible prediction.
- **Proof:**

$$
\begin{aligned}
\mathbb{E}_{p_{\text{target}}(t \mid \mathbf{x})}[(y-t)^2 \mid \mathbf{x}] &= \mathbb{E}[y^2 - 2yt + t^2 \mid \mathbf{x}] \\
&= y^2 - 2y\mathbb{E}[t \mid \mathbf{x}] + \mathbb{E}[t^2 \mid \mathbf{x}] \\
&= y^2 - 2y\mathbb{E}[t \mid \mathbf{x}] + \mathbb{E}[t \mid \mathbf{x}]^2 + \mathrm{Var}[t \mid \mathbf{x}] \\
&= y^2 - 2yy_\star + y_\star^2 + \mathrm{Var}[t \mid \mathbf{x}] \\
&= (y - y_\star)^2 + \mathrm{Var}[t \mid \mathbf{x}]
\end{aligned}
$$

Bayes Optimality

$$\mathbb{E}_{p(t \,|\, \mathbf{x})}[(y - t)^2 \,|\, \mathbf{x}] = (y - y_\star)^2 + \mathrm{Var}[t \,|\, \mathbf{x}]$$

- The first term is nonnegative, and can be made 0 by setting $y = y_\star$.
- The second term is the **Bayes error**, or
  the **noise** or inherent unpredictability of the target t.
  - An algorithm that achieves it is **Bayes optimal**.
  - This term doesn't depend on y.
  - Best we can ever hope to do with any learning algorithm.
- This process of choosing a single value y_* based on $p_{\mathrm{target}}(t \,|\, \mathbf{x})$ is an example of **decision theory**.

Decomposition Continued

- Now let's treat y as a random variable (where the randomness comes from the choice of dataset).
- We can decompose the expected loss further (suppressing the conditioning on $\mathbf{x}$ for clarity):

$$\begin{aligned}
\mathbb{E}_{\mathcal{D}}[\mathbb{E}_{p_{\text{target}}(t)}[(y-t)^2]] &= \mathbb{E}_{\mathcal{D}}[(y-y_\star)^2 + \mathrm{Var}(t)] \\
&= \mathbb{E}_{\mathcal{D}}[(y-y_\star)^2] + \mathrm{Var}(t) \\
&= \mathbb{E}_{\mathcal{D}}[y_\star^2 - 2y_\star y + y^2] + \mathrm{Var}(t) \\
&= y_\star^2 - 2y_\star \mathbb{E}_{\mathcal{D}}[y] + \mathbb{E}_{\mathcal{D}}[y^2] + \mathrm{Var}(t) \\
&= y_\star^2 - 2y_\star \mathbb{E}_{\mathcal{D}}[y] + \mathbb{E}_{\mathcal{D}}[y]^2 \\
&\quad + \underbrace{\mathbb{E}_{\mathcal{D}}[y^2] - \mathbb{E}_{\mathcal{D}}[y]^2}_{\mathrm{Var}(y)} + \mathrm{Var}(t) \\
&= \underbrace{(y_\star - \mathbb{E}_{\mathcal{D}}[y])^2}_{\text{bias}} + \underbrace{\mathrm{Var}(y)}_{\text{variance}} + \underbrace{\mathrm{Var}(t)}_{\text{Bayes error}}
\end{aligned}$$

Bayes Optimality

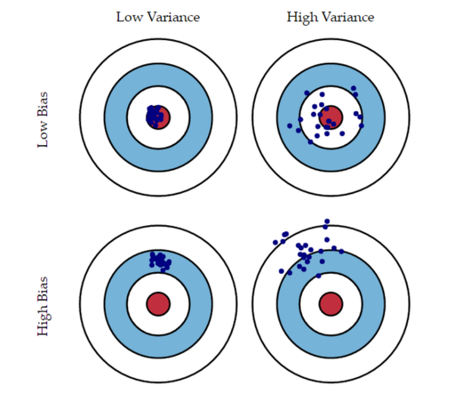
$$\mathbb{E}_{\mathcal{D}}[\mathbb{E}_{p(t)}[(y-t)^2]] = \underbrace{(y_\star - \mathbb{E}_{\mathcal{D}}[y])^2}_{\text{bias}} + \underbrace{\mathrm{Var}(y)}_{\text{variance}} + \underbrace{\mathrm{Var}(t)}_{\text{Bayes error}}$$

We split the expected loss into three terms:

- **bias**: how wrong the expected prediction is (corresponds to underfitting)
- **variance**: the amount of variability in the predictions (corresponds to overfitting)
- **Bayes error**: the inherent unpredictability of the targets

Bias and Variance

- Throwing darts = predictions for each draw of a dataset



- Be careful, what doesn't this capture?
    - We average over points $\mathbf{x}$ from the data distribution.