Math Diagnostic Quiz

Rahul G. Krishnan
CSC311 Fall 2023

September 7, 2023

This assignment will be graded for a good faith effort. To receive full marks, you must attempt all questions listed herein. If you find you are unable to move forward, you need to identify and write down the aspect of the question you find tricky/you are stuck on.

The main purpose is to help you decide whether you have the math background required for the course. This is not meant to be a difficult assignment but is meant to provide a simple reflection for how hard you may have to work to brush up on your foundational prerequisites for this course.

Markus Submission link: `https://markus.teach.cs.toronto.edu/2023-09/`. You will have to submit two files, one pdf with the written answers and another with the requested python script.

1 Linear Algebra

1.1

Consider a set of n-dimensional vectors $\{v_1, \ldots, v_n\}$.

(a) Explain the concept of linear independence in the context of these vectors.

(b) How would you test if they are linearly independent?

(c) What does it mean for these set of vectors to form a basis for $\mathbb{R}^n$? Specify two necessary conditions.

(d) If these vectors were the columns of a square n-dimensional matrix, what does their linear independence imply about the rank of the matrix?

1.2

Write down a three dimensional matrix whose eigenvectors spans $\mathbb{R}^2$.

1.3

Write down a matrix which has the following eigenvalues $-1, 5, 5, 3$.

2 Numerical Methods

2.1

Let $f(a, b)$ be a function of two variables. List two algorithms to find the minima of the function in the second argument (i.e. the minima of f with respect to b).

2.2

Let $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. What are the gradient and Hessian of $x^T Ax$ with respect to x? (Assume A is symmetric).

2.3

The gradient of $f(a, b)$ is $f'(a, b)$. If $f'(a*, b*) = 0$ and $f'(a, b) > 0; \forall a \neq a*, b \neq b*$, what does $a*, b*$ represent for the function f?

3 Probability Theory

3.1

Consider a coin that lands heads with probability p. The entropy of a coin flip is defined as :

$$-p \log_2 p - (1 - p) \log_2(1 - p)$$

. What value of p maximizes the entropy (provide a few line proof of the same)?

What does this result provide in terms of an intuitive explanation of what entropy of a random variable quantifies.

3.2

For each scenario, a random variable is identified. Describe what distribution you might choose to model the behavior of that random variable. There may be more than one right answer for each of the three subparts. (a) You are throwing a from the top floor of a building onto twelve buckets below. The marble always lands in a bucket. The random variable is the location where the marble lands. (b)You are waiting at a bus stop for the next bus. The random variable is the time taken for the bus to arrive since you started waiting. (c) You are throwing darts on an infinite dart board. The random variable is the location where the dart lands.

4 Mathematical Programming

Write a python script to generate two random matrices, multiply them and check whether their result is identical. Your code should use the following three functions in python.

The first function uses loops to multiply two matrices of dimension $N \times M$ and $M \times K$
`def multiply_with_loops(a=matrix1, b=matrix2)`

The second function uses numpy to multiple two matrices of dimension $N \times M$ and $M \times K$
`def multiply_with_numpy(a=matrix1, b=matrix2)`

The third function `def check_equal(a=matrix1, b=matrix2)` returns True if two $N \times K$ matrices are identical (up to numerical precision) and False otherwise.

The script should generate two random matrices of dimensions $N = 3, M = 2, K = 4$, multiply them using the two aforementioned functions and call the third function to check if their result is identical.